CHAPTER VI.

FUNCTIONS OF A COMPLEX VARIABLE.

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ART. 1. DEFINITION OF FUNCTION.

If two or more quantities are such that no one of them, when any values whatsoever are assigned to the others, suffers any restriction in regard to the values which it can assume the quantities are said to be "independent."

If one quantity is so related to another quantity or to several independent quantities, that whenever particular values are assigned to the latter, the former is required to take one or another of a system of completely determined values, the former is said to be a "function" of the latter. The quantity or quantities upon the values of which the value of the function depends, are said to be the "independent variables" of the function.

A function is "one-valued" when to every set of values assigned to the independent variables there corresponds but one value of the function. It is said to be "n-valued" when to every set of values of the independent variables n values of the function correspond.

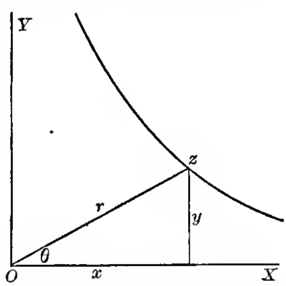
The "Theory of Functions" has among its objects the study of the properties of functions, their classification according to their properties, the derivation of formulas which exhibit the relations of functions to one another or to their independent variables, and the determination whether or not functions exist satisfying assigned conditions.

ART. 2. REPRESENTATION OF COMPLEX VARIABLE.

A variable quantity is capable, in general, of assuming both real and imaginary values. In fact, unless it be otherwise specified, every quantity w is to be regarded as having the "complex" form $u + v\sqrt{-1}$, u and v being real. It is customary to denote $\sqrt{-1}$ by i, and to write the preceding quantity thus: $u + iv$. If v is zero, w is real; if u is zero, w is a "pure imaginary."

A quantity $z = x + iy$ is said to vary "continuously" when between every pair of values which it takes, $c_1 = a_1 + ib_1$ and $c_2 = a_2 + ib_2$, the value of z varies in such a manner that x and y pass through all real values intermediate to a_1 and a_2, b_1 and b_2, respectively.

It is usual to give to a variable quantity $z = x + iy$ a graphical representation by drawing in a plane a pair of rectangular axes and constructing a point whose abscissa and ordinate are respectively equal to x and y. To every value of z will correspond a point; and, conversely, to every point will correspond a value of z. The terms "point" and value, then, may be interchanged without confusion. When z varies continuously the graphical representation of its variation, or its "path," will be a continuous line. This graphical representation is of the highest importance. By means of it some of the most complicated propositions may be given an exceedingly condensed and concrete expression.



By putting $x = r\cos\theta$, $y = r\sin\theta$, where r is a positive real quantity, the point

$$z = r(\cos\theta + i\sin\theta)$$

is referred to polar coördinates. The quantity r is called the absolute value or "modulus" of z. It is often written $|z|$. θ is known as the "argument" of z.

A function is sometimes considered for only such values of each independent variable as are represented graphically by the points of a certain continuous line. In the study of functions of real variables, for example, the path of each variable is represented by a straight line, namely, the axis of real quantities, or $y = 0$.

ART. 3. ABSOLUTE CONVERGENCE.

The representation of functions by means of infinite series is one of the most important branches of the theory of functions. In many problems, in fact, it is only by means of series that it is possible to determine functions satisfying the conditions assigned and to obtain the required numerical results. Frequent use will be made of the following theorem.

Theorem.—If the moduli of the terms of a series form a convergent series, the given series is convergent.

Let the given series be $W = w_0 + w_1 + \ldots + w_n + \ldots$ in which $w_0 = r_0(\cos\theta_0 + i\sin\theta_0)$, $w_1 = r_1(\cos\theta_1 + i\sin\theta_1)\ldots$ By hypothesis the series $R = r_0 + r_1 + \ldots + r_n + \ldots$ is convergent. Its terms being all positive, the sum of its first m terms constantly increases with m, but in such a manner as to approach a limit. The same will be true necessarily of any series formed by selecting terms from R. The sum of the first m terms of the series W is composed of two parts,

$$r_0 \cos\theta_0 + r_1 \cos\theta_1 \quad \ldots + r_{m-1} \cos\theta_{m-1},$$
$$i(r_0 \sin\theta_0 + r_1 \sin\theta_1 + \ldots + r_{m-1} \sin\theta_{m-1}),$$

and each of these in turn may be divided into parts which have all their terms of the same sign. Every one of the four parts thus obtained approaches a limit as m is increased; for the terms of each part have the same sign, and cannot exceed, in absolute value, the corresponding terms of R. Hence, as m is increased, the sum of the first m terms of W approaches a limit; which was to be proved.

A series, the moduli of whose terms form a convergent series, is said to be "absolutely convergent."

Prob. 1. Show that the series $1 + z + z^2 + \ldots + z^n + \ldots$ is absolutely convergent, if $|z| < 1$.

ART. 4. ELEMENTARY FUNCTIONS.

In elementary mathematics the functions are usually considered for only real values of the independent variables. In the case of the algebraic functions, however, there is no difficulty in assuming that the independent variables are complex. The theory of elimination shows that every algebraic equation can be freed from radicals. Every algebraic function, therefore, is defined by an equation which may be put in a form wherein the second member is zero and the first member is rational and entire in the function and its independent variables.

Besides the algebraic functions, the functions most often occurring in elementary mathematics are the trigonometric and exponential functions and the functions inverse to them. The definitions, by which these functions are generally first introduced, have no significance in the case where the independent variables are complex. However, the following familiar series,

$$e^z = \exp z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots,$$

$$\cos z = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots,$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

which have been established for the case where the variables are real, furnish most convenient general definitions for $\exp z$, $\cos z$, and $\sin z$, these series being absolutely convergent for every finite value of z. Defining the logarithmic function by the equation

$$e^{\log z} = \exp(\log z) = z,$$

it follows that

$$a^z = e^{z \log a} = \exp(z \log a).$$

The following equations also are to be regarded as equations of definition:

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = \frac{1}{\cos z}, \qquad \operatorname{cosec} z = \frac{1}{\sin z}.$$

It may be shown that the formulas which are usually obtained on the supposition that the independent variables are real, and which express in that case properties of and relations between the preceding functions, still hold when the independent variables are complex.

Prob. 2. Show that $e^m e^n = e^{m+n}$, m and n being complex.

Prob. 3. Deduce $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$, $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$.

Prob. 4. Deduce $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$,
$\sin(z_1 + z_2) = \cos z_1 \sin z_2 + \sin z_1 \cos z_2$.

ART. 5. CONTINUITY OF FUNCTIONS.

Let a function of a single independent variable have a determinate value for a given value c of the independent variable. If, when the independent variable is made to approach c, whatever supposition be made as to the method of approach, the function approaches as a limit its determinate value at c, the function is said to be "continuous" at c.

This definition may be otherwise expressed as follows: A function of a single independent variable is continuous at the point c, when, being given any positive quantity ϵ, it is possible to construct a circle, with center at c and radius equal to a determinate quantity δ, so small that the modulus of the difference between the value of the function at the center and that at every other point within the circle is less than ϵ.

A function of several independent variables is said to be continuous for a particular set of values assigned to those variables, when it takes for that set of values a determinate value $\bar{c}$, and for every new set of values, obtained by altering the variables by quantities of moduli less than some determinate positive quantity δ, the value of the function is altered by a quantity of modulus less than any previously chosen arbitrarily small positive quantity ϵ.

A function of one independent variable is said to be continuous in a given region of the plane upon which its independent variable is represented, if it is contiuuous at every point in that region.

From the principles of limits, it follows that if two functions are continuous at a given point, their sum, difference, and product are continuous at that point. As an immediate consequence, every rational entire function of z is continuous at every finite point; for every such function can be constructed from z and constant quantities by a finite number of additions, subtractions, and multiplications.

Let a function of a single independent variable be continuous at c, and let it take at that point the value t, different from zero. Suppose also that at any other point $c + \Delta c$ the function takes the value $t + \Delta t$. Then

$$\frac{1}{t + \Delta t} - \frac{1}{t} = -\frac{\Delta t}{t(t + \Delta t)}.$$

If it be assumed that $|\Delta t| < |t|$, the modulus of the preceding difference cannot exceed

$$\frac{|\Delta t|}{|t|(|t| - |\Delta t|)},$$

and will, therefore, be less than ϵ if

$$|\Delta t| < \frac{\epsilon |t|^2}{1 + \epsilon |t|}.$$

Hence if a function is continuous and different from zero at a point c, its reciprocal is also continuous at c. It follows at once that if two functions are both continuous at c, their ratio is continuous at c, unless the denominator reduces to zero at that point. But every rational function of z may be expressed as the ratio of two entire functions. It is therefore continuous for all values of z except those for which its denominator vanishes.

Consider the function $\exp z$,

$$e^{z+\Delta z} - e^z = e^z(e^{\Delta z} - 1) = e^z\left(\Delta z + \frac{\Delta z^2}{2!} + \ldots\right).$$

Hence if $$|\Delta z| < 1,$$

$$|e^{z+\Delta z} - e^z| \overline{\gtrless} |e^z|\left(|\Delta z| + \frac{|\Delta z|^2}{2!} + \ldots\right) \overline{\gtrless} |e^z| \frac{|\Delta z|}{1 - |\Delta z|},$$

but the limit of the second member is zero when $|\Delta z|$ approaches zero. Hence $\exp z$ is continuous for all finite values of z.

Prob. 5. Show that $\cos z$ and $\sin z$ are continuous for all finite values of z.

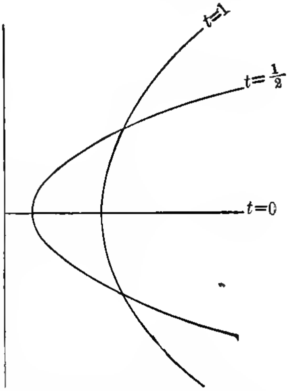
Prob. 6. Show that $\tan z$ is continuous in any circle described about the origin as a center with a radius less than $\frac{1}{2}\pi$.

ART. 6. GRAPHICAL REPRESENTATION OF FUNCTIONS.

It was shown in Art. 2 that a plane suffices for the complete graphical representation of the values of an independent variable. In the same way it is convenient to use a second plane to represent graphically the values of any one-valued function. For example, if $w = f(z)$ be such a function, to each point $x + iy$ of the independent variable will correspond a point $u + iv$ of the function. This point $u + iv$ is called the "image" of the point $x + iy$. If w is a continuous function of z, then every continuous curve in the z-plane will have an image in the w-plane, and this image will be also a continuous curve.

Consider the expression $u + iv = x^2 + y^2 + 2ixy$. Here

$u = x^2 + y^2$ and $v = 2xy$. Since to every value of z correspond determinate values of x and y, and consequently determinate values of u and v, this expression falls under the general definition of a function of z. It is evidently continuous. Every straight line $x = t$ parallel to the axis of y is converted by means of it into a parabola $v^2 = 4t^2(u - t^2)$.



Prob. 7. Find the family of curves into which the straight lines parallel to the axis of y are converted by means of the function $u + iv = x^2 - y^2 + 2ixy$. Show that no two curves of this family intersect.

ART. 7. DERIVATIVES.

Let $w = f(z)$ be a given function of z. If h is an "infinitesimal," that is, a variable having zero as its limit, and if the expression

$$\frac{f(z + h) - f(z)}{h}$$

has a finite determinate limit, remaining the same under all possible suppositions as to the way in which h approaches zero, this limit is said to be the "derivative" of the function $f(z)$ at the point z. In this case $w = f(z)$ is said to be "monogenic" at z. The derivative is written $f'(z)$ or $\frac{dw}{dz}$. A function is said to be monogenic in a region of the plane of the independent variable if it is monogenic at every point of that region.

Consider now the circumstances under which a function $w = u + iv$ may have a derivative at the point $z = x + iy$. If z be given a real increment, x is changed into $x + \Delta x$, while y is unaltered, so that $\Delta z = \Delta x$; and

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{\Delta x} + i\frac{\Delta v}{\Delta x}.$$

If, on the other hand, z is given a purely imaginary increment, $\Delta z = i\Delta y$, and

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u}{i\Delta y} + \frac{\Delta v}{\Delta y}.$$

If the second members of these equations approach determinate limits as Δx and Δy approach zero, and if these limits are equal,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Hence, equating real and imaginary parts,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y},$$

which are necessary conditions for the existence of a derivative.

It can be shown that these conditions are also sufficient. For let the increment of the independent variable be entirely arbitrary, no supposition being made as to the relative magnitudes of its real and imaginary parts. Then the differential of the function, that is, that part of the increment of the function which remains after subtracting the terms of order higher than the first, is

$$du + idv = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy.$$

Hence

$$\frac{du + idv}{dx + idy} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{dy}{dx}}{1 + i\frac{dy}{dx}},$$

which, by virtue of the conditions written above, is equal to either member of the equation

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

The value thus obtained is independent of $\frac{dy}{dx}$, or, what is the same thing, of the direction of approach to the point z. The existence of a derivative of the function w depends, therefore, only on the existence of partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ satisfying the specified equations of conditions.

The same equations of condition express the fact that $w = u + iv$, supposed to be an analytical expression involving x and y, involves z as a whole, that is, may be constructed from z by some series of operations, not introducing x or y except in the combination $x + iy$. In other words, they indicate that x and y may both be eliminated from $w = \phi(x, y)$ by means of the equation $z = x + iy$. This property might have been used to define monogenic function, but such a definition would have had the disadvantage of assuming a priori that the function was capable of analytical expression in terms of the independent variable.

A monogenic function is necessarily continuous; that is, the existence of a derivative involves continuity. For, if

$$\text{limit} \frac{f(z+h) - f(z)}{h} = f'(z),$$

it follows that

$$f(z+h) = f(z) + h\,[f'(z) + \eta],$$

where η approaches zero with h. Hence $f(z)$ is the limit of $f(z+h)$ when h approaches zero, or $f(z)$ is continuous at the point z.

The following pages relate almost exclusively to functions which are monogenic except for special isolated values of z. Functions which are discontinuous for every value of the independent variable, and functions which are continuous but admit no derivatives, have been little studied except in the case of real variables.*

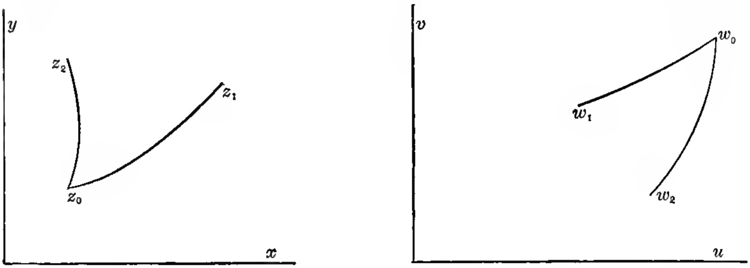
* In this connection see G. Darboux, Sur les fonctions discontinues, Annales de l'École Normale, Series 2, Vol. 4 (1875), pp. 51–112. For a systematic treatment of functions of a real variable, see the German translation of Dini's treatise by Lüroth and Schepp, Leipzig, 1892.

ART. 8. CONFORMAL REPRESENTATION.

Let z start from the point z_0 and trace two different paths forming a given angle at the point z_0, and let z_1 and z_2 be arbitrary points on the first and second paths respectively. Then

$$z_1 - z_0 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1 e^{i\theta_1},$$

where r_1 denotes the length of the straight line joining z_0 and



z_1, and θ_1 denotes the inclination of this line to the axis of reals. In the same way, for the point z_2, there is an equation

$$z_2 - z_1 = r_2(\cos\theta_2 + i\sin\theta_2) = r_2 e^{i\theta_2}.$$

If now w is a one-valued monogenic function of z, in the region of the z-plane considered, to the points z_0, z_1, z_2 correspond points w_0, w_1, w_2; and for these points can be formed the equations

$$w_1 - w_0 = \rho_1 e^{i\theta_1}, \quad w_2 - w_0 = \rho_2 e^{i\theta_2}.$$

From the supposition that w is monogenic, it follows at once that, when z_1 and z_2 are assumed to approach z_0,

$$\text{limit}\ \frac{w_1 - w_0}{z_1 - z_0} = \text{limit}\ \frac{w_2 - w_0}{z_2 - z_0}.$$

If the members of this equation are not equal to zero, it may be put in the form

$$\text{limit}\ \frac{w_1 - w_0}{w_2 - w_0} = \text{limit}\ \frac{z_1 - z_0}{z_2 - z_0},$$

or

$$\text{limit } \frac{\rho_1}{\rho_2} e^{i(\phi_1 - \phi_2)} = \text{limit } \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}.$$

Hence

$$\text{limit } (\phi_1 - \phi_2) = \text{limit } (\theta_1 - \theta_2);$$

and the images in the w-plane of the two paths traced by z form at w_0 an angle equal to that at z_0 in the z-plane. Accordingly, if z be supposed to trace any configuration whatever in a portion of the z-plane in which $\frac{dw}{dz}$ is determinate and not equal to zero, every angle in the image traced by w will be equal to the corresponding angle in the z-plane. If, for example, such a portion of the z-plane be divided into infinitesimal triangles, the corresponding portion of the z-plane will be divided in the same manner, and the corresponding triangles will be mutually equiangular. Such a copy upon a plane, or upon any surface, of a configuration in another surface is called a "conformal representation."

