Probabilités avec CAML

INFO1 - Semaine 22

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1. Outils de simulation
2. Loi binomiale
   - Simulation
   - Calcul des valeurs théoriques
3. Loi géométrique ?...
4. Loi hypergéométrique ?...
5. Loi de Poisson

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```
val int : int -> int
```

Random.int bound returns a random integer between 0 (inclusive) and bound (exclusive). bound must be greater than 0 and less than 2^30.

```
val float : float -> float
```

Random.float bound returns a random floating-point number between 0 and bound (inclusive). If bound is negative, the result is negative or zero. If bound is 0, the result is 0.

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Random.float bound returns a random floating-point number between 0 and bound (inclusive). If bound is negative, the result is negative or zero. If bound is 0, the result is 0.

```
(* vecteur aléatoire de n entiers entre min et max *)
let vec_alea (mini: int) (maxi: int) (taille: int): int list =
 let rec aux_alea (size: int) (acc: int list): int list =
   if size = 0 then acc
   else aux_alea (size - 1) ((mini + Random.int (maxi - mini + 1)) :: acc)
 in aux_alea taille [];;
```

```
# vec_alea 1 6 15;;
- : int list = [5; 3; 2; 4; 1; 2; 6; 6; 3; 5; 4; 1; 6; 1; 6]
```

```
(* vecteur aléatoire de n entiers entre min et max *)
let vec_alea (mini: int) (maxi: int) (taille: int): int list =
 let rec aux_alea (size: int) (acc: int list): int list =
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 in aux_alea taille [];;
```

```
# vec_alea 1 6 15;;
- : int list = [5; 3; 2; 4; 1; 2; 6; 6; 3; 5; 4; 1; 6; 1; 6]
```

```
(* sommes des éléments d'une liste d'entiers *)
let som (liste : int list): int =
  List.fold_left (+) 0 liste;;
```

```
(* simulations du lancer de trois dés *)
let toscane ((): unit): int =
  som (vec_alea 1 6 3);;
```

```
# toscane;;
- : unit -> int = <fun>
```

```
# toscane ();;
- : int = 10
```

```
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```
# toscane;;
- : unit -> int = <fun>
```

```
# toscane ();;
- : int = 10
```

```
(* une liste d'expériences aléatoires *)
let simul_exp (exp: unit -> 'a) (occ: int): 'a list =
  let rec aux_exp = fun n acc ->
    if n = occ then acc
    else aux_exp (n + 1) ((exp ()) :: acc)
  in aux_exp 0 [];;
```

```
# simul_exp toscane 15;;
- : int list = [12; 6; 14; 12; 8; 15; 9; 11; 12; 9; 8; 6; 15; 11; 6]
```

```
(* une liste d’expériences aléatoires *)
let simul_exp (exp: unit -> ’a) (occ: int): ’a list =
  let rec aux_exp = fun n acc ->
    if n = occ then acc
    else aux_exp (n + 1) ((exp ()) :: acc)
  in aux_exp 0 [];;
```

```
# simul_exp toscane 15;;
- : int list = [12; 6; 14; 12; 8; 15; 9; 11; 12; 9; 8; 6; 15; 11; 6]
```

```
(* renvoie la liste des couples (élément, nombre d'occurrences dans l'
    ensemble)
  avant réduction *)
let compte (liste: 'a list) : ('a * float) list =
 let occ = float_of_int (List.length liste) in
 let rec count = fun l accu ->
   match l with
   | []    -> accu
   |t :: q ->
     let c = List.partition (fun x -> x = t) l in
     count (snd c) ((t,(float_of_int (List.length (fst c))) /. occ)::accu)
 in count liste [];;
```

```
# compte  [12; 6; 14; 12; 8; 15; 9; 11; 12; 9; 8; 6; 15; 11; 6];;
- : (int * float) list =
[(11, 0.133333333333333331); (9, 0.133333333333333331);
 (15, 0.133333333333333331); (8, 0.133333333333333331);
 (14, 0.0666666666666666657); (6, 0.2); (12, 0.2)]
```

```
# Printf.sprintf "%.3f" 0.123456789;;
- : string = "0.123"
```

```
# compte  [12; 6; 14; 12; 8; 15; 9; 11; 12; 9; 8; 6; 15; 11; 6];;
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 (15, 0.133333333333333331); (8, 0.133333333333333331);
 (14, 0.0666666666666666657); (6, 0.2); (12, 0.2)]
```

