On the Surfaces with Plane or Spherical Curves of Curvature.

By Prof. Cayley.

The theory is considered in two nearly cotemporaneous papers—Bonnet, "Mémoire sur les surfaces dont les lignes de courbure sont planes ou sphériques," *Jour. de l'Ecole Polyt.* t. XX (1853), pp. 117–306, and Serret, "Mémoire sur les surfaces dont toutes les lignes de courbure sont planes ou sphériques," *Liouville*, t. XVIII (1853), pp. 113–162. I desire to reproduce in a more compact form, and with some additional developments, the chief results obtained in these elaborate memoirs.

The basis of the theory is a theorem by Lancret, 1806. In any curve described upon a surface, the angle between the osculating planes at consecutive points is equal to the difference of the angles between the osculating planes and the corresponding tangent planes of the surface.

This includes as a particular case Joachimsthal's theorem, *Crelle*, t. XXX (1846): If a surface have a plane curve of curvature, then at any point thereof the angle between the plane of the curve and the tangent plane of the surface has a constant value.

Bonnet and Serret each deduce the like theorem for a spherical curve of curvature, viz: If a surface have a spherical curve of curvature, then at any point thereof the angle between the tangent plane of the sphere and the tangent plane of the surface has a constant value. Bonnet (Mémoire, p. 235) says that this follows from Lancret's theorem. Serret (Mémoire, p. 128) obtains it, by the transformation by reciprocal radius vectors, from Joachimsthal's theorem.

I remark that the theorem for a spherical curve of curvature, and (as a particular case thereof) that for a plane curve of curvature, are obtained at once from the most elementary geometrical considerations, viz: if we have (in the same plane or in different planes) the two isosceles triangles NPP', OPP' on a common base PP', then the angle OPN is equal to the angle $OP'N$. For take

P, P' consecutive points on a spherical curve of curvature; then at P, P' the normals of the surface meet in a point N, and the normals (or radii) of the sphere meet in the centre O, and we have angle $OPN =$ angle $OP'N$, that is, at each of these points the inclination of the normal of the surface to the normal of the sphere has the same value; and this value being thus the same for any two consecutive points, must be the same for all points of the curve of curvature. The proof applies to the plane curve of curvature; but in this case the fundamental theorem may be taken to be, a line at right angles to the base PP' of the isosceles triangle NPP' is equally inclined to the two equal sides NP, NP'.

A surface may have one set of its curves of curvature plane or spherical. To include the two cases in a common formula, the equation may be written $k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0$; $k = 1$ in the case of a sphere, $= 0$ in that of a plane; and the expression a sphere may be understood to include a plane. I write in general A, B, C to denote the cosines of the inclinations of the normal of the surface at the point (x, y, z) to the axes of coordinates (consequently $A^2 + B^2 + C^2 = 1$). Hence considering a surface, and writing down the equations

$$k(x^2+y^2+z^2) - 2ax - 2by - 2cz - 2u = 0,$$
$$(kx-a)A + (ky-b)B + (kz-c)C = l,$$

where (a, b, c, u, l) are regarded as functions of a parameter t, the first of these equations is that of a variable sphere; and the second equation expresses that at a point of intersection of the surface with the sphere, the inclination of the tangent plane of the surface to the tangent plane of the sphere has a constant value l, viz: this is a value depending only on the parameter t, and therefore constant for all points of the curve of intersection of the sphere and surface: by what precedes, the curve of intersection is a curve of curvature of the surface, and the surface will thus have a set of spherical curves of curvature.

Supposing the surface defined by means of expressions of its coordinates (x, y, z) as functions of two variable parameters, we may for one of these take the parameter t which enters into the equation of the sphere; and if the other parameter be called θ, then the expressions of the coordinates are of the form $x, y, z = x(t, \theta), y(t, \theta), z(t, \theta)$ respectively; these give equations $dx, dy, dz = a\,dt + a'd\theta,\ b\,dt + b'd\theta,\ c\,dt + c'd\theta$ (where of course (a, b, c, a', b', c') are in general functions of t, θ), and we have A, B, C proportional to

$bc' - b'c$, $ca' - c'a$, $ab' - a'b$, viz: the values are equal to these expressions each divided by the square root of the sum of their squares. In order that the surface may have a set of spherical curves of curvature, the above three equations must be satisfied identically by means of the values of

$$a,\ b,\ c,\ u,\ l,\ A,\ B,\ C,\ x,\ y,\ z$$

as functions of (t, θ); and it may be seen without difficulty that we are thereby led to a partial differential equation of the first order for the determination of the surface. But I do not at present further consider this question of the determination of a surface having one set of its curves of curvature (plane or) spherical.

Suppose now that there is a second set of (plane or) spherical curves of curvature. We have in like manner

$$\begin{aligned} \varkappa(x^2 + y^2 + z^2) - 2\alpha x - 2\beta y - 2\gamma z - 2v &= 0, \\ (\varkappa x - \alpha)A + (\varkappa y - \beta)B + (\varkappa z - \gamma)C - \lambda &= 0, \end{aligned}$$

where $\varkappa$ is $=1$ or $=0$ according as the curves are spherical or plane, and $(\alpha, \beta, \gamma, v, \lambda)$ are functions of a variable parameter θ. We take the t of the former set of equations and the θ of these equations as the two parameters in terms of which the coordinates (x, y, z) are expressed. This being so (the former equations being satisfied as before), if these equations are satisfied identically by the values of $\alpha, \beta, \gamma, v, \lambda, A, B, C, x, y, z$ as functions of (t, θ), then the surface will have its other set of curves of curvature also spherical. It will be recollected that by hypothesis a, b, c, u, l are functions of the parameter t only, and that $\alpha, \beta, \gamma, v, \lambda$ functions of the parameter θ only. The foregoing equations, together with the assumed relations

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

are the "six equations" for the determination of a surface having its two sets of curves of curvature each of them (plane or) spherical.

Assuming now the values of a, b, c, l, u as functions of t, and $\alpha, \beta, \gamma, \lambda, v$ as functions of θ, the question at once arises whether we can then satisfy the six equations. These equations other than $Adx + Bdy + Cdz = 0$, or say the five equations, in effect determine any five of the eight quantities $A, B, C,$

x, y, z, t, θ, in terms of the remaining three, say they determine A, B, C, t, θ as functions of x, y, z: we thus have a differential equation $Adx + Bdy + Cdz = 0$ wherein A, B, C are to be regarded as given functions of (x, y, z). An equation of this form is not in general integrable; and if the equation in question be not integrable, then clearly the system of equations cannot be satisfied by any value of z as a function of (x, y), or, what is the same thing, by any values of (x, y, z) as functions of (t, θ). We thus arrive at the condition, the equation must be integrable, viz: the condition is

$$\nabla, = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right), = 0.$$

If this be satisfied, then we have an integral equation $I = 0$ (containing a constant of integration which is an absolute constant) and which is in fact the equation of the required surface. But it is proper to look at the question somewhat differently. Supposing that the condition $\nabla = 0$ is satisfied, then we have the integral equation $I = 0$, and this equation, together with the five equations, in effect determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two of them, or, what is the same thing, they determine a relation between any three of these quantities. We can, from the five equations and their differentials, and from the equation $Adx + Bdy + Cdz = 0$, obtain a differential equation between any three of the eight quantities: and it has just been seen that corresponding hereto we have an integral relation between the same three quantities; that is (the condition $\nabla = 0$ being satisfied), we can from the six equations obtain between any three of the quantities $A, B, C, x, y, z, t, \theta$ a linear differential equation of the foregoing form (for instance $Zdz + Tdt + \Theta d\theta = 0$, where Z, T, Θ are given functions of z, t, θ) which will *ipso facto* be integrable, furnishing between z, t, θ an integral equation which may be used instead of the before-mentioned integral equation $I = 0$. And we thus have (without any further integration) in all six equations which serve to determine any six of the quantities $A, B, C, x, y, z, t, \theta$ in terms of the remaining two. It is often convenient to seek in this way for the expressions of (A, B, C and) x, y, z as functions of t, θ in preference to seeking for the integral equation $I = 0$ between the coordinates x, y, z.