The modulus of the derivative $\left|\frac{dw}{dz}\right| = \text{limit } \left|\frac{\Delta w}{\Delta z}\right|$ is the "magnification." Its value, which, in general, changes from point to point, may be obtained from the relations

$$\left|\frac{dw}{dz}\right|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

The theory of conformal representation has interesting applications to map drawing.*

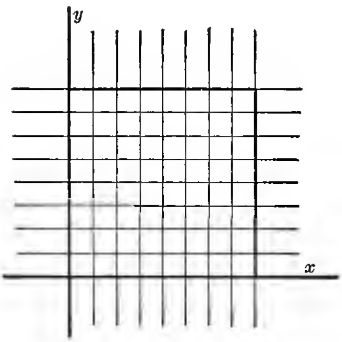
* For the literature of the subject, see Forsyth, Theory of Functions, p. 500, and Holzmüller, Einführing in die Theorie der isogonalen Verwandschaften und der conformen Abbildungen, verbunden mit Anwendungen auf mathematische Physik.

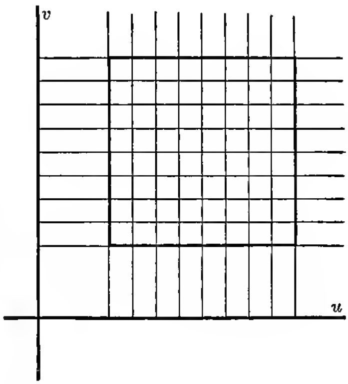
ART. [illegible]. EXAMPLES OF CONFORMAL REPRESENTATION.

Case I.—Let $w = z + c$. This function is formed from the independent variable by the addition of a constant. Putting for w, z, and c, respectively, $u + iv$, $x + iy$ and $a + ib$, one obtains

$$u = x + a, \quad v = y + b.$$

Any configuration in the z-plane appears, therefore, in the w-plane unaltered in magnitude, and is situated with respect to the axes as if it had been moved parallel to the axis of reals through the distance a and parallel to the axis of imaginaries through the distance b. The following diagrams represent the transformation of a network of squares by means of the relation $w = z + c$.

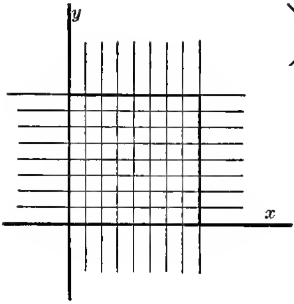


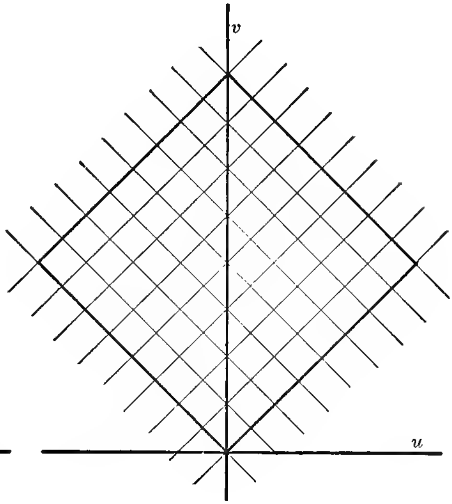


Case II. — Let $w = cz$. Writing $w = \rho e^{i\phi}$, $z = re^{i\theta}$, and $c = r_1 e^{i\theta_1}$, the following equations result :

$$\rho = r_1 r, \quad \phi = \theta_1 + \theta.$$

The origin transforms into the origin, all distances measured from the origin are multiplied by a constant quantity, and all straight lines passing through the origin are turned through a constant angle. See the following diagrams.





Case III.—Let $w = e^z$. Writing $z = x + iy$, the function becomes

$$w = e^x e^{iy} = e^x(\cos y + i \sin y).$$

Every straight line $x = t_1$ parallel to the axis of y is transformed into a circle $\rho = e^{t_1}$ described about the origin as a center, the axis of y becoming the unit circle. Points to the right of the axis of y fall without the unit circle, while points to the left of this axis fall within. Every straight line $y = t_2$ parallel to the axis of x becomes a straight line $v/u = \tan t_2$ passing through the origin. The accompanying diagrams* exhibit in a simple manner the periodicity expressed by the equation

$$\exp(z + 2n\pi i) = \exp(z),$$

where n is any positive or negative integer.

To every point in the w-plane, excluding the origin, correspond an infinite number of points in the z-plane. These points are all situated on a straight line parallel to the axis of

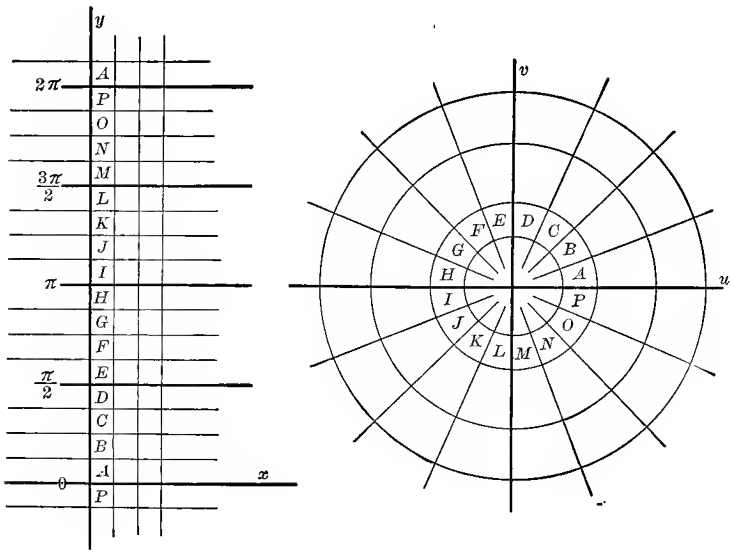
* The figures of this and the following example are taken from Holzmüller's treatise.

y, and divide it into segments, each of length 2π. If z' be one of these points, the general value of the inverse function is

$$\log w = z' + 2ni\pi,$$

where n is any positive or negative integer.

If any straight line beginning at the origin be drawn in the w-plane, there will correspond in the z-plane an infinite number



of straight lines parallel to the axis of x, dividing that plane into strips of equal width. To any curve in the w-plane which does not meet the line just drawn, will correspond in the z-plane an infinite number of curves, of which there will be one in each strip.

Case IV.—Let $w = \cos z$. Writing $w = u + iv, z = x + iy$, and employing as equations of definition $\cos(iy) = \cosh y$, $\sin(iy) = i \sinh y$, the given function takes the form

$$u + iv = \cos x \cosh y - i \sin x \sinh y.$$

Hence $$u = \cos x \cosh y, \quad v = - \sin x \sinh y.$$

Any straight line, $x = t_1$, parallel to the axis of y, is transformed into one branch of a hyperbola,

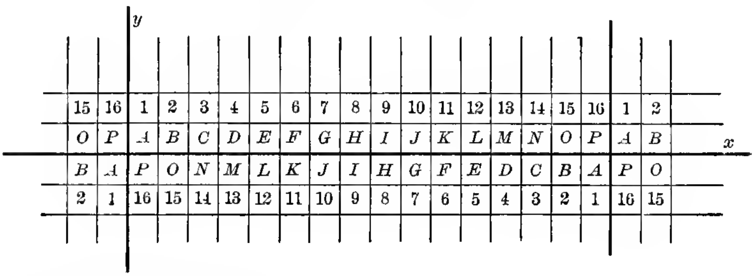
$$\frac{u^2}{\cos^2 t_1} - \frac{v^2}{\sin^2 t_1} = 1,$$

having its foci at the points $+1$ and -1. Any straight line, $y = t_2$, parallel to the axis of x, is transformed into an ellipse,

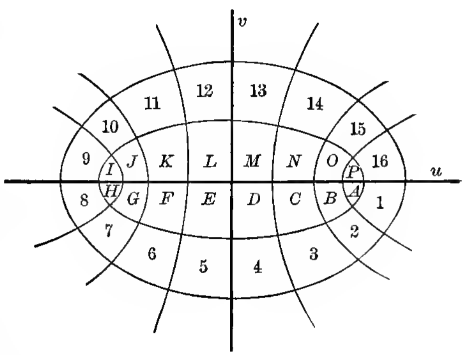
$$\frac{u^2}{\cosh^2 t_2} + \frac{v^2}{\sinh^2 t_2} = 1,$$

having its foci at the same points, any segment of the straight line equal in length to 2π corresponding to the entire curve taken once. By means of these confocal conics, the w-plane is divided into curvilinear rectangles, the conformal representation breaking down only at the foci, where the condition that $\frac{du}{dz}$ should be different from zero is not fulfilled. The periodicity of the function, expressed by the equation

$$\cos(z + 2\pi) = \cos z,$$



is exhibited graphically in the accompanying diagrams.



It is interesting to note in this example, as also in the preceding one, that the conformal representation introduces well-known systems of curvilinear coordinates, the cartesian coordinates, x, y of a point in the

z-plane serving to determine its image in the w-plane as an intersection of orthogonal curves.

Case V.—Let $w = z^3$. Writing $w = u + iv$, $z = x + iy$, the relations

$$u = x^3 - 3xy^2, \quad v = 3x^2y - y^3$$

follow at once. If one of the variables x, y be eliminated from these two equations by means of the equation $lx + my + n = 0$, representing a straight line in the z-plane, equations are obtained representing a unicursal cubic in the w-plane.

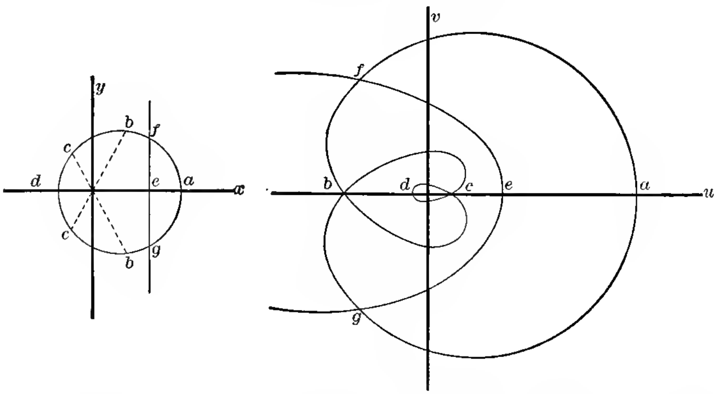
By putting $w = \rho(\cos\phi + i\sin\phi)$, $z = r(\cos\theta + i\sin\theta)$, the relations $\rho = r^3$, $\phi = 3\theta$, are obtained. Hence the circle

$$r^2 - 2ar\cos\theta + a^2 = c^2$$

gives the curve

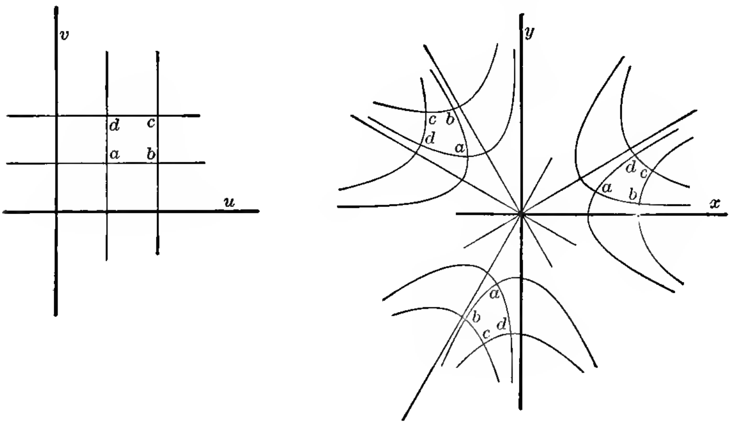
$$\rho^{\frac{2}{3}} - 2a\rho^{\frac{1}{3}}\cos\frac{\theta}{3} + a^2 = c^2,$$

which enwraps three times the point corresponding to the center. The accompanying figure represents this transformation, the straight line feg giving the curve feg.



To each point in the w-plane, excluding the origin, at which $\dfrac{dw}{dz} = 0$ and the conformal representation is not maintained, there correspond three points in the z-plane, having for their arguments $\frac{\phi}{3}, \frac{\phi + 2\pi}{3}, \frac{\phi + 4\pi}{3}$, respectively. Any straight line drawn from the origin in the w-plane will have, therefore, three images in the z-plane, viz., three straight lines diverging from the origin, and dividing the plane into three equal regions. Any continuous curve in the w-plane not meeting the line just drawn will be represented in the z-plane by three curves, of which one will be situated within each of these regions. In the figure here given are exhibited the three conformal representations of a square formed in the w-plane by lines $u = t_1$, $u = t_2$, $v = t_1$, $v = t_2$, parallel to the axes.

If the relation between w and z be reversed, and z be taken as a function of w, z will be a three-valued function, its values giving rise to three branches which will remain distinct and continuous except when w becomes equal to zero.

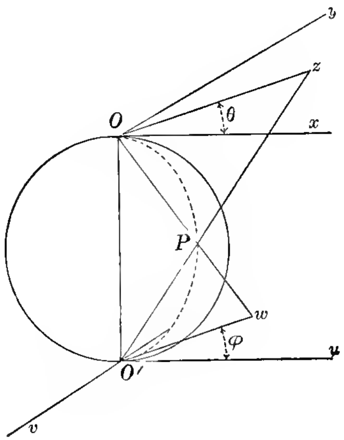


Prob. 8. If $w = z + \frac{1}{z}$, show that circles in the z-plane having a common center at the origin transform into confocal ellipses.

Prob. 9. If $w = \frac{z - i}{z + i}$, show that the axis of reals in the z-plane transforms into the circle $|w| = 1$, and the upper half of the z-plane into the interior of this circle.

ART. 10. CONFORMAL REPRESENTATION OF A SPHERE.

Let OPO' be a sphere having its diameter OO' equal in length to unity. Construct tangent planes at at O and O'. Draw in the tangent plane at O rectangular axes Ox and Oy; and in the other plane draw as axes $O'u$, parallel to Ox and measured in the same direction, and $O'v$ parallel to Oy but measured in a contrary direction. Join any point z in the plane xOy to O' by a straight line, and let $O'z$ meet the sphere in P. Draw OP and produce it to meet the plane $uO'v$ in w.



From the similar triangles $O'Oz$ and $OO'w$

$$\frac{Oz}{OO'} = \frac{OO'}{O'w}, \quad \text{or} \quad Oz \, . \, O'w = \overline{OO'}^2;$$

that is, $$|z| . |w| = r\rho = 1.$$

To an observer standing on the sphere at O' rotation about OO' from $O'u$ toward $O'v$ is positive, while to an observer standing on the sphere at O such a rotation is negative. Hence

$$\angle xOz = - \angle uO'w, \quad \text{or} \quad \theta = - \phi.$$

The following equation results:

$$wz = \rho r e^{i(\phi+\theta)} = 1.$$

The w- and z-planes are therefore conformal representations of one another. Any configuration in one plane can be formed from its image in the other by an inversion with respect to the origin as a center, combined with a reflection in the axis of reals. Such a transformation was termed by Cayley a "quasi-inversion." By it points at a great distance from the origin in one plane are brought near together in the immediate neighborhood of the origin in the other plane.

Since the line $O'Pz$ makes the same angle with the plane tangent to the sphere at P as with the plane xOy, any spherical angle having its vertex at P is projected into an equal angle at z. The sphere is thus seen to be related conformally to the plane xOy, and it must be also so related to the plane $uO'v$.

The representation of the sphere upon a tangent plane in the manner described above is termed a "stereographic projection." When to this representation is applied a logarithmic transformation, that is, one inverse to the transformation described in Case III of the preceding article, the so-called "Mercator's projection" is obtained.

ART. 11. CONJUGATE FUNCTIONS.

The real and imaginary parts of a monogenic function, $w = u + iv$, have been shown to satisfy the partial differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

At any point, therefore, where u and v admit second partial derivatives, one obtains

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0;$$

that is, the functions u and v are solutions of Laplace's equation for two dimensions. Any two real solutions p and q of this equation, such that $p + iq$ is a monogenic function of $x + iy$, are called "conjugate functions."* Thus the examples of Art. 9 furnish the following pairs of conjugate functions:

* Maxwell, Electricity and Magnetism, 1873, vol. 1, p. 227.

$x + a$, $y + b$; $r_1 r \cos(\theta_1 + \theta)$, $r_1 r \sin(\theta_1 + \theta)$; $e^x \cos y$, $e^x \sin y$; $\cos x \cosh y$, $-\sin x \sinh y$; $x^3 - 3xy^2$, $3xy^2 - y^3$. The second pair is expressed in polar coordinates, but may be transformed to cartesian coordinates by means of the relations

$$r = \sqrt{x^2 + y^2}, \quad \cos\theta = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin\theta = \frac{y}{\sqrt{x^2 + y^2}}.$$

If one of two conjugate functions be given, the other is thereby determined except for an additive constant. Let u, for example, be given. Then

$$\begin{aligned} dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \\ &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy, \end{aligned}$$

and therefore the value of v is

$$\int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right).$$

The equations $u = c_1$, $v = c_2$, obtained by assigning constant values to two conjugate functions, represent in the w-plane straight lines parallel to the coordinate axes. It follows that the curves which these equations define in the z-plane intersect at right angles. Consequently, by varying the quantities c_1 and c_2, two orthogonal systems of curves are obtained; and c_1 and c_2 may be taken as orthogonal curvilinear coordinates for the determination of position in the z-plane.

Prob. 10. Show that if p and q are conjugate functions of u and v, where u and v are conjugate functions of x and y, p and q will be conjugate functions of x and y.

Prob. 11. Show that if u and v are conjugate functions of x and y, x and y are conjugate functions of u and v.

ART. 12. APPLICATION TO FLUID MOTION.

Consider an incompressible fluid, in which it is assumed that every element can move only parallel to the z-plane, and has a velocity of which the components parallel to the coordinate axes are functions of x and y alone. The whole motion of the fluid is known as soon as the motion in the z-plane is ascertained. When any curve in the z-plane is given, by the "flux across the curve"* will be meant the volume of fluid which in unit time crosses the right cylindrical surface having the curve as base and included between the z-plane and a parallel plane at a unit distance.

The flux across any two curves joining the points z_0 and z is the same, provided the curves enclose a region covered with the moving fluid. For, corresponding to the enclosed region, there must be neither a gain nor a loss of matter. Let z_0 be fixed, and z be variable. Let ψ denote the flux across any curve z_0z, reckoned from left to right for an observer stationed at z_0 and looking along the curve toward z. If l, m be the direction cosines of the normal (drawn to the right) at any point of the curve, and p, q be the components parallel to the axes of the velocity of any moving element, the value of ψ will be

$$\psi = \int_{z_0}^{z} (lp + mq)ds,$$

where the path of integration is the curve joining z_0 and z. The function ψ is a one-valued function of z in any region within which every two curves joining z_0 to z enclose a region covered with the moving fluid.

If z moves in such a manner that the value of ψ does not vary, it will trace a curve such that no fluid crosses it, i.e., a "stream-line." The curves $\psi =$ const. are all stream-lines, and ψ is called the "stream-function." If p and q are continuous, and if z be given infinitesimal increments parallel to x and y respectively, one obtains

$$\frac{\partial\psi}{\partial x} = -q, \quad \frac{\partial\psi}{\partial y} = p.$$

If now the motion of the fluid be characterized, as is the

* Lamb's Hydrodynamics (1895), p. 69.

case in the so-called "irrotational" motion,* by the existence of a velocity-potential ϕ, so that

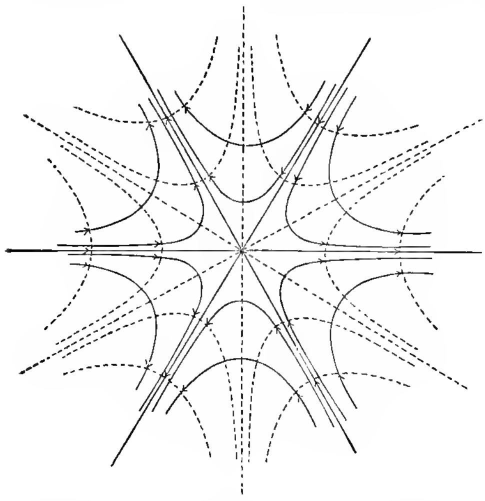
$$p = \frac{\partial \phi}{\partial x}, \quad q = \frac{\partial \phi}{\partial y},$$

the following equations result:

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}.$$

Hence $\phi + i\psi$ is a monogenic function of $x + iy$. The curves $\phi =$ const., which are orthogonal to the stream-lines, are called the "equipotential curves."

Consider, as an example, the motion corresponding to the function† $w = z^3$. The equipotential curves are given by the equations



$$u = x^3 - 3xy^2 = \text{const.},$$

the stream-lines by the equations

$$v = 3x^2y - y^3 = \text{const.}$$

In the following figure the stream-lines are the heavy lines, while the equipotential curves are dotted.

The fluid moves in toward the origin, which is called a "cross-point," from three directions, and flows out again in three other directions. At the cross-point the fluid is at a standstill, since at that point the velocity, for which the general expression is

$$\sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2},$$

* In irrotational motion each element is subject to translation and pure strain, but not to rotation.

† F. Klein: Riemann's Theory of Algebraic Functions; translated by Frances Hardcastle (1893), p. 3.

is equal to zero. The stream-lines in the figure represent the motion of the fluid in each of six different angles, as if the fluid were confined between walls perpendicular to the z-plane.

It is of importance to note that if the function considered be multiplied by i, the equipotential curves and stream-lines are interchanged, since the function $\phi + i\psi$ then becomes $-\psi + i\phi$.

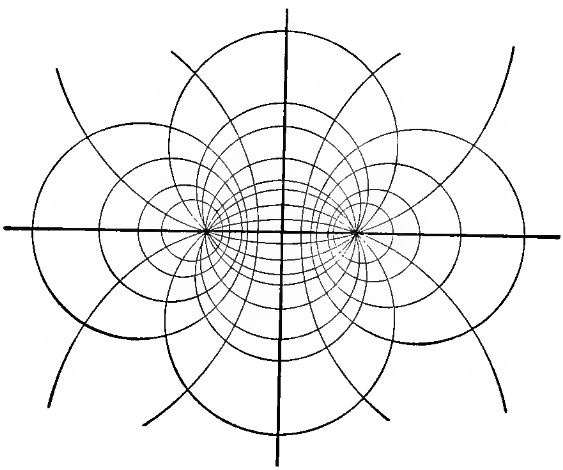
An example of particular interest is

$$w = -\mu \log \frac{z-a}{z+a}.$$

Let $z - a = r_1 e^{i\theta_1}$, $z + a = r_2 e^{i\theta_2}$; then

$$u = -\mu \log \frac{r_1}{r_2}, \quad v = -\mu(\theta_1 - \theta_2).$$

The curves $u =$ const., $v =$ const. form two orthogonal systems of circles, either of which may be regarded as the stream-lines, the other constituting the equipotential curves.



The velocities are everywhere, except at the points $\pm a$, finite and determinate. If the circles $r_1/r_2 =$ const. be taken as the stream-lines, each of the points $\pm a$ is a "vortex-point." If the circles $\theta_1 - \theta_2 =$ const. be taken as the stream-lines, one of the points $\pm a$ is a "source," the other a "sink." In the latter case, besides the hydrodynamical interpretation, a very simple electrical illustration is afforded by attaching the poles of a battery to a conducting plate of indefinite extent at two fixed points of the plate.

As another example may be taken the relation $w = \cos z$. As has been shown, the curves $x =$ const. form a system of confocal hyperbolas, while the curves $y =$ const. form an orthogonal system of ellipses. Either system may be regarded as stream-lines. In one case the motion of the fluid would be such as would occur if a thin wall were constructed along the axis of reals, except between the foci, and the fluid should be impelled through the aperture thus formed. In the other case the fluid would circulate around a barrier placed on the axis of reals and included between the foci.

Besides their application to fluid-motion, conjugate functions have important applications in the theory of electricity and magnetism * and in elasticity.†

ART. 13. CRITICAL POINTS.

Let w be any rational function of z. It can be written in the form

$$w = \frac{f(z)}{\phi(z)},$$

where $f(z)$ and $\phi(z)$ are entire and without common factors. This function is finite and admits an infinite number of successive derivatives for every finite value of z, except the roots of the equation $\phi(z) = 0$. Let a be such a root. Then the reciprocal of the given function is finite and admits an infinite nnmber of successive derivatives at the point a. Such a point

* J. J. Thomson, Recent Researches in Electricity and Magnetism (1893), p. 208.

† Love, Theory of Elasticity (1892), vol. 1, p. 331.

is called a "pole." Any rational function having a pole at a can be put by the method of partial fractions in the form

$$w = \frac{A_1}{z-a} + \dots + \frac{A_k}{(z-a)^k} + \psi(z),$$

where $A_1, \dots, A_k$ are constants, A_k being different from zero, and $\psi(z)$ is finite at the point a. The integer k is said to be the "order" of the pole, and the function is said to have for its value at a infinity of the kth order. In accordance with the definition of a derivative, w does not admit a derivative at a. From the character of the derivative in the immediate neighborhood of a, however, the derivative is sometimes said to become infinite at a.

The trigonometric function $\cot z$ has a pole of the first order at every point $z = m\pi$, m being zero or any integer positive or negative.

The function $w = \log(z - a)$ has for every finite value of z, except $z = a$, an infinite number of values. If $z - a$ is written in the form $Re^{i\Theta}$,

$$w = \log R + i(\Theta + 2m\pi),$$

where $\log R$ is real, and m is zero or any positive or negative integer. If z describes a straight line, beginning at a, Θ will remain fixed, but R will vary. The images in the z-plane will therefore be straight lines parallel to the axis of reals, dividing the plane into horizontal strips of width 2π. If now the z-plane is supposed to be divided along the straight line just drawn, and z varies along any continuous path, subject only to the restriction that it cannot cross this line of division, there will be a continuous curve as the image of the path of z in each strip of the w-plane. Each of these images is said to correspond to a "branch" of the function, or, expressed otherwise, the function is said to have a branch situated in each strip. The line of division in the z-plane, which serves to separate the branches from one another is called a "cut."

At the point $z = a$ no definite value is attached to the function. As z approaches that point the modulus of the real part of the function increases without limit, while the imaginary part is entirely indeterminate.

Let z_0 be an arbitrary point, distinct from a, and let

$$\log R_0 + i\Theta_0 + 2m\pi i$$

be any one of the corresponding values of the function. Suppose that z starts from z_0 and describes a closed path around the point a, the values of the function being taken so as to give a continuous variation. Upon returning to the point z_0 the value of the function will be

$$\log R_0 + i\Theta_0 + 2(m + 1)\pi i,$$

or

$$\log R_0 + i\Theta_0 + 2(m - 1)\pi i,$$

according as the curve is described in a positive or negative direction. By repeating the curve a sufficient number of times it is evidently possible to pass from any value of the function at z_0 to any other. When a point is such that a z-path enclosing it may lead in this manner from one value of a function to another value, it is called a "branch-point." In the case of the function here considered, the point $z = a$ is called a "logarithmic branch-point," or a point of "logarithmic discontinuity."

The function $w = \log \dfrac{f(z)}{\phi(z)}$, where $f(z)$ and $\phi(z)$ are entire, has a point of logarithmic discontinuity at every point where either $f(z)$ or $\phi(z)$ is equal to zero. For, writing

$$f(z) = A(z - a_1)^{p_1}(z - a_2)^{p_2} \ldots$$
$$\phi(z) = B(z - b_1)^{q_1}(z - b_2)^{q_2} \ldots$$

the value of w may be written

$$w = \log \frac{A}{B} + \sum_m p_m \log (z - a_m) - \sum_n q_n \log (z - b_n).$$

Take now the function $w = e^{\frac{1}{z}}$. It has a single finite value for every value of z except $z = 0$. If z is supposed to approach zero, the limit of the value of the function is indeterminate.

For let $p + iq$ be perfectly arbitrary, and write

$$e^{p+iq} = c + id.$$

If now $a + ib$ is the reciprocal of $p + iq$, so that

$$a = \frac{p}{p^2 + q^2}, \qquad b = \frac{-q}{p^2 + q^2},$$

the preceding equation may be written

$$e^{\frac{1}{a+ib}} = c + id.$$

But whatever the value of the integer m, $q + 2m\pi$ may be substituted for q without altering the value of $c + id$, and hence both a and b may be made less than any assignable quantity. The given function $e^{\frac{1}{z}}$ therefore takes the value $c + id$ at points $a + ib$ indefinitely near to the origin. A point such that, when z approaches it, a function elsewhere one-valued tends toward an indeterminate limiting value is called an "essential singularity."

Prob. 12. Show that for the function $e^{\frac{1}{z-a}}$ $z = a$ is an essential singularity.

Prob. 13. The function $e^{-\frac{1}{z^2}}$ considered as a function of a real variable is continuous for every finite value of z, and the same is true of each of its successive derivatives. Show that when it is regarded as a function of a complex variable, $z = 0$ is an essential singularity.

In order to illustrate still another class of special points take the function

$$w = \sqrt{(z - a_1)(z - a_2) \ldots (z - a_n)}.$$

This function has at every finite point, except $a_1, a_2, \ldots, a_n$, two distinct values differing in sign. At these points, however, it takes but a single value, zero. From each of the points $a_1, a_2, \ldots, a_n$ let a straight line of indefinite extent be drawn in such a manner that no one of them intersects any other, and suppose the z-plane to be divided, or cut, along each of these lines. Along any continuous path in the z-plane thus divided the values of the function form two distinct branches.

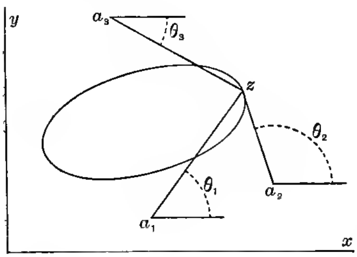
For, writing

$$z - a_1 = r_1 e^{i\theta_1}, \quad z - a_2 = r_2 e^{i\theta_2}, \quad \ldots, \quad z - a_n = r_n e^{i\theta_n},$$

the function takes the form

$$w = \sqrt{r_1 r_2 \ldots r_n} \; e^{i\frac{\theta_1 + \theta_2 + \ldots + \theta_n}{2}}.$$

No closed path in the divided plane will enclose any of the points $a_1, a_2, \ldots, a_n$, and the quantities $\theta_1, \theta_2, \ldots, \theta_n$, after continuous variation along such a path, must resume at the initial point their original values. No such path, therefore, can lead from one value of the function at any point to a new value of the function at the same point. If, however, the cuts are disregarded and z traces in a positive direction, a closed curve including an odd number of the points $a_1, a_2, \ldots, a_n$, and not intersecting itself, then an odd number of



the quantities $\theta_1, \theta_2, \ldots, \theta_n$ are each increased by 2π; and the value of the function is altered by a factor $e^{(2k+1)\pi i}$, and so changed in sign. In the same way any closed path described about one of these points, and enwrapping it an odd number of times, leads from one value of the function to the other. On the other hand, a simple closed path enclosing an even number of these points, or a closed path which encloses but one of the points, enwrapping it an even number of times, leads back to the initial value of the function. It follows that each of the points $a_1, a_2, \ldots, a_n$ is a branch-point. Any point in the z-plane, closed paths about which lead from one to another of a set of different values of a function, the number of values in the set being finite, is called an "algebraic branch-point."

As a further illustration, consider the function

$$w = z^{\frac{1}{2}} + (z - a)^{\frac{1}{3}},$$

which is a root of the equation of the sixth degree,

$$w^6 - 3zw^4 - 2(z - a)w^3 + 3z^2w^2 - 6z(z - a)w + (z-a)^2 - z^3 = 0.$$

The function has at every point, except $z = 0$ and $z = a$, six distinct values. Six branches are thereby formed which can be completely separated from one another by making cuts from the points $z = 0$ and $z = a$ to infinity. Putting ω for the cube root of unity, these six branches can be written

$$\begin{aligned} w_1 &= z^{1/2} + (z - a)^{1/3}, & w_2 &= - z^{1/2} + (z - a)^{1/3}, \\ w_3 &= z^{1/2} + \omega (z - a)^{1/3}, & w_4 &= - z^{1/2} + \omega (z - a)^{1/3}, \\ w_5 &= z^{1/2} + \omega^2(z - a)^{1/3}, & w_6 &= - z^{1/2} + \omega^2(z - a)^{1/3}. \end{aligned}$$

The branches w_1 and w_2, w_3 and w_4, w_5 and w_6 are interchanged by a small closed circuit described about $z = 0$, while a small circuit described about $z = a$ permutes cyclically the branches w_1, w_3, w_5, and also the branches w_2, w_4, w_6.