The condition $\nabla = 0$ is in fact the condition which expresses that at any point of the surface the two curves of curvature intersect at right angles. Serret (and after him Bonnet) in effect obtain the condition by the assumption of this geometrical relation, without showing that the geometrical relation is in fact the necessary condition for the coexistence of the six equations. They give the condition in the form $dx\delta x + dy\delta y + dz\delta z = 0$, where dx, dy, dz are the increments of (x, y, z) along one of the curves of curvature, and δx, δy, δz the increments along the other curve of curvature. The equations give

$$\begin{aligned}(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz &= 0,\\ A dx + B dy + C dz &= 0,\end{aligned}$$

and similarly

$$\begin{aligned}(\varkappa x - \alpha)\,\delta x + (\varkappa y - \beta)\,\delta y + (\varkappa z - \gamma)\,\delta z &= 0,\\ A\delta x + B\delta y + C\delta z &= 0.\end{aligned}$$

We thence have

$$dx : dy : dz =$$
$$B(kz - c) - C(ky - b) : C(kx - a) - A(kz - c) : A(ky - b) - B(kx - a),$$

and

$$\delta x : \delta y : \delta z =$$
$$B(\varkappa z - \gamma) - C(\varkappa y - \beta) : C(\varkappa x - \alpha) - A(\varkappa z - \gamma) : A(\varkappa y - \beta) - B(\varkappa x - \alpha).$$

We have thus the required condition, in a form which is readily changed into

$$\begin{aligned}&(A^2 + B^2 + C^2)\{(kx - a)(\varkappa x - \alpha) + (ky - b)(\varkappa y - \beta) + (kz - c)(\varkappa z - \gamma)\}\\ &-\{A(kx - a) + B(ky - b) + C(kz - c)\}\{A(\varkappa x - \alpha) + B(\varkappa y - \beta) + C(\varkappa z - \gamma)\} = 0,\end{aligned}$$

and writing herein $A^2 + B^2 + C^2 = 1$, this becomes

$$\begin{aligned}&\tfrac{1}{2}\varkappa\{k(x^2 + y^2 + z^2) - 2ax - 2by - 2cz\}\\ &+ \tfrac{1}{2}k\{\varkappa(x^2 + y^2 + z^2) - 2\alpha x - 2\beta y - 2\gamma z\}\\ &+ (a\alpha + b\beta + c\gamma) - l\lambda = 0,\end{aligned}$$

that is,

$$a\alpha + b\beta + c\gamma - l\lambda + \varkappa u + kv = 0.$$

I proceed to show that this is the condition $\nabla = 0$ for the integrability of the differential equation $Adx + Bdy + Cdz = 0$. Writing as before

$$\nabla = A\left(\frac{dB}{dz} - \frac{dC}{dy}\right) + B\left(\frac{dC}{dx} - \frac{dA}{dz}\right) + C\left(\frac{dA}{dy} - \frac{dB}{dx}\right),$$

we have from the six equations

$$AdA + BdB + CdC = 0,$$
$$(kx - a)\,dA + (ky - b)\,dB + (kz - c)\,dC$$
$$= -k\,(Adx + Bdy + Cdz) + (Aa_1 + Bb_1 + Cc_1 + l_1)\,dt,$$
$$(\varkappa x - \alpha)\,dA + (\varkappa y - \beta)\,dB + (\varkappa z - \gamma)\,dC$$
$$= -\varkappa\,(Adx + Bdy + Cdz) + (A\alpha' + B\beta' + C\gamma' + \lambda')\,d\theta,$$
$$(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz = (a_1 x + b_1 y + c_1 z + u_1)\,dt,$$
$$(\varkappa x - \alpha)\,dx + (\varkappa y - \beta)\,dy + (\varkappa z - \gamma)\,dz = (\alpha' x + \beta' y + \gamma' z + \upsilon')\,d\theta,$$

where a_1, b_1, c_1, l_1, u_1 denote derived functions in regard to t and α', β', γ', λ', υ' derived functions in regard to θ. Putting for shortness

$$\Omega = \begin{vmatrix} A, & B, & C, \\ kx - a, & ky - b, & kz - c \\ \varkappa x - \alpha, & \varkappa y - \beta, & \varkappa z - \gamma \end{vmatrix}$$

we readily obtain

$$\Omega dA = [(\varkappa y - \beta)\,C - (\varkappa z - \gamma)\,B]\left\{-k\,(Adx + Bdy + Cdz) + \frac{Aa_1 + Bb_1 + Cc_1 + l_1}{a_1 x + b_1 y + c_1 z + u_1}\{(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz\}\right\}$$
$$- [(ky - b)\,C - (kz - c)\,B]\left\{-\varkappa\,(Adx + Bdy + Cdz) + \frac{A\alpha' + B\beta' + C\gamma' + \lambda'}{\alpha' x + \beta' y + \gamma' z + \upsilon'}\{\varkappa x - \alpha)\,dx + (\varkappa y - \beta)\,dy + (\varkappa z - \gamma)\,dz\}\right\};$$

say this is

$$\Omega dA = [(\varkappa y - \beta)\,C - (\varkappa z - \gamma)\,B]\left\{-k\,(Adx + Bdy + Cdz) + \frac{L}{P}\{(kx - a)\,dx + (ky - b)\,dy + (kz - c)\,dz\}\right\}$$
$$-[(ky - b)\,C - (kz - c)\,B]\left\{-\varkappa\,(Adx + Bdy + Cdz) + \frac{\Lambda}{\Pi}\{(\varkappa x - \alpha)\,dx + (\varkappa y - \beta)\,dy + (\varkappa z - \gamma)\,dz\}\right\},$$

or introducing further abbreviations, and writing down the analogous values of ΩdB and ΩdC, we have

$$\Omega dA = [(\varkappa y - \beta)\,C - (\varkappa z - \gamma)\,B]\,U - [(ky - b)\,C - (kz - c)\,B]\,\Upsilon,$$
$$\Omega dB = [(\varkappa z - \gamma)\,A - (\varkappa x - \alpha)\,C]\,U - [(kz - c)\,A - (kx - a)\,C]\,\Upsilon,$$
$$\Omega dC = [(\varkappa z - \alpha)\,B - (\varkappa y - \beta)\,A]\,U - [(kx - a)\,B - (ky - b)\,A]\,\Upsilon.$$

We hence find

$$\Omega\frac{dB}{dz} = \quad [(\varkappa z-\gamma)A-(\varkappa x-\alpha)C]\Big(-kC+\frac{L}{P}(kz-c)\Big)$$
$$-[(kz-c)A-(kx-a)C]\Big(-\varkappa C+\frac{\Lambda}{\Pi}(\varkappa z-\gamma)\Big),$$

$$-\Omega\frac{dC}{dy} = -[(\varkappa x-\alpha)B-(\varkappa y-\beta)A]\Big(-kB+\frac{L}{P}(ky-b)\Big)$$
$$+[(kx-a)B-(ky-b)A]\Big(-\varkappa B+\frac{\Lambda}{\Pi}(\varkappa y-\beta)\Big),$$

and combining these two terms, in the resulting value of $\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)$ first the term without L or Λ is found to be

$$\begin{aligned}= &-kA\{A(\varkappa x-\alpha)+B(\varkappa y-\beta)+C(\varkappa z-\gamma)\}\\ &-k(\varkappa x-\alpha)(A^2+B^2+C^2)\\ &+\varkappa(kx-\alpha)(A^2+B^2+C^2)\\ &+\varkappa A\{A(kx-a)+B(ky-b)+C(kz-c)\},\end{aligned}$$

which is

$$\begin{aligned}&= -kA\lambda+k(\varkappa x-\alpha)-\varkappa(kx-a)+\varkappa Al,\\ &= \quad A(\varkappa l-k\lambda)-k\alpha+\varkappa a.\end{aligned}$$

Next the coefficient of $\frac{L}{P}$ is

$$\begin{aligned}&A(kz-c)(\varkappa z-\gamma)-C(\varkappa x-\alpha)(kz-c)\\ +&A(ky-b)(\varkappa y-\beta)-B(\varkappa x-\alpha)(ky-b),\end{aligned}$$

which is

$$\begin{aligned}&= A[(kx-a)(\varkappa x-\alpha)+(ky-b)(\varkappa y-\beta)+(kz-c)(\varkappa z-\gamma)]\\ &\qquad -(\varkappa x-\alpha)[A(kx-a)+B(ky-b)+C(kz-c)]\\ &= AM+(\varkappa x-\alpha)l,\end{aligned}$$

if for shortness

$$M=(kx-a)(\varkappa x-\alpha)+(ky-b)(\varkappa x-\beta)+(kz-c)(\varkappa z-\gamma);$$

and similarly the coefficient of $\frac{\Lambda}{\Pi}$ is

$$\begin{aligned}&-A(kz-c)(\varkappa z-\gamma)+C(kx-a)(\varkappa x-\alpha)\\ &-A(ky-b)(\varkappa y-\beta)+B(kx-b)(\varkappa y-\beta)\end{aligned}$$

which is

$$\begin{aligned}&= -A\,[(kx-a)(\varkappa x-\alpha)+(ky-b)(\varkappa y-\beta)+(kz-c)(\varkappa z-\gamma)]\\ &\quad -(kx-a)[A\,(\varkappa x-\alpha)+B\,(\varkappa y-\beta)+C\,(\varkappa z-\gamma)]\\ &= -AM-(kx-a)\,\lambda\,.\end{aligned}$$

We thus obtain

$$\Omega\left(\frac{dB}{dz}-\frac{dC}{dy}\right)=A\,(\varkappa l-k\lambda)-k\alpha+\varkappa a+\frac{L}{P}\{AM+(\varkappa x-\alpha)\,l\}$$
$$-\frac{\Lambda}{\Pi}\{AM+(kx-a)\,\lambda\}\,,$$

and similarly

$$\Omega\left(\frac{dB}{dx}-\frac{dA}{dz}\right)=B\,(\varkappa l-k\lambda)-k\beta+\varkappa b+\frac{L}{P}\,\{BM+(\varkappa y-\beta)\,l\}$$
$$-\frac{\Lambda}{\Pi}\,\{BM+(ky-b)\,\lambda\},$$

$$\Omega\left(\frac{dA}{dy}-\frac{dB}{dx}\right)=C\,(\varkappa l-k\lambda)-k\gamma+\varkappa c+\frac{L}{P}\,\{\,CM+(\varkappa z-\gamma)\,l\}$$
$$-\frac{\Lambda}{\Pi}\,\{\,CM+(kz-c)\,\lambda\}\,,$$

and hence multiplying by A, B, C and adding, we obtain

$$\Omega\nabla=\varkappa l-k\lambda-k\,(A\alpha+B\beta+C\gamma)+\varkappa\,(Aa+Bb+Cc)$$
$$+\frac{L}{P}\,(M-l\lambda)-\frac{\Lambda}{\Pi}\,(M-l\lambda),$$

where the first four terms are together

$$=\varkappa l-k\lambda+k\,\{\varkappa\,(Ax+By+Cz)-\lambda\}-\varkappa\,\{k\,(Ax+By+Cz)-l\}\,,$$

viz. these destroy each other, and the equation becomes

$$\Omega\nabla=\left(\frac{L}{P}-\frac{\Lambda}{\Pi}\right)(M-l\lambda).$$

But we have

$$M-l\lambda=\tfrac{1}{2}\,\varkappa\,\{k\,(x^2+y^2+z^2)-2ax-2by-2cz\}$$
$$+\tfrac{1}{2}\,k\,\{\varkappa\,(x^2+y^2+z^2)-2\alpha x-2\beta y-2\gamma z\}+(a\alpha+b\beta+c\gamma)-l\lambda,$$

which is $\qquad = a\alpha+b\beta+c\gamma-l\lambda+\varkappa \breve{u}+kv,$

or we find $\qquad \Omega\nabla=\left(\dfrac{L}{P}-\dfrac{\Lambda}{\Pi}\right)(a\alpha+b\beta+c\gamma-l\lambda+\varkappa u+kv),$

viz. the condition $\nabla=0$ is

$$a\alpha+b\beta+c\gamma-l\lambda+\varkappa u+kv=0\,,$$

the result which was to be proved.