All of the special points examined above, poles, points of logarithmic discontinuity, essential singularities, and branch-points, are called critical points. In fact, a function, or a branch of a function, is said to have a "critical point" at each point where it fails to have a continuous derivative,* or about which as a center it is impossible to describe a circle of determinate radius within which the function, or branch, is one-valued.

Any point not a critical point is called an "ordinary point."

* Continuity and, therefore, finiteness of the function are implied in the existence of a derivative.

An ordinary point at which a function reduces to zero is called a "zero" of the function.

If in a certain region of the z-plane there are no critical points for a given function, the function is said to be "synectic" or "holomorphic" in that region. If in a certain region the only critical points are poles, the function is said to be "meromorphic" in that region. Under similar conditions a branch of a function is also described as holomorphic or meromorphic.

Prob. 14. When w and z are connected by the relation $w - g = (z - h)^t$ show that if z describes a circle about h as a center, w describes a circle about g as a center, an angle in the z-plane having its vertex at h is transformed into an angle in the w-plane t times as great and having its vertex at g, and that $z = h$ is a branch-point of w except when t is an integer.

ART. 14. POINT AT INFINITY.

In determining the limiting value of a function when the modulus of the independent variable z is increased indefinitely, it is usual to introduce a new independent variable z' by the relation $z = 1/z'$, and consider the function at the point $z' = 0$. This is equivalent to passing from the z-plane to another plane, the z'-plane, related to the former by the geometrical construction described in Art. 10. It is often very convenient, however, to go further and to substitute for the z-plane the surface of the sphere of unit diameter touching the z-plane at the origin. No difficulty is thus introduced since, as explained in the article just cited, any configuration in the z-plane obtains a conformal representation upon the sphere; and the advantage is gained that the entire surface upon which the variation of the independent variable is studied is of finite extent. The point of the sphere diametrically opposite to its point of contact with the z-plane coincides with the point written above as $z' = 0$. It is called the point at infinity, $z = \infty$, since a point on the sphere approaches it at the same time that its image in the z-plane recedes indefinitely from the origin.

The point at infinity may be either an ordinary or a critical point. For the function $e^{\frac{1}{z}}$, for example, it is an ordinary point, since $e^{\frac{1}{z}} = e^{z'}$. For a rational entire function of the nth degree it is a pole of order n. Consider it for the function $\sqrt{(z - a_1)(z - a_2) \ldots (z - a_n)}$, discussed in the preceding article. Let a circle of great radius be described in the z-plane inclosing all the branch-points $a_1, a, \ldots, a_n$. Its conformal representation on the sphere will be a small closed curve surrounding the point $z = \infty$. This point must, therefore, be regarded as a branch-point or not, according as the function changes value or not when the curve surrounding it is described, that is according as n, the number of finite branch-points, is odd or even. When the point at infinity is taken into account, then, the total number of branch-points of this function is always even. The character of the point $z = \infty$ for this function can be determined directly, by changing z into $1/z'$ and considering the point $z' = 0$.

Prob. 15. Show that $z = \infty$ is an ordinary point for $\frac{\phi(z)}{\psi(z)}$, where $\phi(z)$ and $\psi(z)$ are rational and entire if the degree of $\phi(z)$ does not exceed that of $\psi(z)$.

ART. 15. INTEGRAL OF A FUNCTION.

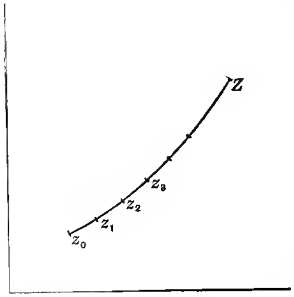
Let $w = f(z)$ be a continuous function of a complex variable z, and suppose z to describe a continuous path L from the point z_0 to the point Z. Let a series of points $z_1, z_2, \ldots, z_n$ be taken on L, and let $t_0, t_1, \ldots, t_n$ be points arbitrarily chosen on the arcs $z_0z_1, z_1z_2, \ldots, z_nZ$ respectively. Form the sum

$$S = (z_1 - z_0)f(t_0) + (z_2 - z_1)f(t_1) + \ldots + (Z - z_n)f(t_n).$$

If now the number of points $z_1, \ldots, z_n$ be increased indefinitely in such a manner that the length* of each of the arcs

* It is assumed in regard to every path of integration that the idea of length may be associated with the portion of it included between any two of its points, or, what is the same thing, that the path is rectifiable. This condition is evidently satisfied if the current coordinates x and y can be expressed in terms of

$z_0z_1, z_1z_2, \ldots, z_nZ$ approaches zero as a limit, the sum S approaches a finite limit which is independent of the choice of the points z_1, $z_2, \ldots, z_n$ and $t_0, t_1, \ldots, t_n$.



For take any other sum

$$S' = (z_1' - z_0)f(t_0') + (z_2' - z_1')f(t_1') + \ldots$$

formed in a similar manner. Suppose for the sake of greater definiteness that the points $z_1, \ldots$ and $z_1', \ldots$ follow one another on the line L in the order

$$z_1,\ z_1',\ z_2',\ z_2,\ z_3,\ z_3', \ldots,$$

and form a third sum

$$S'' = (z_1 - z_0)f(\tau_0) + (z_1' - z_1)f(\tau_1) + (z_2' - z_1')f(\tau_2) + (z_2 - z_2')f(\tau_3) + \ldots,$$

in which both series of points occur. It may be shown that as the number of points in each of the series $z_1, \ldots$ and $z_1', \ldots$ is increased, the differences $S'' - S$ and $S'' - S'$ both approach zero, from which it follows that the difference $S - S'$ has a limit equal to zero. For example, the difference $S'' - S$ has the value

$$(z_1 - z_0)[f(\tau_0) - f(t_0)] + (z_1' - z_1)[f(\tau_1) - f(t_1)] + (z_2' - z_1')[f(\tau_2) - f(t_1)] + \ldots$$

If M denotes the upper extreme of the quantities

$$|f(\tau_0) - f(t_0)|, \qquad |f(\tau_1) - f(t_1)|, \qquad |f(\tau_2) - f(t_1)|, \ldots$$

the modulus of $S'' - S$ will be less than

$$M[|z_1 - z_0| + |z_1' - z_1| + |z_2' - z_1'| + \ldots].$$

any parameter t so that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are continuous. For then the integral $\int \sqrt{dx^2 + dy^2}$ is finite. See, in this connection, Jordan, Cours d'Analyse, 2d Edition, Vol. I., p. 100.

But $|z_1 - z_0|$ is equal to the chord of the arc z_0z_1, and must therefore be less than or equal to this arc, and a similar result holds for each of the quantities $|z_1'-z_1|, |z_2'-z_1'|, \ldots$ Hence

$$|S'' - S| \overline{\overline{<}} Ml,$$

where l denotes the length of the path of integration. When the number of points of division on the line L is increased, the differences

$$f(\tau_0) - f(t_0), \qquad f(\tau_1) - f(t_1), \qquad f(\tau_2) - f(t_1), \ldots$$

decrease indefinitely, for $f(z)$ is continuous. M acccordingly decreases indefinitely and the difference $S'' - S$ approaches zero.

The limit, the existence of which has just been demonstrated, is called the integral of $f(z)$ along the path L. It is written $\int_L f(z)dz$. The definition here given is similar to that given for the integral of a function of a real variable. It is unnecessary to specify the path of integration when the independent variable is restricted to real values, since in that case it must be the portion of the axis of reals included between the limits of integration.

The following well-known principles, applicable to the case of a real independent variable, may be readily extended to the general case:

1. The modulus of the integral cannot exceed the length of the path of integration multiplied by the upper extreme of the modulus of the function along that path.

2. The independent variable may be altered by any equation of transformation, but L', the path of integration in the transformed integral, must be such that it is described by the new variable while z describes L.

3. If $F(z)$ is any one-valued function having everywhere $f(z)$ for its derivative, the equation

$$\int_L f(z)dz = F(Z) - F(z_0)$$

must be true.

To prove the third principle, write $F(Z) - F(z_0)$ in the form

$$F(Z)-F(z_n)+F(z_n)-F(z_{n-1})+\ldots+F(z_2)-F(z_1)+F(z_1)-F(z_0).$$

Since the derivative of $F(z)$ is $f(z)$,

$$F(z_{m+1}) - F(z_m) = [f(z_m) + \eta_m](z_{m+1} - z_m),$$

where η_m has zero for its limit when z_{m+1} is made to approach z_m. Hence

$$F(Z) - F(z_0) = \text{limit } \Sigma f(z_m)(z_{m+1} - z_m) + \text{limit } \Sigma \eta_m(z_{m+1} - z_m);$$

or, since the second term of the right-hand member is equal to zero,

$$F(Z) - F(z_0) = \int_L f(z)dz.$$

If no function $F(z)$ fulfilling the preceding conditions is known, the value of the integral requires further investigation.

Consider as an example the integral $\int \frac{dz}{z^2}$ taken from the point $z = -1$ to the point $z = 1$, the path of integration being the upper half of the circumference of a unit circle described about the origin as a center. Writing $z = \exp(i\theta)$, z will describe the required path while θ varies from π to 0.

The equations $\frac{1}{z^2} = e^{-2i\theta}$, $\quad dz = ie^{i\theta}d\theta$,

$$\frac{dz}{z^2} = ie^{-i\theta}d\theta = i\cos\theta\, d\theta + \sin\theta\, d\theta = id(\sin\theta) - d(\cos\theta),$$

follow at once. Hence for the path specified

$$\int_{-1}^{+1} \frac{dz}{z^2} = i\int_{\pi}^{0} d(\sin\theta) - \int_{\pi}^{0} d(\cos\theta) = -2.$$

The application of the direct and more familiar method gives the same result:

$$\int_{-1}^{+1} \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{z=1} - \left[-\frac{1}{z}\right]_{z=-1} = -2.$$

For a path along the axis of reals between the limits of integration this result is unintelligible. The discontinuity of the differential, $\frac{dz}{z^2}$, at the point $z = 0$, prevents the consideration of such a path; and that the result should be negative when the differential is at every point of the path positive has no significance. The introduction of the complex variable furnishes a perfectly satisfactory explanation of the result.

Prob. 16. Show that the integral of $\frac{dz}{z}$ along any semi-circumference described about the origin as a center is equal to πi.

ART. 16. REDUCTION OF COMPLEX INTEGRALS TO REAL.

The integral

$$\int_L f(z)dz$$

may be written in the form

$$\int_L (u + iv)(dx + idy),$$

or, separating the real and imaginary terms,

$$\int_L (udx - vdy) + i\int_L (vdx + udy).$$

Hence the calculation of the integral may be reduced to the calculation of two real curvilinear integrals.

The equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

which express the condition that $u + iv$ should be monogenic, express also that

$$udx - vdy, \quad vdx + udy$$

are the exact differentials of two real functions of the variables x, y. Consider the case where these functions are one-valued.

Denoting them by $P(x, y)$ and $Q(x, y)$ respectively, the integral may be written

$$[P(X, Y) - P(x_0, y_0)] + i[Q(X, Y) - Q(x_0, y_0)],$$

(x_0, y_0) and (X, Y) being the initial and terminal points respectively of the path of integration.

ART. 17. CAUCHY'S THEOREM.

Cauchy's Theorem furnishes the necessary and sufficient conditions that a one-valued function $f(z)$, having a continuous derivative $f'(z)$, should yield a one-valued integral, that is, an integral the value of which, when the lower limit is fixed, depends simply on the upper limit, and not on the path of integration. It will be more convenient, before considering Cauchy's Theorem, to demonstrate the following lemma:

Lemma.—Let A be a portion of the z-plane, having a boundary S which consists of a closed curve not intersecting itself, or of several closed curves not intersecting themselves or one another. Denote by λ the inclination to the axis of x of the exterior normal at any point of the boundary,* that is, the normal drawn to the right as the boundary is described in a positive direction. If at every point of the region A, including its boundary S, a function W of the real variables x and y is one-valued and continuous and has continuous partial derivatives $\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}$, the relations

$$\int_S W dy = \int\int_A \frac{\partial W}{\partial x} dx\, dy, \qquad (1)$$

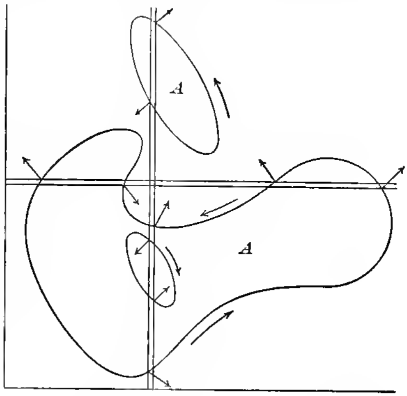
$$\int_S W dx = -\int\int_A \frac{\partial W}{\partial y} dx\, dy \qquad (2)$$

exist, the integrals in the first members being taken along the

* It is assumed that the boundary has a determinate tangent at every point. If the boundary of a given region is not of this sort, the theorem holds for any interior curve of which this assumption is true.

boundary in the positive direction, and those in the second members being taken over the enclosed area.

If any straight line parallel to the axis of x be traced in the direction of increasing values of x, at each point where it passes into the area A, $\cos \lambda$ is negative, and therefore in the first member of (1) $dy = \cos \lambda \, ds$ is negative. At each point where this straight line passes out of the area A, $\cos \lambda$, and therefore dy, in the first member of equation (1), is positive. Hence in the first member of equation (1) the differentials Wdy corresponding



to a given value of y, and taken in the order of increasing values of x, have signs which, compared with those of the corresponding values of W, first differ, then agree, and so on alternately. In order now to compare the integral in the first member of equation (1) with the integral in the second member, it is necessary to take dy as essentially positive. The sum of the differentials in the first member, corresponding to a fixed value of y, must therefore be written in the form

$$dy(-W_1 + W_2 - W_3 + W_4 - \ldots),$$

where W_1, W_2, ... are the corresponding values of W taken in the order of increasing values of x. But performing now in the second member of equation (1) an integration with respect to x, the same result is obtained, so that the two members of equation (1) become identical, and the equation is verified.

To obtain equation (2) the same method is used. It is necessary in this case to observe that if a line parallel to the axis of y is traced in the direction of increasing values of y, at each point where it enters A, dx in the integral of the first member must be taken as positive; and at each point where this line passes out of A, dx in that integral must be taken as negative.

By means of the preceding lemma, Cauchy's Theorem is easily proved. This theorem may be stated as follows:

Theorem.—If, on the boundary of and within a given region A, a one-valued function $w = f(z)$ is monogenic, and its derivative $f'(z)$ is continuous,* the integral $\int_S f(z)dz$ taken along the boundary S is equal to zero.

For writing the integral in the form

$$\int_S wdz = \int_S (udx - vdy) + i\int_S (udy + vdx),$$

the preceding lemma gives

$$\int_S (udx - vdy) = -\int\int_A \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx\, dy,$$

$$\int_S (udy + vdx) = \int\int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy;$$

but since at every point of A

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0, \qquad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$

the given integral reduces to zero.

ART. 18. APPLICATION OF CAUCHY'S THEOREM.

From Cauchy's Theorem it follows that, if two different paths L_1 and L_2 lead from the point z_0 to the point Z, and if along these paths and in the region inclosed between them a given function $f(z)$ has no critical points, the integrals of the function taken along these two paths are equal. For two such paths taken together, one described directly, the other reversed, constitute a closed curve, and the integral taken along

*Otherwise expressed, the one-valued function $f(z)$ has no critical points on the boundary of or within A, or $f(z)$ is holomorphic in A.

it is equal to zero. But, since reversing the direction of the path of integration is equivalent to changing the sign of the integral, the equation

$$\int_{L_1} f(z)dz - \int_{L_2} f(z)dz = 0$$

is obtained.

The result just established may be stated in the following theorem:

Theorem I.—If a function is holomorphic in any simply connected region bounded by a continuous closed curve, the integral of the function, from a fixed lower limit in that region to any point contained therein, is independent of the path of integration, and is a one-valued function of its upper limit.

A region whose boundary is composed of disconnected curves is not necessarily characterized by the property stated in the theorem. Take, for example, the function

$$w = \sqrt{(z-a_1)(z-a_2)\ldots(z-a_n)},$$

and suppose that $0 < |a_1| < |a_2| < \ldots < |a_n|$. With the origin as a center, construct a system of concentric circles C_1, $C_2, \ldots, C_n$, C_1 passing through a_1, C_2 through a_2, and so on. Denote by S_0 the region inclosed within the first circle C_1, by S_1 that inclosed between C_1 and C_2, and so on, the portion of the plane exterior to the last circle C_n being denoted by S_n. At an initial point z_0 interior to one of these regions, assign to w one of the two values possible, and consider the branch of w resulting from a continuous variation. Then however z may vary within any such region, this branch of w will be a monogenic function, and its derivative will be continuous. Having regard to the branch-points $a_1, a_2, \ldots, a_n$, it is evident that in the regions $S_0, S_2, \ldots$ it will be one-valued, and in the regions $S_1, S_3, \ldots$, it will be two-valued. Thus in the former regions $S_0, S_2, \ldots$, the branch fulfils all the conditions required by the theorem above. The theorem is applicable, however, only to S_0, for in any other region two paths may be drawn joining the same two points, and such that the branch is not one-valued throughout the enclosed portion of the z-plane.