If we consider separately the cases where the two sets of curves of curvature are each plane, the first plane and the second spherical, and each of them spherical; or say the cases *PP*, *PS* and *SS*, then in these cases respectively the condition is

$$a\alpha + b\beta + c\gamma - l\lambda = 0,$$
$$a\alpha + b\beta + c\gamma - l\lambda + u = 0,$$
$$a\alpha + b\beta + c\gamma - l\lambda + u + v = 0:$$

we have in each case to take the italic letters functions of t and the greek letters functions of θ, satisfying identically the appropriate equation, but otherwise arbitrary; and then in each case the six equations lead to a differential equation $Adx + Bdy + Cdz = 0$ (or say $Zdz + Tdt + \Theta d\theta = 0$) between three variables, which equation is *ipso facto* integrable; and we thus obtain a new integral equation which with the original five integral equations gives the solution of the problem. The condition is in each case of the form $\Sigma a\alpha = 0$, the number of terms $a\alpha$ being 4, 5 or 6. Considering for instance the form

$$a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + f\phi = 0$$

with 6 terms, it is easy to see how such an equation is to be satisfied by values of a, b, c, d, e, f which are functions of t, and values of $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ which are functions of θ. Suppose that $t_1, t_2, \ldots$ are particular values of t, and $a_1, b_1, \ldots f_1$; $a_2, b_2, \ldots f_2$, etc., the corresponding values of $a, b, \ldots f$, these values being of course absolute constants; we have $\alpha, \beta, \ldots \phi$, functions of θ, satisfying all the equations

$$(a_1, b_1, c_1, d_1, e_1, f_1 \between \alpha, \beta, \gamma, \delta, \varepsilon, \phi) = 0,$$
$$(a_2, b_2, c_2, d_2, e_2, f_2 \between \quad " \quad) = 0,$$
$$\text{etc.},$$

and if 6 or more of these equations were independent, the equations could, it is clear, be satisfied only by the values $\alpha = \beta = \gamma = \delta = \varepsilon = \phi = 0$. To obtain a proper solution, only some number less than 6 of these equations can be independent. Suppose for instance only two of the equations are independent; we then have $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ functions of θ satisfying these equations, but otherwise arbitrary; or, what is the same thing, we may take $\alpha, \beta, \gamma, \delta, \varepsilon, \phi$ linear functions of $6 - 2, = 4$ arbitrary functions, say P, Q, R, S of θ; say we have

$$\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3 \between P, Q, R, S),$$
$$\beta = (\beta_0, \ldots\ldots \between \quad " \quad),$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\phi = (\phi_0, \ldots\ldots \between \quad " \quad),$$

where the suffixed greek letters denote absolute constants; and this being so, in order to satisfy the proposed equation $a\alpha + b\beta + c\gamma + d\delta + e\varepsilon + f\phi = 0$, we must have

$$\begin{aligned}(\alpha_0, \beta_0, \gamma_0, \delta_0, \varepsilon_0, \phi_0 \between a, b, c, d, e, f) &= 0,\\ (\alpha_1, \ldots\ldots\ldots\ldots \between \quad \text{"} \quad) &= 0,\\ (\alpha_2, \ldots\ldots\ldots\ldots \between \quad \text{"} \quad) &= 0,\\ (\alpha_3, \ldots\ldots\ldots\ldots \between \quad \text{"} \quad) &= 0,\end{aligned}$$

viz. a, b, c, d, e, f will then be functions of t satisfying these four equations, but otherwise arbitrary. The above is a solution for the partition $2+4$ of the number 6. We have in like manner a solution for any other partition of 6; or if we disregard the extreme cases $a = b = c = d = e = f = 0$ and $\alpha = \beta = \gamma = \delta = \varepsilon = \phi = 0$, then we have in this manner solutions for the several partitions 15, 24, 33, 42 and 51 of the number 6.

But applying this theory to the actual problem, there is a good deal of difficulty as regards the enumeration of the really distinct cases. I use the letters P, S to denote that a set of curves of curvature is plane or spherical as the case may be, the surfaces to be considered are thus PP, PS, and SS. First, for the PP problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda = 0$, the two systems (a, b, c, l) and $(\alpha, \beta, \gamma, \lambda)$ are symmetrically related to each other, and instead of the solutions 13, 22 and 31, it is sufficient to consider the solutions 13 and 22. But here (a, b, c, l) are not a system of four symmetrically related functions, (a, b, c) are a symmetrical system, and l is a distinct term: and the like for the system $(\alpha, \beta, \gamma, \lambda)$. In the PS problem, where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u = 0$, and thus the systems (a, b, c, l, u), $(\alpha, \beta, \gamma, \lambda, 1)$ are of different forms, we should consider the solutions 14, 23, 32 and 41: but here again in each of the systems separately the terms are not symmetrically related to each other. Lastly, in the SS problem where the equation is $a\alpha + b\beta + c\gamma - l\lambda + u + v = 0$, the systems $(a, b, c, l, u, 1)$ and $(\alpha, \beta, \gamma, \lambda, 1, v)$ are of the same form, it is enough to consider the solutions 15, 24 and 33; but in this case also in each of the systems separately the terms are not symmetrically related to each other. I do not at present further consider the question, but simply adopt Serret's enumeration.

It is to be remarked that for a developable (but not for a skew surface) the generating lines may be curves of curvature, and regarding the generating lines as plane curves we might have developables PP or PS; but a straight line is not a curve in a *determinate* plane, and it is better to consider the case apart from the general theory. Again, the curves of curvature of one set or those of each set may be circles; and a circle may be regarded either as a plane or a spherical curve; regarding it, however, as a spherical curve, it is a curve not in a *determinate* sphere. The cases in question, of the curves of curvature of the one set or of those of each set being circles, are therefore also to be considered apart from the general theory. The surfaces referred to present themselves for consideration among Serret's cases PP 1^0, 2^0, 3^0; PS 1^0, 2^0, 3^0, 4^0, 5^0, 6^0, 7^0; and SS 1^0, 2^0, 3^0, 4^0; but they are excluded from his enumeration, and he in fact reckons in his "Conclusion," pp. 161, 162, two kinds of surfaces PP, three kinds PS, and two kinds SS.

It is very easily seen that if a surface has a plane or spherical curve of curvature, then on any parallel surface the corresponding curve is a plane or spherical curve of curvature: and thus if a surface be PP, PS, or SS, then the parallel surfaces are respectively PP, PS or SS. The solutions obtained include for the most part all the parallel surfaces, and thus there is no occasion to make use of this theorem; but see in the continuation of the present paper the case considered under the subheading *post*, $PS4^0 =$ Serret's third case of PS.

If a surface have a plane or spherical curve of curvature, then transforming the surface by reciprocal radius vectors (or inverting in regard to an arbitrary point), then in the transformed surface the corresponding curve is a spherical curve of curvature. Hence if a surface be PP, PS or SS, the transformed surface is SS. Conversely, as shown by Bonnet and Serret, and as will appear, every surface SS is in fact an inversion of a surface PP or PS.

I proceed to the enumeration, developing the theory only in regard to the two, three, and two, cases PP, PS and SS respectively.

PP, THE TWO SETS OF CURVES OF CURVATURE EACH PLANE.

The six equations are

$$\begin{aligned}
A^2 + B^2 + C^2 &= 1, \\
ax + by + cz + u &= 0, \\
Aa + Bb + Cc + l &= 0, \\
\alpha x + \beta y + \gamma z + v &= 0, \\
A\alpha + B\beta + C\gamma + \lambda &= 0, \\
Adx + Bdy + Cdz &= 0,
\end{aligned}$$

the condition is

$$a\alpha + b\beta + c\gamma - l\lambda = 0$$

(not containing u or v, so that these remain arbitrary functions of t, θ respectively). The cases are

	a	b	c	l	α	β	γ	λ
$PP1^0$	1	0	c	0	0	1	0	λ
$PP2^0$	0	1	0	$-m$	α	$-m\lambda$	γ	λ
$PP3^0$	1	0	c	mc	0	1	$m\lambda$	λ

m is an arbitrary constant, and in the body of the table c is an arbitrary function of t, and α, γ, λ arbitrary functions of θ.