Theorem II.—If $f(z)$ is holomorphic in any simply connected region S bounded by a continuous closed curve, the integral $\int f(z)dz$, taken from a fixed lower limit z_0 in that region to any point Z contained therein, is a holomorphic function of its upper limit.

Let L be any path from z_0 to Z. When the upper limit is at the point $Z + dZ$, L followed by a straight line from Z to $Z + dZ$ can be taken as the path of integration. Hence

$$\int_{z_0}^{Z+dZ} f(z)dz - \int_{z_0}^{Z} f(z)dz = \int_{Z}^{Z+dZ} f(z)dz$$

$$= f(Z)\int_{Z}^{Z+dZ} dz + \int_{Z}^{Z+dZ} [f(z) - f(Z)]dz.$$

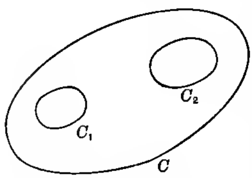
The first term is equal to $f(Z)dZ$. The modulus of second term is equal to or less than $M|dZ|$, where M is the upper extreme of $|f(z) - f(Z)|$ along the line joining Z to $Z + aZ$. But since $f(z)$ is continuous, the limit of M when $Z + dZ$ approaches Z is zero. Hence

$$\int_{z_0}^{Z+dZ} f(z)dz - \int_{z_0}^{Z} f(z)dz = [f(Z) + \eta]dZ,$$

where η approaches zero with dZ. The integral therefore has $f(Z)$ for a derivative, and is holomorphic in S.

In the case of a region bounded by several disconnected closed curves, of which one is exterior to all the others, Cauchy's Theorem may be stated in the following form:

Theorem III.—Let a function $f(z)$ be holomorphic in a region A bounded by a closed curve C and one or more closed curves C_1, C_2, . . . interior to C. The integral of $f(z)$ taken along C will be equal to the sum of its integrals taken in the same direction along the curves C_1, C_2, . . .



For the integral of $f(z)$ taken in a positive direction completely around the boundary of A is equal to zero. But the curves C_1, C_2, . . . are then described in the direction opposite to that in which C is described. Hence if all the curves are described in the same direction, the result may be written

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \dots$$

If there is but one interior curve, so that the region A is included between two curves C and C_1, the integral taken along every closed curve containing C_1 but interior to C has the same value, viz., the common value corresponding to the paths C and C_1.

ART. 19. THEOREMS ON CURVILINEAR INTEGRALS.

Theorem I.—If $f(z)$ be continuous in a given region except at the point a, the integral $\int f(z)dz$, taken around a small circle c, having its center at a, will approach zero as a limit simultaneously with the radius r of the circle c, provided only

$$\lim (z - a)f(z) = 0 \quad \text{when} \quad z = a.$$

For let the upper extreme of the modulus of $(z - a)f(z)$ on the circle c be denoted by M. Then at every point of c,

$$\text{mod } f(z) \leqq \frac{M}{|z - a|} \leqq \frac{M}{r},$$

and consequently

$$\text{mod} \int_c f(z)dz \leqq \frac{M}{r}\int ds \leqq 2\pi M.$$

Theorem II.—The integral $\int \frac{dz}{(z - a)^n}$, taken around any closed curve C containing the point a, is equal to zero, except when $n = 1$. When $n = 1$, this integral is equal to $2\pi i$.

For the value of the integral will be the same if any circle described about a as a center be taken as the path of integration. Let then $z - a = re^{i\theta}$, where r is a constant and θ varies from 0 to 2π. The integral becomes

$$\frac{i}{r^{n-1}}\int_0^{2\pi} e^{-(n-1)i\theta}\, d\theta$$

which reduces to zero except when $n = 1$. If $n = 1$, its value is $2\pi i$, whence

$$\int \frac{dz}{z - a} = 2\pi i.$$

Theorem III.—If $f(z)$ is a function holomorphic in a given region S, C a closed curve the interior of which is wholly within S, and a a point situated within C, then

$$\int_C \frac{f(z)}{z - a} dz = 2\pi i f(a).$$

For describing about a as a center a small circle c of radius r, the equation

$$\int_C \frac{f(z)}{z - a} dp = \int_c \frac{f(z)}{z - a} dz$$

is obtained. But at every point of c,

$$f(z) = f(a) + \eta,$$

where, by choosing r sufficiently small, the modulus of η may be made less than any fixed positive quantity. Hence

$$\int_C \frac{f(z)}{z - a} dz = \int_c \frac{f(a)}{z - a} dz + \int_c \frac{\eta}{z - a} dz,$$

but by the preceding theorems the first term of the right-hand member is equal to $2\pi i f(a)$, and the second term is equal to zero.

If the equation of the theorem just established be differentiated with respect to a, the following important formulas, expressing the successive derivatives of a holomorphic function at a given point, are obtained:

$$\int_C \frac{f(z)}{(z - a)^2} dz = 2\pi i f'(a),$$

$$1 \cdot 2 \int_C \frac{f(z)}{(z - a)^3} dz = 2\pi i f''(a),$$

$$\cdots\cdots\cdots\cdots$$

$$1 \cdot 2 \ldots n \int_C \frac{f(z)}{(z - a)^{n+1}} dz = 2\pi i f^{(n)}(a).$$

The integrals in the first members of these equations are all finite and determinate for every position of a within the curve C. Therefore any function holomorphic in a given region admits an infinite number of successive derivatives at every interior point. Each of these derivatives being monogenic must be continuous. Hence the following:

Theorem IV.—If $f(z)$ is holomorphic within a given region, there exists an infinite number of successive derivatives of $f(z)$, which are all holomorphic within the same region.

Denote by r the shortest distance from the point a to the curve C. Then at every point of this curve $|z-a| \geqq r$. Let M be the upper extreme of the modulus $f(z)$ on C, and l the length of C. Then

$$\operatorname{mod}\int_C \frac{f(z)}{(z-a)^{n+1}}dz \leqq \int_C \frac{M}{r^{n+1}}ds \leqq \frac{Ml}{r^{n+1}},$$

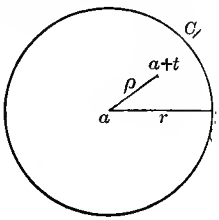
and consequently $\operatorname{mod} f^{(n)}(a) \leqq \dfrac{1 \cdot 2 \ldots n}{2\pi} \cdot \dfrac{Ml}{r^{n+1}}$.

In particular, if C is a circle having a for its center,

$$\operatorname{mod} f^{(n)}(a) \leqq \frac{1 \cdot 2 \ldots n \cdot M}{r^n}.$$

ART. 20. TAYLOR'S SERIES.

Theorem.—Let $f(z)$ be holomorphic in a region S, and let C be any circle situated in the interior of S. If a be the center and $a+t$ any other point interior to C,



$$f(a+t) = f(a) + tf'(a) + \frac{t^2}{1 \cdot 2}f''(a) + \ldots$$
$$+ \frac{t^n}{1 \cdot 2 \ldots n}f^{(n)}(a) + \ldots$$

From the preceding article, denoting a variable point on C by ζ,

$$f(a+t) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - a - t}d\zeta$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{\zeta - a}\left[1 + \frac{t}{\zeta - a} + \ldots + \frac{t^n}{(\zeta-a)^n} + \frac{t^{n+1}}{(\zeta-a)^n(\zeta-a-t)}\right]$$

$$= f(a) + tf'(a) + \frac{t^2}{1 \cdot 2}f''(a) + \ldots + \frac{t^n}{1 \cdot 2 \ldots n}f^{(n)}(a) + R,$$

where

$$R = \frac{1}{2\pi i}\int_C \frac{t^{n+1}f(\zeta)}{(\zeta - a)^{n+1}(\zeta - a - t)}d\zeta.$$

By taking n sufficiently great the modulus of R may be made less than any given positive quantity. Let M be the upper extreme of the modulus of $f(z)$ on the circle C, ρ the modulus of t, and r the modulus of $\zeta - a$ or radius of C. Then

$$|R| \leqq \frac{1}{2\pi}\int_C M\frac{\rho^{n+1}}{r^{n+1}(r-\rho)}ds \leqq \frac{Mr}{r-\rho}\left(\frac{\rho}{r}\right)^{n+1},$$

which, since $\rho < r$, has zero for its limit when $n = \infty$.

Writing now z for $a + t$, Taylor's Series becomes

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{1 \cdot 2}f''(a)+\ldots+\frac{(z-a)^n}{1 \cdot 2 \ldots n}f^{(n)}(a)+\ldots$$

The series is convergent and the equality is maintained for every point z included within a circle described about a as a center with a radius less than the distance from a to the nearest critical point of $f(z)$.

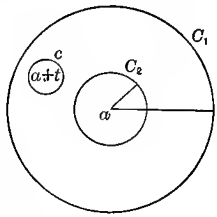
When a is equal to zero, Taylor's Series takes the form

$$f(z) = f(0) + zf'(0) + \frac{z^2}{1 \cdot 2}f''(0) + \ldots + \frac{z^n}{1 \cdot 2 \ldots n}f^{(n)}(0) + \ldots,$$

expressing $f(z)$ in terms of powers of z. This form is known as Maclaurin's Series.

ART. 21. LAURENT'S SERIES.

Theorem.—Let S, a portion of the z-plane bounded by two concentric circles C_1 and C_2, be situated in the interior of the region E, in which a given function $f(z)$ is holomorphic. If a be the common center of the two circles, and $a+t$ a point interior to S, $f(a+t)$ can be expressed in a convergent double series of the form



$$f(a+t) = \sum_{m=-\infty}^{m=\infty} A_m t^m.$$

With $a+t$ as a center construct a circle c sufficiently small to be contained within the region S. If then C_1 be the greater of the two given circles, it follows from Article 18 that

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-a-t} = \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta-a-t} + \frac{1}{2\pi i}\int_{c}\frac{f(\zeta)d\zeta}{\zeta-a-t}.$$

But from Article 19,

$$\frac{1}{2\pi i}\int_{c}\frac{f(\zeta)d\zeta}{\zeta-a-t} = f(a+t),$$

whence

$$f(a+t) = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-a-t} - \frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta-a-t}.$$

The two integrals of the right-hand member may be written:

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-a}\left[1+\frac{t}{\zeta-a}+\ldots+\frac{t^n}{(\zeta-a)^n}\right]+R_1,$$

$$-\frac{1}{2\pi i}\int_{C_2}f(\zeta)d\zeta\left[\frac{1}{t}+\frac{\zeta-a}{t^2}+\ldots+\frac{(\zeta-a)^n}{t^{n+1}}\right]+R_2,$$

where

$$R_1 = \frac{1}{2\pi i}\int_{C_1}\frac{t^{n+1}f(\zeta)d\zeta}{(\zeta-a)^{n+1}(\zeta-a-t)},$$

$$R_2 = \frac{1}{2\pi i}\int_{C_2}\frac{(\zeta-a)^{n+1}f(\zeta)d\zeta}{t^{n+1}(\zeta-a-t)}.$$

But $|t|<|\zeta-a|$ at every point of C_1, and $|t|>|\zeta-a|$ at every point of C_2, so that R_1 and R_2 both have zero for a limit when $n = \infty$. The value of $f(a+t)$ can therefore be expressed in the form

$$f(a+t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \dots$$
$$+ \frac{A_{-1}}{t} + \frac{A_{-2}}{t^2} + \frac{A_{-3}}{t^3} + \dots$$

Since in the region S the function $f(z)/(z-a)^{m+1}$ is holomorphic for both positive and negative values of m, A_m may be written

$$A_m = \frac{1}{2i\pi}\int \frac{f(\zeta)}{(\zeta - a)^{m+1}} d\zeta,$$

where C is any circle concentric with C_1 and C_2 and included between them.

The series thus obtained is convergent at every point $a+t$ contained within the region S. It is important to notice, however, that when the positive and negative powers of t are considered separately, the two resulting series have different regions of convergence. The series containing the positive powers of t converges over the whole interior of the circle C_1; while the series of negative powers of t converges at every point exterior to the circle C_2. The region S can be regarded, therefore, as resulting from an overlapping of two other regions in which different parts of Laurent's Series converge.

Writing z for $a+t$, Laurent's Series takes the form

$$f(z) = A_0 + A_1(z-a) + A_2(z-a)^2 + \dots$$
$$+ A_{-1}(z-a)^{-1} + A_{-2}(z-a)^{-2} + \dots$$

Consider as a special numerical example the fraction

$$\frac{1}{(z-1)(z-2)(z-3)} = \frac{1}{2(z-1)} - \frac{1}{z-2} + \frac{1}{2(z-3)}$$

If $|z| < 1$, all three terms of the second member, when developed in powers of z, give only positive powers. If $1 < |z| < 2$, the first term of the second member gives a series of negative descending powers, but the others give the same series as before. If $2 < |z| < 3$, the first and second terms both give negative powers. If $|z| > 3$, all three terms give negative powers, and the development of the given fraction can contain no positive powers. Thus a system of concentric annular regions is obtained in each of which the given fraction is expressed by a convergent power-series. Laurent's Series gives analogous results for every function which is holomorphic except at isolated points of the z-plane.

ART. 22. FOURIER'S SERIES.

Let $w = f(z)$ be holomorphic in a region S_0, and let it be periodic, having a period equal to ω, so that $f(z + n\omega) = f(z)$, where n is any positive or negative integer. Denote by S_n the region obtained from S_0 by the addition of $n\omega$ to z; and suppose that the regions $\ldots, S_{-n}, \ldots, S_{-1}, S_0, S_1, \ldots, S_n, \ldots$ meet or overlap in such a manner as to form a continuous strip S, in which, of course, the function w will be holomorphic. Draw two parallel straight lines, inclined to the axis of reals at an angle equal to the argument of ω, and contained within the strip S. The band T included between these parallels will be wholly interior to S.

By means of the transformation $z' = e^{\frac{2\pi i z}{\omega}}$ the band T in the z-plane becomes in the z'-plane a ring T' bounded by two concentric circles described about the origin as a center, z and $z + n\omega$ falling at the same point z'. Since w is holomorphic in a region including T, and

$$\frac{dw}{dz'} = \frac{dw}{dz}\frac{dz}{dz'} = \frac{\omega}{2\pi i} e^{-\frac{2\pi i z}{\omega}} \frac{dw}{dz},$$

w regarded as a function of z' will be holomorphic in T'. Hence, by Laurent's Theorem,

$$w = \sum_{m=-\infty}^{m=\infty} A_m z'^m,$$

the quantity a in the general formula of the preceding article being in this case equal to zero. Substituting for z' its value, the preceding equation becomes

$$w = \sum_{m=-\infty}^{m=\infty} A_m e^{\frac{2m\pi i z}{\omega}},$$

where

$$A_m = \frac{1}{2\pi i}\int_C \frac{w\,dz'}{z'^{m+1}} = \frac{1}{\omega}\int_z^{z+\omega} e^{-\frac{2m\pi iz}{\omega}}\,w\,dz.$$

In the latter integral the path is rectilinear. Denoting its independent variable by ζ for the purpose of avoiding confusion, the value of w becomes

$$f(z) = \frac{1}{\omega}\sum_{m=-\infty}^{m=\infty}\int_\zeta^{\zeta+\omega} e^{\frac{2m\pi i}{\omega}(z-\zeta)} f(\zeta)d\zeta$$

$$= \frac{1}{\omega}\int_\zeta^{\zeta+\omega} f(\zeta)d\zeta + \frac{2}{\omega}\sum_{m=1}^{m=\infty}\int_\zeta^{\zeta+\omega} \cos\frac{2m\pi}{\omega}(z-\zeta)f(\zeta)d\zeta$$

$$= \frac{1}{\omega}\int^{\zeta+\omega} f(\zeta)d\zeta + \frac{2}{\omega}\sum_{m=1}^{m=\infty}\cos\frac{2m\pi z}{\omega}\int_\zeta^{\zeta+\omega}\cos\frac{2m\pi\zeta}{\omega}f(\zeta)d\zeta$$

$$+ \frac{2}{\omega}\sum_{m=1}^{m=\infty}\sin\frac{2m\pi z}{\omega}\int_\zeta^{\zeta+\omega}\sin\frac{2m\pi\zeta}{\omega}f(\zeta)d\zeta.$$

ART. 23. UNIFORM CONVERGENCE.