$PP1^0$ is Serret's first case of PP, included in his second case.

$PP2^0$ gives developable.

$PP3^0$ is Serret's second case of PP.

I consider the case

$PP3^0$ = SERRET'S SECOND CASE OF PP.

Writing for greater symmetry $m = g$, $\frac{1}{m} = f$, so that $fg = 1$; also $m\lambda = \gamma$, and consequently $\lambda = f\gamma$, we take c and γ for the two parameters respectively, or write $c = t$, $\gamma = \theta$; also changing the letters u, v, we write

	a	b	c	l	u	α	β	γ	λ	v
=	1,	0,	t	gt	P	0	1	θ	$f\theta$	Π

and the six equations thus are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ x + tz - P &= 0, \\ A + tC - gt &= 0, \\ y + \theta z - \Pi &= 0, \\ B + \theta C - f\theta &= 0, \\ Adx + Bdy + Cdz &= 0. \end{aligned}$$

We seek for the differential equation in z, t, θ. We have

$$A^2 + B^2 + C^2 = 1, \quad A = t(g - C), \quad B = \theta(f - C),$$

and thence $$t^2(g-C)^2+\theta^2(f-C)^2+C^2=1,$$
that is $$C^2(1+t^2+\theta^2)+2C(gt^2+f\theta^2)=1-g^2t^2-f^2\theta^2,$$
or multiplying by $1+t^2+\theta^2$ and completing the square,

$$\begin{aligned}\{(1+t^2+\theta^2)C-gt^2-f\theta^2\}^2&=(1-g^2t^2-f^2\theta^2)(1+t^2+\theta^2)-(gt^2+f\theta^2)^2\\ &=\{f+(f-g)t^2\}\{g+(g+(g-f)\theta^2\}\\ &=\frac{1}{T^2\Theta^2},\end{aligned}$$

if
$$\frac{1}{T^2}=f+(f-g)t^2,$$
$$\frac{1}{\Theta^2}=g+(g-f)\theta^2,$$

and thence giving a determinate sign to the square root, say

$$(1+t^2+\theta^2)C=gt^2+f\theta^2-\frac{1}{T\Theta},$$

an equation which may also be written

$$C=\frac{fT-g\Theta}{f-g}.$$

In fact, observing that $\frac{1}{T^2}-\frac{1}{\Theta^2}=(f-g)(1+t^2+\theta^2)$, we deduce from the original form

$$\begin{aligned}\left(\frac{1}{T^2}-\frac{1}{\Theta^2}\right)C&=(f-g)(gt^2+f\theta^2)-\frac{f-g}{T\Theta},\\ &=g\left(\frac{1}{T^2}-f\right)-f\left(\frac{1}{\Theta^2}-g\right)-\frac{f-g}{T\Theta}\\ &=\left(\frac{g}{T}-\frac{f}{\Theta}\right)\left(\frac{1}{T}+\frac{1}{\Theta}\right),\end{aligned}$$

or throwing out the factor $\frac{1}{T}+\frac{1}{\Theta}$, and reducing, we have the required value; and thence forming the values of A and B, we have

$$A=-tT\frac{f-g}{T-\Theta},\quad B=-\theta\Theta\frac{f-g}{T-\Theta},\quad C=\frac{fT-g\Theta}{T-\Theta};$$

we have, moreover,

$$x+tz=P,\quad y+\theta z=\Pi,$$

or differentiating, and writing P_1 and Π' for the derived functions in regard to t and θ respectively,

$$dx=-tdz-zdt+P_1dt,\quad dy=-\theta dz-zd\theta-\Pi'd\theta.$$

The equation $Adx + Bdy + Cdz = 0$ thus becomes

$$-Ttdx - \Theta\theta dy + \frac{fT - g\Theta}{f-g}\,dz = 0,$$

viz. this is

$$[-tT(-tdz - zdt + P_1dt) - \theta\Theta(-\theta dz - zd\theta + \Pi' dt)](f-g) + (fT - g\Theta)\,dz = 0,$$

or collecting,

$$[(f + (f-g)t^2)\,T - (g + (g-f))\,\Theta]\,dz + (tTzdt + \theta\Theta zd\theta)(f-g)$$
$$-(tTP_1dt + \theta\Theta\Pi' d\theta)(f-g) = 0,$$

that is,

$$\left(\frac{1}{\Theta} - \frac{1}{T}\right)dz + (f-g)\,z\,(tTdt + \theta\Theta d\theta) - (f-g))(tTP_1dt + \theta\Theta\Pi' d\theta) = 0,$$

which is an integrable form as it should be, viz. the equation is

$$d\left(\frac{1}{T} - \frac{1}{\Theta}\right)z - (f-g)(tTP_1dt + \theta\Theta\Pi' d\theta) = 0,$$

and we obtain

$$\left(\frac{1}{T} - \frac{1}{\Theta}\right)z - (f-g)\int(tTP_1dt + \theta\Theta\Pi' d\theta) = 0,$$

the constant of integration being considered as included in the integral. But it is proper to alter the form of the second term. Take F, Φ arbitrary functions of t, θ respectively; and writing F_1, Φ' for the derived functions, assume $P = \frac{gF_1}{T^3}$, $\Pi = \frac{f\Phi'}{\Theta^3}$; we have

$$\int(tTP_1dt + \theta\Theta\Pi' d\theta) = \int\left(gtT\left(\frac{F_1}{T^3}\right)_1 dt + f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)' d\theta\right)$$
$$= -F + \frac{gtF_1}{T^2} - \Phi + \frac{f\theta\Phi'}{\Theta^2}.$$

In fact this will be true if only

$$\left(-F + \frac{gtF_1}{T^2}\right)_1 = gtT\left(\frac{F_1}{T^3}\right)_1, \quad \left(-\Phi + \frac{f\theta\Phi'}{\Theta^2}\right)' = f\theta\Theta\left(\frac{\Phi'}{\Theta^3}\right)',$$

which are equations of like form in t, θ respectively, and it will be sufficient to verify the first of them. Effecting the differentiation, the terms in F_{11} destroy each other, and there remain only terms containing the factor F_1, and throwing this out we obtain

$$-1 + \frac{g}{T^2} + \frac{gtT_1}{T^3} = 0,$$

viz. this is

$$-1 + g(f + (f-g)t^2) - gt^2(f-g) = 0,$$

which is identically true, and the equation is thus verified.

The foregoing result is

$$\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}=0;$$

we then have

$$x+tz-\frac{gF_1}{T^3}=0,\quad y+\theta z-\frac{f\Phi'}{\Theta^3}=0,$$

and hence, repeating also the equation for z,

$$\begin{aligned}
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)x+(f-g)\left\{-t(F+\Phi)+\frac{ft\theta\Phi'}{\Theta^2}\right\}+\left(-1+\frac{g}{\Theta T}\right)\frac{F_1}{T^2}=0,\\
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)y+(f-g)\left\{-\theta(F+\Phi)+\frac{gt\theta F_1}{T^2}\right\}+\left(1-\frac{f}{\Theta T}\right)\frac{\Phi'}{\Theta^2}=0,\\
&\left(\frac{1}{T}-\frac{1}{\Theta}\right)z+(f-g)\left\{F+\Phi-\frac{gtF_1}{T^2}-\frac{f\theta\Phi'}{\Theta^2}\right\}=0,
\end{aligned}$$

equations which give the values of the coordinates x, y, z in terms of the parameters t, θ. It will be recollected that $fg=1$ (f or g being arbitrary), that the values of T, Θ are

$$\frac{1}{T^2}=f+(f-g)t^2,\quad \frac{1}{\Theta^2}=g+(g-f)\theta^2,$$

and that F, Φ denote arbitrary functions of t, θ respectively. I repeat also the foregoing equations

$$A,\ B,\ C=-tT\frac{f-g}{T-\Theta},\quad -\theta\Theta\frac{f-g}{T-\Theta},\quad \frac{fT-g\Theta}{T-\Theta}.$$

The equations may be presented under a different form; we have

$$\begin{aligned}
&-tTx-\theta\Theta y+\frac{fT-g\Theta}{f-g}z+F+\Phi=0,\\
&-fT^3(x+tz)+F_1=0,\\
&-g\Theta^3(y+\theta z)+\Phi'=0,
\end{aligned}$$

where it will be observed that the second and third equations are the derivatives of the first equation in regard to t and θ respectively. We thus have the required surface as the envelope of the plane represented by the first equation, regarding therein t, θ as variable parameters. Moreover, the second equation (which contains only the parameter t) represents the planes of the curves of curvature of the one set; and the third equation (which contains only the parameter θ) represents the planes of the curves of curvature of the other set. It is to be observed that from the equations for l, λ (viz. $A + tC = gt$ and $B + \theta C = f\theta$) it appears that for any plane of the first set the inclination to a tangent plane of the surface is $= \cos^{-1} \dfrac{gt}{\sqrt{1+t^2}}$, and that for any plane of the second set the inclination is $= \cos^{-1} \dfrac{f\theta}{\sqrt{1+\theta^2}}$.

It may be remarked that the last mentioned results may be arrived at by the consideration of an equation $Ax + By + Cz + D = 0$, where the coefficients are functions of t and θ (A a function of t only, and B a function of θ only) such that the derived equations $A_1x + C_1z + D_1 = 0$ and $B'x + C'z + D' = 0$ depend the former of them upon t only, and the latter of them upon θ only.

A very simple case of the equation is when $f = g = 1$; here $T = \Theta = 1$, and the surface is the envelope of the plane $z - tx - \theta y + F + \Phi = 0$.

Returning to the general form

$$- tTx - \theta\Theta y + \frac{fT - g\Theta}{f-g} z + F + \Phi = 0,$$

I transform this by introducing therein in place of t, θ two variable parameters α, β which are such that $k\alpha = - tT$, $k\beta = \theta\Theta$ $\left(k \text{ a constant which is presently put } = \dfrac{1}{\sqrt{f-g}}\right)$, we find

$$t^2 = \frac{fk^2\alpha^2}{1 - (f-g)k^2\alpha^2}, \quad \theta^2 = \frac{gk^2\beta^2}{1 - (g-f)k^2\beta^2},$$

and thence

$$T = \frac{1}{\sqrt{f}} \sqrt{1 - (f-g)k^2\alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}} \sqrt{1 - (g-f)k^2\beta^2},$$

or putting $k = \dfrac{1}{\sqrt{f-g}}$, these last values are

$$T = \frac{1}{\sqrt{f}} \sqrt{1-\alpha^2}, \quad \Theta = \frac{1}{\sqrt{g}} \sqrt{1+\beta^2},$$

and we hence obtain

$$\begin{aligned} \frac{fT - g\Theta}{f-g} &= \frac{\sqrt{f}}{f-g} \sqrt{1-\alpha^2} - \frac{\sqrt{g}}{f-g} \sqrt{1+\beta^2}, \\ &= \frac{1}{\sqrt{f-g}} \left\{ \frac{\sqrt{f}}{\sqrt{f-g}} \sqrt{1-\alpha^2} - \frac{\sqrt{g}}{\sqrt{f-g}} \sqrt{1+\beta^2} \right\}, \end{aligned}$$

say this is $= k\{\lambda\sqrt{1-\alpha^2} - \mu\sqrt{1+\beta^2}\}$,

where $\lambda = \dfrac{\sqrt{f}}{\sqrt{f-g}}$, $\mu = \dfrac{\sqrt{g}}{\sqrt{f-g}}$, and therefore $\lambda^2 - \mu^2 = 1$ or $\mu = \sqrt{\lambda^2 - 1}$.

Hence writing $F + \Phi = k\,(A + B)$, k times the sum of two arbitrary functions of α and β respectively, the equation becomes

$$\alpha x - \beta y + z\,\{\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2 - 1}\,\sqrt{1+\beta^2}\} + A + B = 0$$

viz. the surface is given as the envelope of this plane considering α, β as two variable parameters. This is the solution given by Darboux, "Leçons sur la théorie générale des surfaces, etc.," Paris, 1887, pp. 128–131. He obtains it in a very elegant manner, starting from the following theorem: Take A, A_1, etc., functions of the parameter α, and B, B_1, etc., functions of the parameter β; then if we have identically

$$(A_1 - B_1)^2 + (A_2 - B_2)^2 + (A_3 - B_3)^2 = (A_4 - B_4)^2,$$

the required surface will be obtained as the envelope of the plane

$$(A_1 - B_1)\,x + (A_2 - B_2)\,y + (A_3 - B_3)\,z = A - B,$$

where A, B are two new functions of α, β respectively.

The foregoing identity is the condition in order that each sphere of the one series $(x - A_1)^2 + (y - A_2)^2 + (z - A_3)^2 = A_4^2$ may touch each sphere of the other series $(x - B_1)^2 + (y - B_2)^2 + (z - B_3)^2 = B_4^2$; the two series of spheres thus envelope one and the same surface which will have its curves of curvature of each set circles: viz. this will be the surface of the fourth order called Dupin's Cyclide, the normals whereof pass through an ellipse and hyperbola which are focal curves one of the other, and which contain the centres of all the spheres touching the surface along its curves of curvature. The equations of the ellipse and hyperbola may be taken to be

$$\left(x^2 + \frac{z^2}{\lambda^2} = 1,\ y = 0\right) \text{ and } \left(y^2 - \frac{z^2}{\lambda^2 - 1} = -1,\ x = 0\right)$$

respectively, and we thence obtain the required PP surface as the envelope of the plane

$$\alpha x - \beta y + (\lambda\sqrt{1-\alpha^2} - \sqrt{\lambda^2 - 1}\sqrt{1+\beta^2})\,z + A + B = 0.$$

THE CASE $PP1^0$ = SERRET'S FIRST CASE OF PP.

We deduce this from the second case by writing therein $m=0$, that is, $g=0$, $f=\infty$; but it is necessary to make also a transformation upon the parameter θ, viz. in place thereof we introduce the new parameter ϕ, where $\theta^2 = \frac{g\phi^2}{f-g\phi^2}$, this gives

$$\frac{1}{\Theta^2} = g + (g-f)\theta^2 = g\left(1 + \frac{(g-f)\phi^2}{f-g\phi^2}\right) = \frac{gf(1-\phi^2)}{f-g\phi^2}, \quad \theta^2 = \frac{g\phi^2}{f-g\phi^2}$$

and thence

$$\theta\Theta = \frac{\theta}{\sqrt{g+(g-f)\theta^2}} = \frac{\phi}{\sqrt{f}\sqrt{1-\phi^2}}; \quad \frac{fT-g\Theta}{f-g} \text{ for } g=0 \text{ is } = T.$$

We have also $T = \frac{1}{\sqrt{f+(f-g)t^2}}, = \frac{1}{\sqrt{f}\sqrt{1+t^2}}$ when $g=0$, and substituting these values, considering Φ as a function of ϕ, and for $F+\Phi$ writing as we may do $\frac{F+\Phi}{\sqrt{f}}$, the equation becomes

$$\frac{-t}{\sqrt{f}\sqrt{1+t^2}}x - \frac{\phi y}{\sqrt{f}\sqrt{1-\phi^2}} + \frac{z}{\sqrt{f}\sqrt{1+t^2}} + \frac{F+\Phi}{\sqrt{f}} = 0,$$

where the divisor $\sqrt{f}$ is to be omitted. Hence finally, instead of ϕ restoring the original letter θ, and again considering Φ as a function of θ, the equation is

$$\frac{z-tx}{\sqrt{1+t^2}} - \frac{\theta y}{\sqrt{1-\theta^2}} + F + \Phi = 0,$$

viz. here F, Φ are arbitrary functions of t, θ respectively, and the surface is the envelope of this plane considering t, θ as variable.

We obtain an imaginary special form of $PP1^0$ by writing in this equation $k\theta$ for θ and then putting $k=\infty$; the Φ remains an arbitrary function of the new θ, and the equation is

$$\frac{z-tx}{\sqrt{1+t^2}} + iy + F + \Phi = 0$$

($i=\sqrt{-1}$ as usual). This is in fact the equation which is obtained from $PP3^0$ by simply writing therein $g=0$ without the transformation upon θ.

PS. THE SETS OF CURVES OF CURVATURE, THE FIRST PLANE, THE SECOND SPHERICAL.

The six equations are

$$\begin{aligned}
&A^2+B^2+C^2 &&=1,\\
&ax+by+cz+u &&=0,\\
&Aa+Bb+Cc+l &&=0,\\
&x^2+y^2+z^2-2\alpha x-2\beta y-2\gamma z-2v &&=0,\\
&A(x-\alpha)+B(y-\beta)+C(z-\gamma)-\lambda &&=0,\\
&Adx+Bdy+Cdz &&=0.
\end{aligned}$$

The condition is

$$a\alpha+b\beta+c\gamma-l\lambda+u=0,$$

(not containing v so that this remains an arbitrary function of θ). The cases are

	a	b	c	l	u	α	β	γ	λ
$PS1^0$	a	b	c	0	0	0	0	0	λ
$PS2^0$	a	b	c	l	ml	0	0	0	m
$PS3^0$	a	b	c	$-mc$	0	0	0	γ	$\frac{1}{m}\gamma$
$PS4^0$	a	b	0	l	ml	0	0	γ	m
$PS5^0$	a	b	0	0	0	0	0	γ	λ
$PS6^0$	0	b	0	l	ml	α	0	γ	m
$PS7^0$	a	b	0	ma	0	α	0	γ	$-\frac{1}{m}\alpha$

where m is an arbitrary constant and in the body of the table the other italic letters are arbitrary functions of t, and the greek letters arbitrary functions of θ.

$PS1^0$ is Serret's first case of *PS*, included in his second case.

$PS2^0$ gives developable.

$PS3^0$ is Serret's second case of *PS*.

$PS4^0$ is Serret's third case of *PS*.

$PS5^0$ gives circular sections (surfaces of resolution).

$PS6^0$ gives circular sections (tubular surfaces).

$PS7^0$ gives circular sections.

I consider

$PS3^0$ = SERRET'S SECOND CASE OF PS.