Let the series $W = w_0 + w_1 + w_2 + \ldots + w_n + \ldots$, each term of which is a function of z, be convergent at every point of a given region S. Denote by W_n the sum of the first n terms of W. If it is possible, whatever the value of the positive quantity ϵ, to determine an integer ν, such that whenever $n > \nu$

$$|W - W_n| < \epsilon$$

at every point of S, the series W is said to be uniformly convergent in the region S.

Uniformly convergent series can in many respects be treated in exactly the same manner as sums containing a finite number of terms.

Theorem I.—A uniformly convergent series, the terms of which are continuous functions of z, is itself a continuous function of z.

For at any point z, W may be written in the form

$W = W_n + R$; and at a neighboring point z', $W' = W_n' + R'$. Hence

$$W - W' = W_n - W_n' + R - R',$$

and
$$|W - W'| \leqq |W_n - W_n'| + |R| + |R'|.$$

But by choosing n sufficiently great, $|R|$ and $|R'|$ may both be made less than any given positive quantity $\frac{\epsilon}{3}$. Having chosen n thus, W_n becomes the sum of a finite number of continuous functions. It is then continuous, and, by making $|z' - z|$ less than a suitable quantity δ, $|W - W_n'|$ may be made less than $\frac{\epsilon}{3}$. But, under these suppositions,

$$|W - W'| < \epsilon.$$

W is, therefore, continuous at the point z.

Theorem II.—If all the terms of a uniformly convergent series

$$W = w_0 + w_1 + \ldots + w_n + \ldots$$

are continuous, the integral of the series, for any path L situated in the region of uniform convergence, is the sum of the integrals of its terms:

$$\int_L W dz = \int_L w_0 dz + \int_L w_1 dz + \ldots + \int_L w_n dz + \ldots$$

For, writing $W = W_n + R$, it is possible to choose n so that, however small ϵ may be, $|R| < \epsilon$ at every point of L. If n be so chosen,

$$\int_L W dz = \int_L W_n dz + \int_L R dz.$$

But, by Article 15, denoting by l the length of the path L,

$$\text{mod} \int_L R dz < \epsilon l,$$

which, when $n = \infty$, has zero for its limit. Hence

$$\int_L W dz = \lim_{n=\infty} \int_L W_n dz.$$

Theorem III.—If the series $W = w_0 + w_1 + \ldots + w_n + \ldots$ is convergent, and the series

$$W' = \frac{dw_0}{dz} + \frac{dw_1}{dz} + \ldots + \frac{dw_n}{dz} + \ldots$$

is uniformly convergent in a region S, and if further the terms of W' are continuous in S, W' will be the derivative of W.

For, integrating W' from a to z along a path L contained in S,

$$\int_L W'dz = w_0(z) - w_0(a) + \ldots + w_n(z) - w_n(a) + \ldots$$
$$= W(z) - W(a).$$

But the derivative of the first member is W', which must also be the derivative of the second member, and therefore of W.

An immediate consequence of the preceding theorems is the following:

Theorem IV.—If the terms of the convergent series

$$W = w_0 + w_1 + \ldots + w_n + \ldots$$

are holomorphic in a given region S, contained in the region of convergence, and if the series

$$W' = \frac{dw_0}{dz} + \frac{dw_1}{dz} + \ldots + \frac{dw_n}{dz} + \ldots$$

is uniformly convergent, W will be holomorphic in the region S, and will have W' for its derivative.

To illustrate by an example that uniformity of convergence is essential to the preceding theorems, take the series

$$W = \frac{1}{1+z} + \sum_1^\infty \frac{z^n(1-z)}{(1+z^n)(1+z^{n+1})}.$$

At the point $z = 1$ each term is continuous, and the series is convergent, having the value $1/2$. The series is, however, discontinuous at $z = 1$. For, writing it in the form

$$W = \frac{1}{1+z} + \left(\frac{1}{1+z^2} - \frac{1}{1+z}\right) + \left(\frac{1}{1+z^3} - \frac{1}{1+z^2}\right) + \ldots,$$

the sum of the first n terms is

$$W_n = \frac{1}{1 + z^n}.$$

But W is the limit of W_n when $n = \infty$, and is therefore unity at every point z for which $|z| < 1$, and zero at every point for which $|z| > 1$.

If now this series be considered for the points within and upon a circle described about the origin as a center with an assigned radius less than unity, the remainder after n terms, or $1 - W_n = \frac{z^n}{1 + z^n}$ can, by a suitable choice of n, be made less in absolute value than any given quantity. In such a region, then, the series converges uniformly, and, by Theorem I, can have no point of discontinuity. A similar result holds for the region exterior to any circle described about the origin as a center with an assigned radius greater than unity.

By means of Theorem II given above it can be shown that Laurent's Series is unique. For, assuming the notation used in the determination of the series, the series is uniformly convergent in the region included between any two given circles concentric with C_1 and C_2, both being interior to C_1 and exterior to C_2.

Suppose, now, that two such series are possible:

$$f(a + t) = \sum_{m = \infty}^{m = \infty} A_m t^m = \sum_{m = -\infty}^{m = \infty} A_m' t^m.$$

Divide by t^{n+1}, and integrate along any circle described about a as a center and included in the region of uniform convergence. The integral $\int t^{m-n-1} dt$ for such a path is zero, except when $m = n$; the integral $\int t^{-1} dt = 2i\pi$.

Hence for such a path,

$$\int \frac{f(a + t)dt}{t^{n+1}} = 2i\pi A_n = 2i\pi A_n';$$

from which it follows that $A_n = A_n'$, and the two series are identical.

ART. 24. ONE-VALUED FUNCTIONS WITH CRITICAL POINTS.

Theorem I.—A function holomorphic in a region S and not equal to a constant, can take the same value only at isolated points of S.

For in the neighborhood of any point a interior to S, by Taylor's theorem,

$$f(z) - f(a) = (z-a)f'(a) + \frac{(z-a)^2}{1 \cdot 2} f''(a) + \ldots$$

Unless $f(z)$ is constant over the entire circle of convergence of this series, the derivatives $f'(a)$, $f''(a)$, ... cannot all be equal to zero. Let $f^{(n)}(a)$ be the first which is not equal to zero. Then

$$f(z) - f(a) = (z-a)^n\left[\frac{f^{(n)}(a)}{1 \cdot 2 \ldots n} + \frac{f^{(n+1)}(a)}{1 \cdot 2 \ldots (n+1)}(z-a) + \ldots\right]$$

If $|z-a|$ be given a finite value sufficiently small, the modulus of the first term of the series within the brackets will exceed the sum of the moduli of all the other terms, and the same result will hold for every still smaller value of $|z-a|$. For values of z, then, distant from a by less than a certain finite amount, $f(z) - f(a)$ is different from zero.

If, on the other hand, the function is constant over the entire circle, described about a as a center, within which Taylor's series converges, it will be possible, by giving in succession new positions to the point a, to show that the value of the function is constant over the whole region S.

Theorem II.—Two functions which are both holomorphic in a given region S and are equal to each other for a system of points which are not isolated from one another, are equal to each other at every point of S.

For let $f(z)$ and $\phi(z)$ be two such functions. By the preceding theorem, the difference $f(z) - \phi(z)$ must be equal to zero at every point of S.

Theorem III.—A function which is holomorphic in every part of the z-plane, even at infinity, is constant.

For, a being any given point, whatever the value of z,

$$f(z) = f(a) + (z - a)f'(a) + \ldots + \frac{(z-a)}{1 \cdot 2 \ldots n} f^{(n)}(a) + \ldots$$

But by Article 20, r being the radius of any arbitrary circle having its center at a, and M being the upper extreme of the modulus of $f(z)$ on the circumference of this circle,

$$\text{mod}\, f^{(n)}(a) \leqq \frac{1 \cdot 2 \ldots nM}{r^n}.$$

But M is always finite, and r may be made indefinitely great. Hence $f^{(n)}(a) = 0$ for all values of n, and

$$f(z) = f(a).$$

Theorem IV.—If a function $f(z)$, holomorphic in a region S, is equal to zero at the point a situated within S, the function can be expressed in the form

$$f(z) = (z - a)^m \phi(z),$$

where m is a positive integer, and $\phi(z)$ is holomorphic in S and different from zero at a.

For in the neighborhood of the point a, by Taylor's Theorem,

$$f(z) = f(a) + (z - a)f'(a) + \ldots$$

Let $f^{(m)}(a)$ be the first of the successive derivatives at a which is not equal to zero. Then

$$f(z) = (z-a)^m \left[\frac{f^{(m)}(a)}{1 \cdot 2 \ldots m} + \frac{f^{(m+1)}(a)}{1 \cdot 2 \ldots (m+1)}(z-a) + \ldots \right],$$

which is the required form. The point a is a zero of $f(z)$, and m is its order.

Theorem V.—If the point a is a critical point of a given function $f(z)$, but is interior to a region S, in which the reciprocal of $f(z)$ is holomorphic, the function can be expressed in the form

$$f(z) = \frac{\chi(z)}{(z-a)^m},$$

where m is a positive integer, and $\chi(z)$ is holomorphic in the neighborhood of a.

For by the preceding theorem

$$\frac{1}{f(z)} = (z-a)^m \phi(z),$$

where $\phi(z)$ is holomorphic and not equal to zero at $z = a$. Hence

$$f(z) = \frac{1}{(z-a)^m} \cdot \frac{1}{\phi(z)} = \frac{\chi(z)}{(z-a)^m}.$$

Further, since in a region of finite extent including the point a

$$\chi(z) = A_0 + A_1(z-a) + \ldots,$$

$$f(z) = \frac{A_0}{(z-a)^m} + \ldots + \frac{A_{m-1}}{z-a} + \psi(z),$$

a being an ordinary point for $\psi(z)$.

The point a is a pole of $f(z)$ and m is its order.

Theorem VI.—A function, not constant in value, and having no finite critical points except poles, must take values arbitrarily near to every assignable value.

For suppose that $f(z)$ is such a function, but that it takes no value for which the modulus of $f(z) - A$ is less than a given positive quantity ϵ. Then the function

$$\frac{1}{f(z) - A}$$

will be holomorphic in every part of the z-plane, which, by Theorem III, is impossible unless $f(z)$ is a constant.

Theorem VII.—A function $f(z)$, having no critical point except a pole at infinity, is a rational entire function of z.

For the only critical point of $f\left(\frac{1}{z}\right)$ is a pole at the origin. Hence

$$f\left(\frac{1}{z}\right) = \frac{A_m}{z^m} + \ldots + \frac{A_1}{z} + \phi(z),$$

where $\phi(z)$ is holomorphic over the entire plane, including the point at infinity. $\phi(z)$ is consequently equal to a constant A_0. The given function therefore can be written in the form

$$f(z) = A_m z^m + \ldots + A_1 z + A_0.$$

Theorem VIII.—A function $f(z)$ whose only critical points are poles is a rational function of z.

The poles must be at determinate distances from one another; otherwise the reciprocal of $f(z)$ would be equal to zero for points not isolated from one another. The number of poles cannot increase indefinitely as $|z|$ is increased; for then the reciprocal of $f\left(\frac{1}{z}\right)$ would be an infinite number of zeros indefinitely near to the origin. The total number of poles is therefore finite. Let $a, b, \ldots$ denote them. In the neighborhood of a the function can be expressed in the form

$$\frac{A_m}{(z-a)^m} + \ldots + \frac{A_1}{z-a} + \phi(z),$$

a being an ordinary point for $\phi(z)$. In the neighborhood of b, $\phi(z)$ can be expressed in the form

$$\frac{B_n}{(z-b)^n} + \ldots + \frac{B_1}{z-b} + \psi(z),$$

a and b being both ordinary points for $\psi(z)$. Proceeding in this way the given function will be expressed as the sum of a finite number of rational fractions and a term which can have no critical point except a pole at infinity. This term is a rational entire function.

Theorem IX.—If the function $f(z)$ has no zeros and no critical points for finite values of z, it can be expressed in the form $f(z) = e^{g(z)}$, where $g(z)$ is holomorphic in every finite region of the z-plane.

For $\dfrac{f'(z)}{f(z)}$ can have no critical points except at infinity, since in every finite region of the z-plane $f(z)$ and $f'(z)$ are holomorphic and $f(z)$ is different from zero. Hence, choosing an arbitrary lower limit z_0, the integral

$$\int_{z_0} \frac{f'(z)}{f(z)} = h(z)$$

is holomorphic in every finite region. The function $f(z)$ consequently must take the form

$$f(z) = f(z_0)e^{h(z)} = e^{g(z)},$$

where $$g(z) = h(z) + \log f(z_0).$$

Theorem X.—If two functions $f(z)$ and $\phi(z)$ have no critical points in the finite portion of the z-plane except poles, and if these poles are identical in position and in order for the two functions, and their zeros are also identical in position and order, there must exist a relation of the form

$$f(z) = \phi(z)e^{g(z)},$$

where $g(z)$ is holomorphic in every finite region of the z-plane.

For the ratio of the two functions has no zeros and no critical points in the finite portion of the z-plane.

ART. 25. RESIDUES.

If a one-valued function has an isolated critical point a, it is expressible by Laurent's series in the region comprised between any two concentric circles described about a with radii less than the distance from a to the nearest critical point. Hence in the neighborhood of a

$$f(z) = A_0 + A_1(z-a) + A_2(z-a)^2 + \ldots$$
$$+ B_1(z-a)^{-1} + B_2(z-a)^{-2} + \ldots$$

The coefficient of $(z-a)^{-1}$ in this expansion is called the "residue" of $f(z)$ at the point a.

If any closed curve C including the point a be drawn in the region of convergence of this series, and $f(z)$ be integrated along C in a positive direction, the result will be

$$\int_C (fz)dz = 2\pi i B_1.$$

The following may be regarded as an extension of Cauchy's theorem:

Theorem I.—If in a region S the only critical points of the one-valued function $f(z)$ are the interior points a, a', . . , the integral $\int f(z)dz$ taken around its boundary C in a positive direction is equal to

$$\int_C f(z)dz = 2\pi i(B + B' + \ldots),$$

where B, B', . . . are the residues of $f(z)$ at the critical points. For the integral taken along C is equal to the sum of the integrals whose paths are mutually exterior small circles described about the points a, a', . . .

The following theorems are immediate consequences of the preceding:

Theorem II.—If in a region having a given boundary C the only critical points of the one-valued function $f(z)$ are poles interior to C, an equation

$$\int_C \frac{f'(z)}{f(z)}dz = 2i\pi(M - N)$$

exists, M denoting the number of zeros and N the number of poles within C, each such point being taken a number of times equal to its order.

For in the neighborhood of the point a

$$f(z) = (z - a)^m \phi(z)$$

where $\phi(z)$ is finite and different from zero at a, and m is a positive integer if a is a zero, a negative integer if a is a pole. Hence

$$\frac{f'(z)}{f(z)} = \frac{m}{z - a} + \frac{\phi'(z)}{\phi(z)}.$$

The integrand, therefore, has a pole at every zero and pole of $f(z)$, and its residue is the order, taken positively for a zero, and negatively for a pole.

Theorem III.—Every algebraic equation of degree n has n roots.

For let $f(z)$ represent the first member of the equation $z^n + a_1 z^{n-1} + \ldots + a_n = 0$. Since $f(z)$ has no poles in the finite part of the z-plane, the number of roots contained within any closed curve C will be given by the integral

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz.$$

But taking for C a circle described about the origin as a center with a very great radius, this integral is

$$\frac{1}{2\pi i}\int_C \frac{nz^{n-1}+(n-1)a_1z^{n-2}+\ldots}{z^n+a_1z^{n-1}+\ldots}dz = \frac{1}{2\pi i}\int_C \frac{ndz}{z}(1+\epsilon)$$

where ϵ has zero for a limit when $|z|=\infty$. Hence the limit of the preceding integral, as $|z|$ is increased, is n.

Prob. 17. Show that if $z=\infty$ is an ordinary point of $f(z)$, that is, if $f(z)$ is expressible for very great value of z by a series containing only negative powers of z, the integral of $f(z)$ around an infinitely great circle is equal to $2\pi i$ into the coefficient of $\frac{1}{z}$. This coefficient is called the residue for $z=\infty$.

Prob. 18. Show that the sum of all the residues of $f(z)$, of the preceding problem, including the residue at infinity, is equal to zero.

Prob. 19. If $\frac{\phi(z)}{\psi(z)}$ is a rational function of which the numerator is of degree lower by 2 than the denominator, and if the zeros $a_1, a_2, \ldots, a_n$ of the denominator are of the first order, show that

$$\sum_1^n \frac{\phi(a_\nu)}{\psi'(a_\nu)} = 0.$$

ART. 26. INTEGRAL OF A ONE-VALUED FUNCTION.