The six equations are

$$\begin{aligned} A^2 + B^2 + C^2 &= 1, \\ ax + by + cz &= 0, \\ Aa + Bb + Cc &= -cm, \\ x^2 + y^2 + (z - m\phi)^2 &= \theta + m^2\phi^2, \\ Ax + By + C(z - m\phi) &= \phi, \\ Adx + Bdy + Cdz &= 0, \end{aligned}$$

(where a, b, c are assumed such that $a^2 + b^2 + c^2 = 1$). We easily obtain

$$(1 - c^2)A = -ac(C + m) - b\sqrt{\Omega},$$
$$(1 - c^2)B = -bc(C + m) + a\sqrt{\Omega},$$

and thence

$$aB - bA = \sqrt{\Omega},$$

where

$$\Omega = (1 - c^2)(1 - C^2) - c^2(C + m)^2, \ = 1 - c^2 - C^2 - 2c^2Cm - c^2m^2;$$

also

$$\begin{aligned} x\sqrt{1 - c^2m^2} &= A\phi\sqrt{1 - c^2m^2} + (bC - cB)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ y\sqrt{1 - c^2m^2} &= B\phi\sqrt{1 - c^2m^2} + (cA - aC)\sqrt{\theta + (m^2 - 1)\phi^2}, \\ z\sqrt{1 - c^2m^2} &= (C + m)\phi\sqrt{1 - c^2m^2} + (aB - bA)\sqrt{\theta + (m^2 - 1)\phi^2}. \end{aligned}$$

We seek for the differential equation in C, t, θ. From the equation

$$Ax + By + (C - m\phi)z = \phi,$$

and attending to $Adx + Bdy + Cdz = 0$, we deduce

$$xdA + ydB + (z - m\phi)dC - (1 + Cm)\phi' d\theta = 0,$$

and we have herein to substitute for dA, dB their values in terms of dC, dt, $d\theta$. We have

$$AdA + BdB = -CdC,$$
$$adA + bdB = -cdC - Q,$$

if for shortness $Q = Ada + Bdb + (C + m)dc$. Hence

$$\sqrt{\Omega}\,dA = (-cB + bC)dC - BQ,$$
$$\sqrt{\Omega}\,dB = (-aC + cA)dC + AQ.$$

We find without difficulty,

$$(1 - c^2)Q = (C + m)dc + (adb - bda)\sqrt{\Omega},$$

and consequently,

$$(1-c^2)\sqrt{\Omega}\,dA = \{\quad b(C+c^2m) - ac\sqrt{\Omega}\}\,dC \\ \qquad\qquad - B\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\},$$

$$(1-c^2)\sqrt{\Omega}\,dB = \{-a(C+c^2m) - bc\sqrt{\Omega}\}\,dC \\ \qquad\qquad + A\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\}.$$

Substituting these values we have

$$\{(bx-ay)(C+c^2m) - (ax+by)\,c\sqrt{\Omega}\}\,dC \\ -(Bx-Ay)\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\} \\ +(1-c^2)\sqrt{\Omega}\{(z-m\phi)\,dC - (1+Cm)\,\phi' d\theta\} = 0,$$

viz. this is

$$\{(bx-ay)(C+c^2m) - (ax+by)\,c\sqrt{\Omega} + (1-c^2)(z-m\phi)\sqrt{\Omega}\}\,dC \\ -(Bx-Ay)\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\} \\ -(1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta = 0.$$

The coefficient of dC contains a term $-(ax+by+cz)\,c\sqrt{\Omega}$ which is $=0$. Moreover, we have

$$bx - ay = -\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2},$$

and then

$$(1-c^2)(Bx-Ay) = -c(C+m)(bx-ay) - cz\sqrt{\Omega} \\ = -c(C+m)\left\{-\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right\} \\ -c\sqrt{\Omega}\left\{(C+m)\phi + \frac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\right\},$$

which, observing that the terms in $C\phi\sqrt{\Omega}$ destroy each other, and that we have $(C+m)(C+c^2m)+\Omega = (1-c^2)(1+Cm)$, gives

$$Bx - Ay = \frac{-c(1+Cm)}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2},$$

and the equation becomes

$$\left\{\left(-\phi\sqrt{\Omega} + \frac{C+c^2m}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}\right)(C+c^2m) + z\sqrt{\Omega} \\ -(1-c^2)\,m\phi\sqrt{\Omega}\right\}dC \\ -c(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc + (adb-bda)\sqrt{\Omega}\right\} \\ -(1-c^2)\sqrt{\Omega}(1+Cm)\,\phi' d\theta = 0.$$

Here the coefficient of dC is $= [z-(C+m)\phi]\sqrt{\Omega}+\dfrac{(C+c^2m)^2}{\sqrt{1-c^2m^2}}\sqrt{\theta+(m^2-1)\phi^2}$,

or substituting for $z-(C+m)\sqrt{\Omega}$ its value $=\dfrac{\sqrt{\Omega}\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}$, and observing that $\Omega+(C+c^2m)^2=(1-c^2m^2)(1-c^2)$, this coefficient is found to be $=\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}$, and we have

$$\sqrt{1-c^2m^2}\,(1-c^2)\sqrt{\theta+(m^2-1)\phi^2}\,dC$$
$$-c\,(1+Cm)\frac{\sqrt{\theta+(m^2-1)\phi^2}}{\sqrt{1-c^2m^2}}\left\{(C+m)\,dc+(adb-bda)\sqrt{\Omega}\right\}$$
$$-(1-c^2)(1+Cm)\sqrt{\Omega}\phi' d\theta=0,$$

or as this may be written

$$\frac{1}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}dC}{1+Cm}-\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\}-\frac{c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$$
$$-\frac{\phi' d\theta}{\sqrt{\theta+(m^2-1)\phi^2}}=0,$$

where from the foregoing value of Ω we have identically

$$\Omega\,(1-m^2)=(1-c^2)(1+Cm)^2-(1-c^2m^2)(C+m)^2.$$

Here a, b, c are functions of t and we have thus the required differential equation in C, t, θ.

It is convenient to multiply by the constant factor $\sqrt{1-m^2}$. The first term is an exact differential, viz. writing

$$\sin\zeta=\frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm},\ \text{and therefore}\ \cos\zeta=\frac{\sqrt{1-m^2}\sqrt{\Omega}}{\sqrt{1-c^2}\,(1+Cm)},$$

we have

$$d\zeta=\frac{\sqrt{1-m^2}}{\sqrt{\Omega}}\left\{\frac{\sqrt{1-c^2m^2}\,dC}{1+Cm}+\frac{(C+m)\,cdc}{(1-c^2)\sqrt{1-c^2m^2}}\right\},$$

as may easily be verified. And the second and third terms are obviously the differentials of a function of t and a function of θ respectively. But to obtain the integral functions, a transformation of each term is required.

First, for the term $\dfrac{\sqrt{1-m^2}\,c\,(adb-bda)}{(1-c^2)\sqrt{1-c^2m^2}}$; we take a, b, c functions of t which are such $a^2+b^2+c^2=1$; and then writing a_1, b_1, c_1 for the derived functions so that $aa_1+bb_1+cc_1=0$, we assume a', b', $c'=Va_1$, Vb_1, Vc_1 where $\dfrac{1}{V^2}=a_1^2+b_1^2+c_1^2$; we have therefore $aa'+bb'+cc'=0$, and $a'^2+b'^2+c'^2=1$;

and then writing a$''$, b$''$, c$''$ $=$ bc$'$ $-$ b$'$c, ca$'$ $-$ c$'$a, ab$'$ $-$ a$'$b respectively, we have $\mathrm{aa''} + \mathrm{bb''} + \mathrm{cc''} = 0$, $\mathrm{a'a''} + \mathrm{b'b''} + \mathrm{c'c''} = 0$, $\mathrm{a''^2} + \mathrm{b''^2} + \mathrm{c''^2} = 1$; thus a, b, c, a$'$, b$'$, c$'$, a$''$, b$''$, c$''$ are a set of rectangular coefficients. We then write

$$a, b, c = \frac{1}{\rho}(\mathrm{a'} + m\mathrm{b''}),\ \frac{1}{\rho}(\mathrm{b'} - m\mathrm{a''}),\ \frac{1}{\rho}\mathrm{c'},$$

determining ρ so that $a^2 + b^2 + c^2 = 1$ as above, viz. we thus have

$$\rho^2 = (1 + \mathrm{c}m)^2 + \mathrm{c'^2}m^2.$$

Observe that we thus have $\rho^2(1 - c^2) = \rho^2 - \mathrm{c'^2}$ and $\rho^2(1 - c^2m^2) = (1 + \mathrm{c}m)^2$.

Writing now

$$T = \tan^{-1}\frac{\mathrm{c} + m}{\mathrm{c''}\sqrt{1 - m^2}},\ \text{and therefore}\ \sin T = \frac{\mathrm{c} + m}{\sqrt{\rho^2 - \mathrm{c'^2}}},\ \cos T = \frac{\mathrm{c''}\sqrt{1 - m^2}}{\sqrt{\rho^2 - \mathrm{c'^2}}},$$

we find that

$$dT = \frac{\sqrt{1 - m^2}\, c\,(adb - bda)}{(1 - c^2)\sqrt{1 - c^2m^2}}.$$

The verification is somewhat long, but it is very interesting. We have

$$dT = \frac{\sqrt{1 - m^2}\,\{\mathrm{c''}d\mathrm{c} - (\mathrm{c} + m)\,d\mathrm{c''}\}}{\rho^2 - \mathrm{c'^2}},$$

or observing that $\mathrm{c''} = \mathrm{ab'} - \mathrm{a'b}$, $= V(\mathrm{ab_1} - \mathrm{a_1b})$, $d\mathrm{c} = \mathrm{c_1}dt$, this is

$$dT = \frac{\sqrt{1 - m^2}}{\rho^2 - c^2}\{V(\mathrm{ab_1} - \mathrm{a_1b})\,\mathrm{c_1} - (\mathrm{c} + m)[V_1(\mathrm{ab_1} - \mathrm{a_1b}) + V(\mathrm{ab_{11}} - \mathrm{a_{11}b})]\,dt,$$