It was shown in Article 18 that, if a function $f(z)$ is holomorphic in a given region S, its integral taken from a fixed lower limit contained in S to a variable upper limit z is a one-valued function of z within S. If $F(z)$ is a function which takes a determinate value $F(z_0)$ at $z=z_0$ and is one-valued while z remains within S, having at every point $f(z)$ for its derivative, the integral of $f(z)$ from z_0 to z is equal to $F(z)-F(z_0)$. If $F_1(z)$ is another function fulfilling these conditions, so that the integral of $f(z)$ can be written also in the form $F_1(z) - F_1(z_0)$, the functions $F(z)$ and $F_1(z)$ differ only by a constant term; for

$$F_1(z) = F(z) + [F_1(z_0) - F(z_0)].$$

Suppose now that $f(z)$ is still one-valued in S, but that it has isolated critical points $a_1, a_2, \ldots$ interior to S. Any two paths from z_0 to z, which inclose between them a region containing none of the points $a_1, a_2, \ldots$, will give integrals identical in value. Let the two paths L_1, L include between them a single critical point a_κ; and consider the integrals along these two paths. The integral along L_1 will be equal to the integral along the composite path $L_1L^{-1}L$, where the exponent -1 indicates that the corresponding path is reversed; for the integral along $L^{-1}L$ is equal to zero. But L_1L^{-1} is a closed curve, or "loop," including the critical point a_κ, and, assuming that it is described in a positive direction about a_κ, the integral along it is equal to $2\pi i B_\kappa$, where B_κ is the residue of $f(z)$ at a_κ. Hence

$$\int_{L_1} f(z)dz = 2\pi i B_\kappa + \int_L f(z)dz.$$

If now the two paths L_1, L from z_0 to z include between them several critical points $a_\kappa, a_\lambda, a_\mu, \ldots$, draw intermediate paths $L_2, \ldots, L_m$, so that the region between any two consecutive paths contains only one critical point. The integral along L_1 will be equal to the integral along the composite path $L_1L_2^{-1}L_2 \ldots L_m^{-1}L_mL^{-1}L$, since the integrals corresponding to $L_2^{-1}L_2, \ldots, L_m^{-1}L_m$, $L^{-1}L$ are all equal to zero. But $L_1L_2^{-1}$, $L_2L_3^{-1}, \ldots, L_mL^{-1}$ are all closed paths or loops, each including a single critical point, so that, assuming that each is described in a positive direction and that $B_\kappa, B_\lambda, B_\mu, \ldots$ denote the residues of $f(z)$ at the critical points,

$$\int_{L_1} f(z)dz = 2\pi i(B_\kappa + B_\lambda + B_\mu + \ldots) + \int_L f(z)dz.$$

It has been assumed in the preceding that neither of the paths L_1, L intersects itself. In the case where a path. for example L_1, intersects itself in several points $c_1, c_2, \ldots$, it is possible to consider L_1 as made up of a path L_1' not intersecting itself, together with a series of loops attached to L_1' at the points $c_1, c_2, \ldots$ Each of these loops encloses a single critical point a_κ and, if described in a positive direction, adds to the integral a term $2\pi i B_\kappa$ Each such loop described in a negative direction adds a term of the form $-2\pi i B_\kappa$. It is evident that the form of each loop and the point at which it is attached to L_1' may be altered arbitrarily without altering the value of the integral, provided no critical point be introduced into or removed from the loop. In fact all the loops may be regarded as attached to L_1' at z_0.

It can be proved by similar reasoning that the most general path that can be drawn from z_0 to z will be equivalent, so far as the value of the integral is concerned, to any given path L preceded by a series of loops, each of which includes a single critical point and is described in either a positive or negative direction. The value of the integral is therefore of the form

$$\int_L f(z)dz + 2\pi i(m_1 B_1 + m_2 B_2 + \ldots),$$

where $m_1, m_2, \ldots$ are any integers positive or negative.

As an example consider the integral $\int_{z_0}^{z} \frac{dz}{z-a}$. The only critical point is $z = a$. Any path whatsoever from z_0 to z is equivalent to a determinate path, for example, a rectilinear path, preceded by a loop containing a and described a certain number of times in a positive or negative direction. If w denote the integral for a selected path, the general value of the integral will be $w + 2n\pi i$. If now a straight line be drawn joining z_0 to a, and if along its prolongation from a to infinity the z-plane be cut or divided, the integral in the z-plane thus divided is one-valued. But, with the variation of z thus restricted, any branch of the function $\log(z-a)$ is one-valued. Select that branch, for example, which reduces to zero when $z = a + 1$. It takes a determinate value for $z = z_0$, and its derivative for every value of z is $\frac{1}{z-a}$. Hence, denoting it by Log $(z-a)$,

$$\int_{z_0}^{z}\frac{dz}{z-a} = \text{Log}\,(z-a) - \text{Log}\,(z_0-a) = \text{Log}\,\frac{z-a}{z_0-a}.$$

For a path not restricted in any way, the value of the integral is

$$\int_{z_0}^{z}\frac{dz}{z-a} = \text{Log}\,\frac{z-a}{z_0-a} \pm 2n\pi i = \log\frac{z-a}{z_0-a}.$$

Prob. 20. If $\frac{\phi(z)}{\psi(z)}$ is a rational function of z of which the numerator is of degree lower by 2 than the denominator, and if the zeros $a_1, a_2, \ldots, a_n$ of the denominator be of the first order, show that

$$\int_{z_0}^{z}\frac{dz}{z-a} = \sum_{1}^{n}\frac{\phi(a_\nu)}{\psi'(a_\nu)}\log\frac{z-a_\nu}{z_0-a_\nu},$$

where $\sum_{1}^{n}\phi(a_\nu)/\psi'(a_\nu) = 0$. (See Prob. 18, Art. 25.)

ART. 27. WEIERSTRASS'S THEOREM.

Any rational entire function of z, having its zeros at the points $a_1, a_2, \ldots, a_m$, can be put in the form

$$A(z-a_1)^{n_1}(z-a_2)^{n_2}\ldots(z-a_m)^{n_m},$$

where A is a constant and $n_1, n_2, \ldots, n_m$ are positive integers. More generally, any function which has no critical point in the finite portion of the z-plane and has the points $a_1, \ldots, a_m$ as its zeros, is of the form

$$e^{g(z)}(z-a_1)^{n_1}\ldots(z-a_m)^{n_m},$$

where $g(z)$ is holomorphic in every finite region.

The extension of this result to the case where a function without finite critical points has an infinite number of zeros is due to Weierstrass. It is effected by means of the following theorem:

Theorem.—Given an infinite number of isolated points a_1,

$a_2, \ldots, a_n, \ldots$, a function can be constructed holomorphic except at infinity and equal to zero at each of the given points only.*

For the given points can be taken so that

$$|a_1| \leqq |a_2| \leqq \ldots |a_n| \leqq \ldots,$$

$|a_n|$ increasing indefinitely with n. Consider the infinite product

$$\phi(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{P_n(z)},$$

where $P_n(z)$ denotes the rational entire function

$$P_n(z) = \frac{z}{a_n} + \ldots + \frac{z^n}{n a_n^n}.$$

Any factor may be written in the form

$$\left(1 - \frac{z}{a_n}\right) e^{P_n(z)} = e^{\log\left(1 - \frac{z}{a_n}\right) + P_n(z)}.$$

But since

$$\log\left(1 - \frac{z}{a_n}\right) = -\int_0^z \frac{dz}{a_n - z} = -\frac{z}{a_n} - \ldots - \frac{z^n}{n a_n^n} - \int_0^z \frac{z^n dz}{a_n^n (a_n - z)},$$

the path of integration being arbitrary except that it avoids the points $a_1, a_2, \ldots$, the product may be expressed as

$$\prod_1^\infty e^{\psi_n(z)}, \text{ in which } \psi_n(z) = -\int_0^z \frac{z^n dz}{a_n^n (a_n - z)}.$$

In any given finite region of the z-plane it will be possible to assume that $|z| \leqq \rho < a_m$, since $|a_n|$ increases indefinitely with n. Divide the product into two parts,

$$\prod_1^{m-1} \left(1 - \frac{z}{a_n}\right) e^{P_n(z)} . \prod_m^\infty e^{\psi_n(z)}.$$

The second part is equal to

$$e^{\sum_m^\infty \psi_n(z)}.$$

* The following proof is taken from Jordan, Cours d'Analyse, 2d edition, Vol. II.

Consider the series $\sum\limits_m^\infty \psi_n(z)$ and $\sum\limits_m^\infty \psi_n'(z)$, each term of the second being the derivative of the corresponding term of the first. In the given region

$$|\psi_n'(z)| = \left| -\frac{z^n}{a_n^n(a_n - z)} \right| \overline{\overline{<}} \frac{\rho^n}{|a_m|^n(|a_m| - \rho)}.$$

Each term of $\sum\limits_m^\infty \psi_n'(z)$ is accordingly less in absolute value than the corresponding term of a convergent geometrical progression independent of z. The series $\sum\limits_m^\infty \psi_n'(z)$, therefore, converges uniformly. The series $\sum\limits_m^\infty \psi_n(z)$ also converges, since

$$|\psi_n(z)| = \text{mod} \int_0^z \psi_n'(z)dz \overline{\overline{<}} \frac{\rho^n l}{|a_m|^n(|a_m| - \rho)},$$

where l denotes the length of the path of integration.

By Theorem IV, of Article 23, the series $\sum\limits_m^\infty \psi_n(z)$ represents in the given region a holomorphic function. The exponential

$$e^{\sum\limits_m^\infty \psi_n(z)}$$

also must be holomorphic. The other part of the product

$$\prod_1^{m-1}\left(1 - \frac{z}{a_n}\right)e^{P_n(z)}$$

containing only a finite number of factors is everywhere holomorphic, vanishing at all of the points $a_1, a_2, \ldots$, which are situated within the given finite region. But this region may be extended arbitrarily. The product therefore fulfils the required conditions.

In the preceding demonstration it was tacitly assumed that none of the given points $a_1, a_2, \ldots$ was situated at the origin. To introduce a zero at the origin it is necessary merely to multiply the result by a power of z.

The most general function without finite critical points having its only zeros at the given points $a_1, a_2, \ldots, a_n \ldots$, can be expressed in the form

$$f(z) = e^{g(z)}\prod_1^\infty\left(1 - \frac{z}{a_n}\right)e^{P_n(z)},$$

where $g(z)$ is holomorphic except at infinity; for the ratio of any two functions satisfying the required conditions is neither infinite nor zero at any finite point.

By means of Weierstrass's theorem it is possible to express any function, $F(z)$, whose only finite critical points are poles as the ratio of two functions holomorphic except at infinity. For, construct a function $\psi(z)$ having the poles of $F(z)$ as its zeros. The product $F(z) \cdot \psi(z) = \phi(z)$ will have no finite critical point. The given function can, therefore, be written

$$F(z) = \frac{\phi(z)}{\psi(z)},$$

which is the required form.

In applying Weierstrass's theorem to particular examples, it will rarely be found necessary to include in the polynomials $P_n(z)$ so many terms as were employed in the demonstration given above. It is quite sufficient, of course, to choose these polynomials in any way which will make the product converge for finite values of z to a holomorphic function. Factors of the form

$$\left(1 - \frac{z}{a_n}\right)e^{P_n(z)},$$

where $P_n(z)$ is chosen in such a manner, are called "primary factors."

As an application of Weierstrass's Theorem take the resolution of sin z into primary factors. The zeros of sin z are 0, $\pm\pi, \pm 2\pi, \ldots, \pm n\pi, \ldots$. Consider factors of the form

$$\left(1 - \frac{z}{n\pi}\right)e^{\frac{z}{n\pi}}$$

so that $P_n(z)$ contains only one term $\frac{z}{n\pi}$, and

$$\psi_n(z) = -\int_0^z \frac{zdz}{n\pi(n\pi - z)}.$$

The series $\sum_m \psi_n'(z)$ will converge uniformly in any region at every point of which $|z| \leqq \rho < m\pi$; for, since

$$|\psi_n'(z)| = \left| - \frac{z}{n\pi(n\pi - z)} \right| \leqq \frac{\rho}{n^2\pi^2\left(1 - \frac{\rho}{|n\pi|}\right)} \leqq \frac{\rho}{n^2\pi^2\left(1 - \frac{\rho}{m\pi}\right)},$$

each term is less in absolute value than the corresponding term of the series

$$\frac{\rho}{\pi^2\left(1 - \frac{\rho}{m\pi}\right)} \sum_m^\infty \frac{1}{n^2}.$$

A similar result holds for the series $\sum_{-m}^{-\infty} \psi_n'(z)$. The two series

$$\sum_m^\infty \psi_n(z), \quad \sum_{-m}^{-\infty} \psi_n(z)$$

are also convergent; for $|\psi_n(z)|$ cannot exceed the upper extreme of $|\psi_n'(z)|$ multiplied by l, the length of the path of integration from the origin to the point z. These series accordingly represent holomorphic functions in the region for which $|z| \leqq \rho$. Hence the expression required is

$$\sin z = e^{g(z)} \prod_{-\infty}^{+\infty} \left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}}.$$

It will be shown in the course of the next article that $e^{g(z)} = 1$.

Prob. 21. If ω_1 and ω_2 be two quantities not having a real ratio, the doubly infinite series of which the general term is $\frac{1}{(m\omega_1 + n\omega_2)^p}$ is absolutely convergent if $p > 2$. Hence show that the product

$$\sigma(z) = z\Pi\left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{z^2}{2\omega^2}},$$

where $\omega = m\omega_1 + n\omega_2$, defines a holomorphic function in any finite region of the z-plane. This function is Weierstrass's sigma function, and is the basis of his system of elliptic functions.

ART. 28. MITTAG-LEFFLER'S THEOREM.

Any one-valued function $f(z)$ with isolated critical points $a_1, a_2, \ldots$ can be represented in the neighborhood of one of these points by Laurent's series; viz.:

$$f(z) = A_0 + A_1(z - a_n) + A_2(z - a_n)^2 + \ldots \\ + B_1(z - a_n)^{-1} + B_2(z - a_n)^{-2} + \ldots.$$

Hence $$f(z)^n = \phi(z) + G_n\left(\frac{1}{z - a_n}\right),$$

where $\phi(z)$ is holomorphic in a region containing the point a_n, and $G\left(\frac{1}{z - a_n}\right)$ is holomorphic over the whole plane excluding the point a_n. If a_n is a pole of $f(z)$, $G_n\left(\frac{1}{z - a_n}\right)$ contains a finite number of terms; otherwise it is an infinite series. If the number of critical points is finite, and the function $G_n\left(\frac{1}{z - a_n}\right)$ is formed at each such point, by subtracting the sum of these functions from $f(z)$ a remainder will be obtained which has no critical point in the finite part of the plane. This remainder can be expressed as a series of ascending powers $G(z)$ converging for every finite value of z. The function $f(z)$ can therefore be written in the form

$$f(z) = G(z) + \Sigma G_n\left(\frac{1}{z - a_n}\right),$$

analogous to the expression of a rational function by means of partial fractions.

The extension of this result to the case where the number of critical points is infinite is due to Mittag-Leffler. Let a_1, $a_2, \ldots, a_n, \ldots$ be the critical points of the one-valued function $f(z)$, and suppose that

$$|a_1| \gtreqless |a_2| \gtreqless \ldots |a_n| \gtreqless \ldots,$$

$|a_n|$ increasing without limit when n is increased indefinitely. Let, further, $G_n\left(\frac{1}{z - a_n}\right)$ be the series of negative powers of

$z-a_n$ contained in the expansion of $f(z)$ according to Laurent's Series in the neighborhood of a_n.

The function $G_n\left(\frac{1}{z-a_n}\right)$, having no critical point except at a_n, may be developed by Maclaurin's series in the form

$$G_n\left(\frac{1}{z-a_n}\right) = A_0^{(n)} + A_1^{(n)}z + \ldots + A_\nu^{(n)}z + \ldots,$$

and the series will converge uniformly within a circle described about the origin as a center with any determinate radius $\rho_n < |a_n|$. Within the same circle Maclaurin's series, applied to $G_n'\left(\frac{1}{z-a_n}\right)$, the derivative with respect to z of $G_n\left(\frac{1}{z-a_n}\right)$, converges uniformly. Hence, for any point within the circle $|z| = \rho_n$,

$$G_n\left(\frac{1}{z-a_n}\right) = F_n(z) + R, \quad G_n'\left(\frac{1}{z-a_n}\right) = F_n'(z) + R',$$

$F_n(z)$ representing the first $\nu + 1$ terms of the development of $G_n\left(\frac{1}{z-a_n}\right)$ by Maclaurin's theorem, $F_n'(z)$ its derivative, and R, R' remainders which by a suitable choice of ν may be made less in absolute value than any given quantity.