where we have

$$\frac{1}{V^2} = \mathrm{a_1^2} + \mathrm{b_1^2} + \mathrm{c_1^2},\ \text{and therefore}\ -\frac{V_1}{V^3} = \mathrm{a_1a_{11}} + \mathrm{b_1b_{11}} + \mathrm{c_1c_{11}};$$

also from $\mathrm{aa_1} + \mathrm{bb_1} + \mathrm{cc_1} = 0$, we have $\mathrm{a_1^2} + \mathrm{b_1^2} + \mathrm{c_1^2} + \mathrm{aa_{11}} + \mathrm{bb_{11}} + \mathrm{cc_{11}} = 0$, and we thence obtain

$$dT = \frac{\sqrt{1 - m^2}\,V^3dt}{\rho^2 - \mathrm{c'^2}}\{-(\mathrm{ab_1} - \mathrm{a_1b})\,\mathrm{c_1}(\mathrm{aa_{11}} + \mathrm{bb_{11}} + \mathrm{cc_{11}})$$
$$- (\mathrm{c} + m)[-(\mathrm{a_1a_{11}} + \mathrm{b_1b_{11}} + \mathrm{c_1c_{11}})(\mathrm{ab_1} - \mathrm{a_1b}) + (\mathrm{a_1^2} + \mathrm{b_1^2} + \mathrm{c_1^2})(\mathrm{ab_{11}} - \mathrm{ba_{11}})]\},$$

the term in [] is found to be $= -\mathrm{c_1}\{\mathrm{a_{11}}(\mathrm{bc_1} - \mathrm{b_1c}) + \mathrm{b_{11}}(\mathrm{ca_1} - \mathrm{c_1a}) + \mathrm{c_{11}}(\mathrm{ab_1} - \mathrm{a_1b})\}$, hence $\mathrm{c_1}$ appears as a factor of the whole expression, and reducing the part independent of m, we find

$$dT = \frac{\sqrt{1 - m^2}\,V^3\mathrm{c_1}dt}{\rho^2 - \mathrm{c'^2}}\{(\mathrm{a_1b_{11}} - \mathrm{a_{11}b_1}) + m\,[\mathrm{a_{11}}(\mathrm{bc_1} - \mathrm{b_1c})$$
$$+ \mathrm{b_{11}}(\mathrm{ca_1} - \mathrm{c_1a}) + \mathrm{c_{11}}(\mathrm{ab_1} - \mathrm{a_1b})]\}.$$

Next calculating the value of $adb - bda$, we have

$$a = \frac{V}{\rho}\{\mathrm{a_1} + m(\mathrm{ca_1} - \mathrm{c_1a})\},\ b = \frac{V}{\rho}\{\mathrm{b_1} - m(\mathrm{bc_1} - \mathrm{b_1c})\},$$

or as these may be written

$$a = \frac{V}{\rho}\{\mathrm{a}_1(1+\mathrm{c}m) - \mathrm{ac}_1 m\}, \quad b = \frac{V}{\rho}\{\mathrm{b}_1(1+\mathrm{c}m) - \mathrm{bc}_1 m\},$$

and we thence easily obtain

$$adb - bda = \frac{V^2 dt}{\rho^2}(1+\mathrm{c}m)\{\mathrm{a}_1\mathrm{b}_{11} - \mathrm{a}_{11}\mathrm{b}_1)$$
$$+ m\,[\mathrm{a}_{11}(\mathrm{bc}_1 - \mathrm{b}_1\mathrm{c}) + \mathrm{b}_{11}(\mathrm{ca}_1 - \mathrm{c}_1\mathrm{a}) + \mathrm{c}_{11}(\mathrm{ab}_1 - \mathrm{a}_1\mathrm{b})]\},$$

viz: the factor in { } has the same value as in the expression for dT, and we thus have

$$\frac{dT}{adb - bda} = \frac{\sqrt{1-m^2}\,V\mathrm{c}_1\rho^2}{(1+\mathrm{c}m)(\rho^2 - \mathrm{c}'^2)} = \frac{c\sqrt{1-m^2}}{\sqrt{1-c^2m^2}\,(1-c^2)},$$

that is,

$$dT = \frac{\sqrt{1-m^2}\,c\,(adb - bda)}{(1-c^2)\sqrt{1-c^2m^2}},$$

the required equation.

Secondly, for the term $\dfrac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}}$, we introduce Φ a function of θ, such that writing Φ' for the derived function we have

$$\phi = \frac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4(1-m^2)\Phi\Phi' + 4(1-m^2)\theta\Phi'^2}}, \quad = \frac{\Phi - 2\theta\Phi'}{\sqrt{M}} \text{ suppose},$$

whence also

$$\sqrt{\theta + (m^2-1)\phi^2} = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}}, \quad \frac{\phi}{\sqrt{\theta + (m^2-1)\phi^2}} = \frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}}.$$

Then writing

$$\sin\Theta = \frac{\Phi\sqrt{1-m^2}}{\sqrt{\theta}}, \quad \cos\Theta = \frac{\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{\theta}},$$
$$\sin\Theta_0 = \frac{\phi\sqrt{1-m^2}}{\sqrt{\theta}}, \quad \cos\Theta_0 = \frac{\sqrt{\theta + (m^2-1)\phi^2}}{\sqrt{\theta}},$$

we find

$$\cos\Theta\, d\Theta = -\frac{\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\Phi - 2\theta\Phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\Phi^2}},$$

and similarly

$$\cos\Theta_0\, d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta}},$$

that is,

$$d\Theta_0 = \frac{-\frac{1}{2}\sqrt{1-m^2}\,(\phi - 2\theta\phi')\,d\theta}{\theta\sqrt{\theta + (m^2-1)\phi^2}}.$$

Hence

$$-d\Theta + d\Theta_0 = \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\Phi - 2\theta\Phi'}{\sqrt{\theta + (m^2-1)\Phi^2}} - \frac{\phi - 2\theta\phi'}{\sqrt{\theta + (m^2-1)\phi^2}}\right\} d\theta$$

$$= \frac{\sqrt{1-m^2}}{2\theta}\left\{\frac{\phi - (\phi - 2\theta\phi')}{\sqrt{\theta + (m^2-1)\phi^2}}\right\} d\theta, \; = \frac{\sqrt{1-m^2}\,\phi' d\theta}{\sqrt{\theta + (m^2-1)\phi^2}},$$

the required equation.

We find, moreover,

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{1-m^2}\sqrt{\theta + (m^2-1)\Phi^2}}{\sqrt{M}}, \quad \cos(\Theta - \Theta_0) = \frac{1 - 2(1-m^2)\Phi\Phi'}{\sqrt{M}},$$

which will be presently useful.

The differential equation now is $d\zeta - dT + d\Theta - d\Theta_0 = 0$, hence the integral equation (taking the constant of integration $= 0$) is $\zeta = T - \Theta + \Theta_0$, or say

$$\sin\zeta = \sin(T - \Theta + \Theta_0),$$

viz. substituting for $\sin T$ and $\cos T$ their values, and observing that

$$\sin\zeta = \frac{\sqrt{1-c^2m^2}}{\sqrt{1-c^2}}\,\frac{C+m}{1+Cm}, \; = \frac{1+cm}{\sqrt{\rho^2 - c'^2}}\,\frac{C+m}{1+Cm},$$

the factor $\frac{1}{\sqrt{\rho^2 - c'^2}}$ multiplies out, and we have

$$(1+cm)\frac{C+m}{1+Cm} = (c+m)\cos(\Theta - \Theta_0) - c''\sqrt{1-m^2}\sin(\Theta - \Theta_0).$$

And I further remark here that a former equation is

$$\Omega(1-m^2) = (1-c^2)(1+Cm)^2 - (1-c^2m^2)(C+m)^2,$$

that is,

$$\Omega\frac{1-m^2}{(1+Cm)^2} = (1-c^2)\left\{1 - \frac{(1-c^2m^2)(C+m)^2}{(1-c^2)(1+Cm)^2}\right\} = (1-c^2)\cos^2\zeta.$$

We thus have

$$\sqrt{\Omega} = \frac{1+Cm}{\sqrt{1-m^2}}\,\frac{\sqrt{\rho^2-c'^2}}{\rho}\cos\zeta,$$

$$= \frac{1+Cm}{\rho\sqrt{1-m^2}}\{c''\sqrt{1-m^2}\cos(\Omega - \Omega_0) + (c+m)\sin(\Omega - \Omega_0)\}.$$

We have thus C, and consequently also A, B, x, y, z all of them given as functions of t, θ; but the formulae admit of further development.

Write $$s = \frac{C+m}{1+Cm}, \text{ whence also } C = \frac{s-m}{1-ms}.$$

We have $C(1-m8)+m=8$, and hence $(1+cm)\{C(1-m8)+m\}$, $=(1+cm)8$, $=(c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0)$. Using the value of C given by this equation, and calculating from it those of A, B; then writing for shortness

$$\begin{aligned}X &= a\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(a''-mb')\ \sin(\Theta-\Theta_0),\\ Y &= b\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(b''+ma')\ \sin(\Theta-\Theta_0),\\ Z &= \quad (c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0),\end{aligned}$$

we have

$$\begin{aligned}A(1-m8)(1+cm) &= \sqrt{1-m^2}X,\\ B(1-m8)(1+cm) &= \sqrt{1-m^2}Y,\\ C(1-m8)(1+cm) &= \qquad Z-m(1+cm),\end{aligned}$$

to which I join $\qquad 8(1+cm) = \qquad Z.$

By way of verification observe that $A^2+B^2+C^2=1$, and that the equations give

$$(1-m8)^2(1+cm)^2=(1-m^2)(X^2+Y^2+Z^2)+m^2Z^2-2mZ(1+cm)+m^2(1+cm)^2;$$

we have $X^2+Y^2+Z^2=(1+cm)^2$, $Z=8(1+cm)$, and hence the identity $(1-m8)^2(1+cm)^2=(1-m^2+m^2 8^2-2m8+m^2)(1+cm)^2$.