Choose the positive quantities $E_1, E_2, \ldots, E_n, \ldots$ so that the series $E_1 + E_2 + \ldots + E_n + \ldots$ is convergent. Choose also in connection with each of the points $a_1, a_2, \ldots, a_n, \ldots$, an integer ν such that

$$\operatorname{mod}\left[G_1\left(\frac{1}{z-a_1}\right) - F_1(z)\right] < E_1, \quad \operatorname{mod}\left[G_1'\left(\frac{1}{z-a_1}\right) - F_1'(z)\right] < E_1,$$

if $|z| \leqq \rho_1 < |a_1|$;

$$\operatorname{mod}\left[G_2\left(\frac{1}{z-a_2}\right) - F_2(z)\right] < E_2, \quad \operatorname{mod}\left[G_2'\left(\frac{1}{z-a_2}\right) - F_2'(z)\right] < E_2,$$

if $|z| \leqq \rho_2 < |a_2|$; and, in general,

$$\operatorname{mod}\left[G_n\left(\frac{1}{z-a_n}\right) - F_n(z)\right] < E_n, \quad \operatorname{mod}\left[G_n'\left(\frac{1}{z-a_n}\right) - F_n'(z)\right] < E_n,$$

if $|z| \leqq \rho_n < |a_n|$.

Consider now the series

$$\sum_1^\infty\left[G_n\left(\frac{1}{z-a_n}\right)-F_n(z)\right],\quad \sum_1^\infty\left[G_n'\left(\frac{1}{z-a_n}\right)-F'(z)\right]$$

in any finite region of the plane, the points $a_1, a_2, \ldots, a_n, \ldots$ being excluded. Since $|a_n|$ increases indefinitely with n, it is possible, in any finite region of the z-plane, to assume that $|z| \leqq \rho_m < |a_m|$. Separate from each of these two series its first $m-1$ terms. These terms will have in each case a finite sum. The remaining terms of either series taken in order will be less in absolute value than $E_m, E_{m+1}, \ldots$ respectively, $|z|$ being less than each of the quantities $\rho_m, \rho_{m+1}, \ldots$. Accordingly, each of the series

$$\sum_1^\infty\left[G_n\left(\frac{1}{z-a_n}\right)-F_n(z)\right],\quad \sum_1^\infty\left[G_n'\left(\frac{1}{z-a_n}\right)-F_n'(z)\right]$$

is absolutely convergent for every value of z except $a_1, a_2, \ldots, a_n, \ldots$. It is evident, further, that in any given finite region, from which the points $a_1, a_2, \ldots, a_n, \ldots$ are excluded, the two series converge uniformly. In such a region any term of either series is holomorphic; and, therefore, by Theorem IV of Article 23, the first of these series defines a holomorphic function.

The point a_n is an ordinary point for the difference

$$f(z)-\left[G_n\left(\frac{1}{z-a_n}\right)-F_n(z)\right]=\left[f(z)-G_n\left(\frac{1}{z-a_n}\right)\right]+F_n(z),$$

since in its neighborhood this difference may be developed as a convergent series containing only positive powers of $z-a_n$. In the same way each of the points $a_1, a_2, \ldots, a_n, \ldots$ is an ordinary point for the function

$$f(z)-\sum_1^\infty\left[G_n\left(\frac{1}{z-a_n}\right)-F_n(z)\right].$$

This function, therefore, can have no critical point except at infinity, and must be expressible as a series $G(z)$ containing only positive powers of z and converging uniformly in any finite region of the z-plane. Hence the function $f(z)$ may be put in the form

$$f(z) = G(z) + \sum_{1}^{\infty} \left[G_n\left(\frac{1}{z - a_n}\right) - F_n(z) \right],$$

in which the character of each critical point is exhibited.

As an application of Mittag-Leffler's theorem consider cot z. Its critical points are $z = 0, \pm \pi, \pm 2\pi, \ldots$. In the neighborhood of $z = 0$, $\cot z - \frac{1}{z}$ is holomorphic; and in the neighborhood of $z = n\pi$, n being any positive or negative integer, $\cot z - \frac{1}{z - n\pi}$ is holomorphic. The series

$$\sum_{m}^{+\infty} \frac{1}{z - n\pi},$$

in which m is an arbitrary positive integer, is not convergent for finite values of z, even when $|z| < m$. The series

$$\sum_{m}^{+\infty} \left[\frac{1}{z - n\pi} + \frac{1}{n\pi}\right] = \sum_{m}^{+\infty} \frac{z}{n\pi(z - n\pi)} = \sum_{m}^{+\infty} \frac{-z}{n^2\pi^2\left(1 - \frac{z}{n\pi}\right)}$$

is, however, absolutely convergent at every point for which $|z| < m$. For the modulus of any term is equal to

$$\frac{|z|}{n^2\pi^2\left|1 - \frac{z}{n\pi}\right|} \overline{\overline{<}} \frac{|z|}{n^2\pi^2\left(1 - \frac{|z|}{n\pi}\right)},$$

and, therefore, less than the corresponding term in the series

$$\frac{|z|}{\pi^2\left(1 - \frac{|z|}{m\pi}\right)} \sum_{m}^{\infty} \frac{1}{n^2}.$$

A similar result holds for the series

$$\sum_{m}^{\infty} \left[\frac{1}{z + n\pi} - \frac{1}{n\pi}\right].$$

It is easy to see now that the reasoning employed in the demonstration of Mittag-Leffler's theorem may be applied to show that the series

$$\frac{1}{z}+\sum_{-\infty}^{+\infty}\left[\frac{1}{z-n\pi}+\frac{1}{n\pi}\right],$$

where the summation does not include $n=0$, defines a function holomorphic in any finite region of the z-plane, the points $0, \pm\pi, \pm 2\pi, \ldots$ being excluded. The difference

$$\cot z-\frac{1}{z}-\sum_{-\infty}^{+\infty}\left[\frac{1}{z-n\pi}+\frac{1}{n\pi}\right]$$

can have no critical point except at infinity. It must, therefore, be expressible as a series $G(z)$ of positive powers of z, having an infinite circle of convergence. Hence

$$\cot z=G(z)+\frac{1}{z}+\sum_{-\infty}^{+\infty}\left[\frac{1}{z-n\pi}+\frac{1}{n\pi}\right].$$

The next step is to determine $G(z)$. It is to be observed that, if $G(z)$ is a constant, its value must be zero, since $\cot(-z)=-\cot z$. If $G(z)$ is not a constant, differentiation of the preceding expression for $\cot z$ gives

$$-\frac{1}{\sin^2 z}=G'(z)-\frac{1}{z^2}-\sum_{-\infty}^{+\infty}\frac{1}{(z-n\pi)^2}.$$

It follows, by changing z into $z+\pi$, that

$$G'(z+\pi)=G'(z).$$

Hence $G'(z)$ is periodic, having a period equal to π; and as the point z traces a line parallel to the axis of reals, $G'(z)$ passes again and again through the same range of values. But $G'(z)$, being the derivative of $G(z)$, is holomorphic for every finite value of z. It can, therefore, become infinite, if at all, only when the imaginary part of z is infinite. If z be written in the form $x+iy$, the value of $G'(z)$ may be expressed as

$$G'(z)=\frac{1}{(x+iy)^2}+\sum_{-\infty}^{+\infty}\frac{1}{(x+iy-n\pi)^2}-\left(\frac{2ie^y(\cos x+i\sin x)}{(\cos 2x+i\sin 2x)-e^{2y}}\right)^2.$$

When $y=\pm\infty$ the first and last terms of the second member vanish. In regard to the series it can be proved that, for any given region is which y is finite and different from zero, an integer ν can be found such that the sum of the moduli of those terms for which $|n| > \nu$ is less in absolute value than any previously assigned quantity ϵ. As $|y|$ is increased the modulus of each of these terms is diminished. The modulus of their sum, therefore, cannot exceed ϵ when $y = \pm\infty$. But when $y = \pm\infty$ the sum of any finite number of terms of the series is zero. Hence the limit of the whole series is zero. $G'(z)$, therefore, never becomes infinite. Hence, by Theorem III, Article 24, it is constant, and is equal to zero. It follows that $G(z)$ is equal to zero.

The expression for cot z is accordingly

$$\cot z = \frac{1}{z} + \sum_{-\infty}^{+\infty}\left[\frac{1}{z - n\pi} + \frac{1}{n\pi}\right].$$

The logarithmic derivative of the product expression for sin z, given in the preceding article as an example of Weierstrass's theorem, is

$$\cot z = g'(z) + \frac{1}{z} + \sum_{-\infty}^{+\infty}\left[\frac{1}{z - n\pi} + \frac{1}{n\pi}\right].$$

Hence $g(z)$ in that expression is a constant. Making $z = 0$, its value is seen to be unity.

Prob. 22. From the expression for cot z deduce the equation

$$\operatorname{cosec}^2 z = \sum_{-\infty}^{+\infty}\frac{1}{(z - n\pi)^2},$$

where the summation does not exclude $n = 0$.

Prob. 23. Show that the doubly infinite series

$$\wp(z) = \frac{1}{z^2} + \sum\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right],$$

where $\omega = m\omega_1 + n\omega_2$, defines a function whose only finite critical points are $z = \omega$. This function is Weierstrass's $\wp$-function. (Compare Problem 21.)

Prob. 24. Prove that

$$\wp(z) = -\frac{d^2}{dz^2}\log\sigma(z).$$

Prob. 25. Prove that $\wp'(z) = -2\Sigma\frac{1}{(z-\omega)^3}$, where the summation does not exclude $\omega = 0$.

ART. 29. CRITICAL LINES AND REGIONS.

The functions whose properties have been considered in the preceding articles have been assumed to have only isolated critical points. That an infinite number of critical points may be grouped together in the neighborhood of a single finite point is evident, however, from the consideration of such examples as

$$w = \cot\frac{1}{z}, \qquad w = e^{\operatorname{cosec}\frac{1}{z-a}}.$$

In the former an infinite number of poles are grouped in the neighborhood of the origin. In the latter an infinite number of essential singularities are situated in the vicinity of the point $z = a$.

It is easy to illustrate by an example the occurrence of lines and regions of discontinuity. Take the series *

$$\theta(z) = \frac{1}{1-z} + \frac{z}{z^2-1} + \frac{z^2}{z^4-1} + \frac{z^4}{z^8-1} + \ldots$$

The sum of its first n terms is

$$-\frac{1}{z^{2^{n-1}}-1},$$

which converges to unity if $|z| < 1$, and to zero if $|z| > 1$. Hence the circle $|z| = 1$ is a line of discontinuity for this series.

Consider now any two regions S_1 and S_2, the former situated within, the latter without, the unit circle. Let $\phi(z)$ and $\psi(z)$ be two arbitrary functions both completely defined in these regions. The expression

$$\phi(z)\theta(z) + \psi(z)[1 - \theta(z)]$$

* This series is due to J. Tannery. See Weierstrass, Abhandlungen aus der Functionenlehre (1886), p. 102.

will be equal to $\phi(z)$ in S_1 and $\psi(z)$ in S_2. In regions completely separated from one another by a critical line, the same literal expression may thus represent entirely independent functions.

For a single continuous region, however, in the interior of which exist only isolated critical points, the character of the function in one part determines its character in every other part. Let S be such a region, and assume that its boundary is a critical line. In the neighborhood of any interior point a, not a critical point, the given function is expressible as a power series, viz.:

$$f(z) = f(a) + (z-a)f'(a) + \ldots + \frac{(z-a)^n}{1 \cdot 2 \ldots n} f^{(n)}(a) + \ldots$$

This series will converge uniformly over a circle described about a as a center with any determinate radius less than the distance from a to the nearest critical point. It serves for the calculation of $f(z)$ and all its successive derivatives at any point b interior to this circle. From the preceding power series, accordingly, can be obtained another

$$f(z) = f(b) + (z-b)f'(b) + \ldots + \frac{(z-b)^n}{1 \cdot 2 \ldots n} f^{(n)}(b) + \ldots,$$

representing the $f(z)$ within a circle described about b as a center. In general, the point b can be so chosen that a portion of this new circle will lie without the circle of convergence of the former power series. At any new point c within the circle whose center is b, the value of the function and all its successive derivatives can be calculated; and so, as before, a power series can be obtained convergent in a circle described about c as a center and, in general, including points not contained in either of the preceding circles. By continuing in this manner it will be possible, starting from a given point a with the expression of $f(z)$ in ascending powers, to obtain an expression of the same character at any other point k which can be connected with a by a continuous line everywhere at a finite distance from the nearest critical point. It follows that the character of the function everywhere within S can be determined completely from its expression in ascending power series in the neighborhood of a single interior point.

It will be impossible by the process just explained to derive any information in regard to the function at points exterior to S. The example given above, furthermore, shows that a complete definition of $f(z)$ within S may carry with it the definition of an entirely independent function without S.

As an example of a function having a critical region consider the function defined by the series

$$1 + 2z + 2z^4 + 2z^9 + \ldots,$$

which represents a function without critical points in the interior of the circle $|z| = 1$. For points on or without this circle the series is divergent; and, further, it is impossible to obtain from it an expression converging when $|z| \geqq 1$. The function thus defined, consequently, exists only in the region interior to the unit circle. By changing z into $1/z$ a series

$$1 + \frac{2}{z} + \frac{2}{z^4} + \frac{2}{z^9} + \ldots$$

is obtained, representing a function which has no existence in the interior of the unity circle. Functions in connection with which such regions arise are called "lacunary functions." *

ART. 30. FUNCTIONS HAVING n VALUES.

Let the function $w = f(z)$ take at the point z_0 of a given region S a value $w^{(0)}$. Suppose that along any continuous path, beginning at z_0, and subject only to the conditions that it shall remain in the interior of S and shall not pass through certain isolated points $a_1, a_2, \ldots, w$ is continuous and has a continuous derivative. If it is impossible, when z traces such a path, to return to the point z_0 so as to obtain there a value of w different from $w^{(0)}$, w is one-valued in the region S. On the other

* Poincaré, American Journal of Mathematics, Vol. XIV; Harkness and Morley, Theory of Functions (1893), p. 119

hand, certain paths may lead back to z_0 with new values of w.

Suppose that at each point of S, except $a_1, a_2, \ldots, w$ has n different values, and that starting from such a point z_0 and tracing any continuous curve not passing through $a_1, a_2, \ldots,$ the several values of w give rise to n branches $w_1, w_2, \ldots, w_n$, each of which is characterized by a continuous derivative. In the neighborhood of a_k any one of the points $a_1, a_2, \ldots$ these branches are said to be distinct or not, according as small closed curves described about this point lead from each value of w back to the same value again, or cause some of the branches to interchange values. In the latter case the point is a branch point.

About any branch point a_k as a center describe a small circle; and suppose that, starting from any point of it with the value w_α corresponding to a certain branch, the values $w_\beta, w_\gamma \ldots$ are obtained by successive revolutions about a_k, the original value being reproduced after p revolutions. Introduce now a new independent variable z' such that

$$z' = (z - a_k)^{\frac{1}{p}}.$$

It can be shown that when z makes one revolution about a_k, z' makes only one pth part of a revolution about the origin of the z'-plane, and that to a complete revolution of z' about the origin of the z'-plane correspond p revolutions of z about a_k. Considering then the branch w_α as a function of z', the origin cannot be a branch point, for whenever z' describes a small circle about it, the value w_α is reproduced. The branch w_α must accordingly be expressible by Laurent's series in the form

$$w_\alpha = \sum_{-\infty}^{+\infty} A_m z'^m,$$

or, substituting for z' its value,

$$w_\alpha = A_0 + A_1(z - a_k)^{\frac{1}{p}} + A_2(z - a_k)^{\frac{2}{p}} + \ldots$$
$$+ A_{-1}(z - a_k)^{-\frac{1}{p}} + A_{-2}(z - a_k)^{-\frac{2}{p}} + \ldots$$

This expression makes plain the relation between the different branches of a function in the neighborhood of a branch point. When the development of a branch in the neighborhood of one of its branch points gives rise to only a finite number of terms containing negative powers, the branch point is called a "polar branch point."

Consider the functions

$$P_1 = w_1 + w_2 + \ldots + w_n ,$$
$$P_2 = w_1 w_2 + w_1 w_3 = \ldots + w_{n-1} w_n ,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$P_n = w_1 w_2 \ldots w_n .$$

Each of these functions is unchanged in value when several or all of the quantities $w_1, w_2, \ldots, w_n$ are interchanged, and is consequently a one-valued function of z within S. Hence w must satisfy an equation of the nth degree,

$$w^n + P_1 w^{n-1} + P_2 w^{n-2} + \ldots + P^n = 0,$$

the coefficients of which are one-valued functions of z having only isolated critical points within S. When the entire z-plane can be taken as the region S, and those branch points at which the branches do not all remain finite are polar branch points, the only other critical points being poles for one or more branches, the functions $P_1, P_2, \ldots, P_n$ are rational functions of z. In this case w is an algebraic function of z.