Proceeding to calculate the values of x, y, z, recollecting that $\sqrt{1-c^2m^2}$ $=\frac{1}{\rho}(1+cm)$, we have

$$\begin{aligned}x(1+cm) &= A\phi(1+cm)+\rho(bC-cB)\sqrt{\theta+(m^2-1)\phi^2},\\ &= A\phi(1+cm)+\{(b'-ma'')C-c'B\}\sqrt{\theta+(m^2-1)\phi^2},\end{aligned}$$

that is,

$$\begin{aligned}x(1+cm)(1-m8) &= \phi\sqrt{1-m^2}X+\frac{1}{1+cm}\{b'-ma'')(Z-m(1+cm))\\ &\qquad -c'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &= \phi\sqrt{1-m^2}X+\frac{1}{1+cm}\{(b'-ma'')Z-c'\sqrt{1-m^2}Y\}\sqrt{\theta+(m^2-1)\phi^2}\\ &\qquad -m(b'-ma'')\sqrt{\theta+(m^2-1)\phi^2},\end{aligned}$$

where the term $(b'-ma'')Z-c'\sqrt{1-m^2}Y$ contains the factor $1+cm$; in fact this is

$$\begin{aligned}&= (b'-ma'')\{\quad (c+m)\cos(\Theta-\Theta_0)-c''\sqrt{1-m^2}\sin(\Theta-\Theta_0)\}\\ &\quad -c'\sqrt{1-m^2}\{b\sqrt{1-m^2}\cos(\Theta-\Theta_0)-(b''+ma')\ \sin(\Theta-\Theta_0)\}.\end{aligned}$$

The coefficient of the cosine is $(b'-ma'')(c+m)-bc'(1-m^2)$, which is

$$\begin{aligned}&= b'c-bc'+m(b'-ca'')+m^2(-a''+bc'),\ =-a''+m(b'-ca'')+m^2(-b'c),\\ &\qquad = (1+cm)(-a''+mb'),\end{aligned}$$

and similarly the coefficient of $\sqrt{1-m^2}$ sine is $-\mathrm{c}''(\mathrm{b}'-m\mathrm{a}'')+\mathrm{c}'(\mathrm{b}''+m\mathrm{a}')$, $=-\mathrm{b}'\mathrm{c}''+\mathrm{b}''\mathrm{c}'+m(\mathrm{a}'\mathrm{c}'+\mathrm{a}''\mathrm{c}'')$, $=-\mathrm{a}+m(-\mathrm{ac})$, $=(1+\mathrm{c}m)(-\mathrm{a})$. Calculating in like manner the values of y and z, and putting for shortness

$$X_1=(-\mathrm{a}''+m\mathrm{b}')\quad\cos(\Theta-\Theta_0)-\mathrm{a}\sqrt{1-m^2}\sin(\Theta-\Theta_0),$$
$$Y_1=(-\mathrm{b}''-m\mathrm{a}')\quad\cos(\Theta-\Theta_0)-\mathrm{b}\sqrt{1-m^2}\sin(\Theta-\Theta_0),$$
$$Z=(-\mathrm{c}''\sqrt{1-m^2})\cos(\Theta-\Theta_0)+\quad(\mathrm{c}+m)\sin(\Theta-\Theta_0),$$

we have

$$x=\quad\phi\sqrt{1-m^2}X+X_1\sqrt{\theta+(m^2-1)\phi^2}-m(\mathrm{b}'-m\mathrm{a}'')\sqrt{\theta+(m^2-1)\phi^2},$$
$$y=\quad\phi\sqrt{1-m^2}Y+Y_1\sqrt{\theta+(m^2-1)\phi^2}+m(\mathrm{a}'+m\mathrm{b}'')\sqrt{\theta+(m^2-1)\phi^2},$$
$$z=\sqrt{1-m^2}\{\phi\sqrt{1-m^2}Z+Z_1\sqrt{\theta+(m^2-1)\phi^2}\},$$

which are the required expressions of x, y, z in terms of t and θ. It will be noticed that X, X_1, Y, Y_1, Z, Z_1, each contain a term with $\cos(\Theta-\Theta_0)$ and one with $\sin(\Theta-\Theta_0)$, but as the terms in X_1, Y_1, Z_1 are each multiplied by $\sqrt{\theta+(m^2-1)\phi^2}$, the cosine and sine terms of X, X_1, of Y, Y_1 and of Z, Z_1 do not in any case unite into a single term.

I remark that we have identically

$$aX+bY+c\sqrt{1-m^2}Z=0,$$
$$aX_1+bY_1+c\sqrt{1-m^2}Z_1=0.$$

The foregoing values of x, y, z thus satisfy $ax+by+cz=0$, which is one of the six equations. The others of them might be verified without difficulty. I recall that we have $a, b, c=\frac{1}{\rho}(\mathrm{a}'+m\mathrm{b}'')$, $\frac{1}{\rho}(\mathrm{b}'-m\mathrm{a}'')$, $\frac{1}{\rho}\mathrm{c}'$; the six equations might therefore be written

$$A^2+B^2+C^2=1,$$
$$(\mathrm{a}'+m\mathrm{b}'')x+(\mathrm{b}'-m\mathrm{a}'')y+\mathrm{c}'z=0,$$
$$(\mathrm{a}'+m\mathrm{b}'')A+(\mathrm{b}'-m\mathrm{a}'')B+\mathrm{c}'C=-\mathrm{c}'m,$$
$$x^2+y^2+(z-m\phi)^2=\theta+m^2\phi^2,$$
$$Ax+By+C(z-m\phi)=\phi,$$
$$Adx+Bdy+Cdz=0.$$

THE CASE $PS1^0$ = SERRET'S FIRST CASE OF PS.

This is at once deduced from $PS3^0$ by writing therein $m=0$; the formulae are a good deal more simple. We introduce as before the rectangular coefficients a, b, c, a′, b′, c′, a″, b″, c″, and the values of a, b, c then are a′, b′, c′. The six equations, using therein these values for a, b, c, are

$$\begin{aligned} A^2 + B^2 \quad + C^2 &= 1, \\ a'x + b'y \quad + c'z &= 0, \\ a'A + b'B + c'C &= 0, \\ x^2 + y^2 \quad + z^2 &= \theta, \\ Ax + By \quad + Cz &= \phi. \\ Adx + Bdy + Cdz &= 0. \end{aligned}$$

The function Φ is such that $\phi = \dfrac{\Phi - 2\theta\Phi'}{\sqrt{1 - 4\Phi\Phi' + 4\theta\Phi'^2}} = \dfrac{\Phi - 2\theta\Phi'}{\sqrt{M}}$. We have

$$\sin\Theta = \frac{\Phi}{\sqrt{\theta}} \quad \cos\Theta = \frac{\sqrt{\theta - \Phi^2}}{\sqrt{\theta}},$$

$$\sin\Theta_0 = \frac{\phi}{\sqrt{\theta}} \quad \cos\Theta_0 = \frac{\sqrt{\theta - \phi^2}}{\sqrt{\theta}},$$

and thence

$$\sin(\Theta - \Theta_0) = \frac{2\Phi'\sqrt{\theta - \Phi^2}}{\sqrt{M}}; \quad \cos(\Theta - \Theta_0) = \frac{1 - 2\Phi\Phi'}{\sqrt{M}}.$$

Also

$$\sin\zeta = \frac{C}{\sqrt{1 - c^2}}, \quad \cos\zeta = \frac{\sqrt{1 - c^2 - C^2}}{\sqrt{1 - c^2}}; \quad \sin T = \frac{c}{\sqrt{1 - c'^2}}, \quad \cos T = \frac{c''}{\sqrt{1 - c^2}},$$

$$\zeta = T - \Theta + \Theta_0, \quad C = c\cos(\Theta - \Theta_0) - c''\sin(\Theta - \Theta_0),$$

$$\sqrt{1 - c^2 - C^2} = c''\cos(\Theta - \Theta_0) + c\sin(\Theta - \Theta_0).$$

We have

$$\begin{aligned} A = X &= a\cos(\Theta - \Theta_0) - a''\sin(\Theta - \Theta_0); \quad X_1 = a''\cos(\Theta - \Theta_0) + a\sin(\Theta - \Theta_0), \\ B = Y &= b\cos(\Theta - \Theta_0) - b''\sin(\Theta - \Theta_0); \quad Y_1 = b''\cos(\Theta - \Theta_0) + b\sin(\Theta - \Theta_0), \\ C = Y &= c\cos(\Theta - \Theta_0) - c''\sin(\Theta - \Theta_0); \quad Z_1 = c''\cos(\Theta - \Theta_0) + c\sin(\Theta - \Theta_0), \end{aligned}$$

and then

$$\begin{aligned} x &= X\phi + X_1\sqrt{\theta - \phi^2}, \\ y &= Y\phi + Y_1\sqrt{\theta - \phi^2}, \\ z &= Z\phi + Z_1\sqrt{\theta - \phi^2}, \end{aligned}$$

which are the expressions of the coordinates in terms of the parameters t and θ.

(*To be continued.*)