```
# Printf.sprintf "%.3f" 0.123456789;;
- : string = "0.123"
```

```
let pretty_dic (ft: ('i -> 's, unit, string) format) (dic: ('a * float)
    list): ('a * string) list =
  List.map (fun cpl -> (fst cpl, Printf.sprintf ft (snd cpl))) dic;;
```

```
# pretty_dic "%.3f" (compte  [12; 6; 14; 12; 8; 15; 9; 11; 12; 9; 8; 6;
    15; 11; 6]);;
- : (int * string) list =
[(11, "0.133"); (9, "0.133"); (15, "0.133"); (8, "0.133"); (14, "0.067");
 (6, "0.200"); (12, "0.200")]
```

```
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    15; 11; 6]);;
- : (int * string) list =
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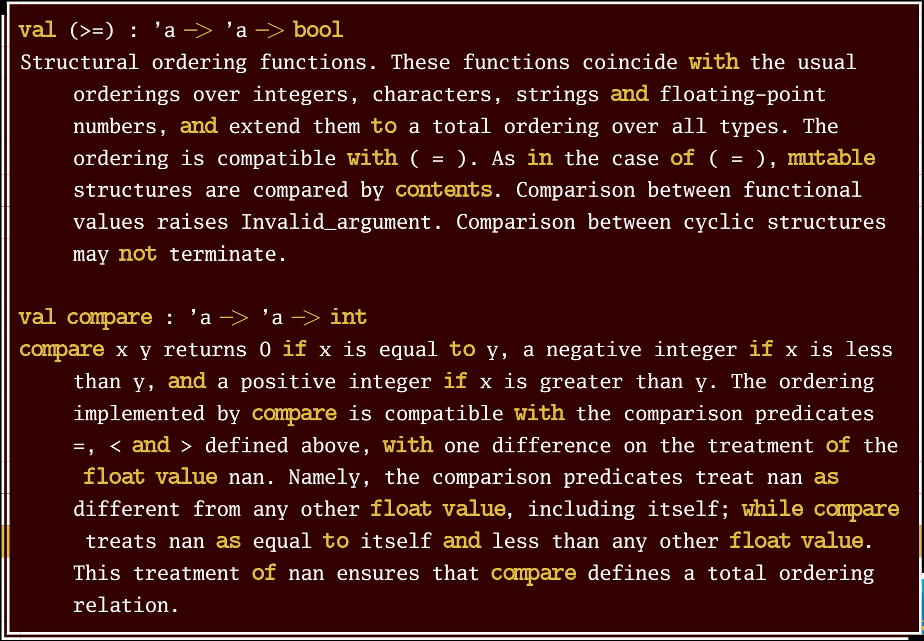
```



```
val (>=) : 'a -> 'a -> bool
Structural ordering functions. These functions coincide with the usual
    orderings over integers, characters, strings and floating-point
    numbers, and extend them to a total ordering over all types. The
    ordering is compatible with ( = ). As in the case of ( = ), mutable
    structures are compared by contents. Comparison between functional
    values raises Invalid_argument. Comparison between cyclic structures
    may not terminate.

val compare : 'a -> 'a -> int
compare x y returns 0 if x is equal to y, a negative integer if x is less
    than y, and a positive integer if x is greater than y. The ordering
    implemented by compare is compatible with the comparison predicates
    =, < and > defined above, with one difference on the treatment of the
     float value nan. Namely, the comparison predicates treat nan as
    different from any other float value, including itself; while compare
     treats nan as equal to itself and less than any other float value.
    This treatment of nan ensures that compare defines a total ordering
    relation.
```

```
# type truc = Lego | Playmobil | Schtroumpf | Hobbit;;
type truc = Lego | Playmobil | Schtroumpf | Hobbit
# Lego < Playmobil;;
- : bool = true
# compare Lego Hobbit;;
- : int = -1
# compare Hobbit Lego;;
- : int = 1
# compare Lego Lego;;
- : int = 0
```

```
let simul_dic (exp: unit -> 'a) (occ: int): ('a * float) list =
  List.sort compare (compte (simul_exp exp occ));;
```

```
# pretty_dic "%.3f" (simul_dic toscane 100_000);;
- : (int * string) list =
[(3, "0.005"); (4, "0.015"); (5, "0.028"); (6, "0.045"); (7, "0.071"); (8,
     "0.097"); (9, "0.114"); (10, "0.125"); (11, "0.126"); (12, "0.115");
     (13, "0.098"); (14, "0.070"); (15, "0.047"); (16, "0.027"); (17, "
    0.014"); (18, "0.005")]
```

```
let simul_dic (exp: unit -> 'a) (occ: int): ('a * float) list =
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     "0.097"); (9, "0.114"); (10, "0.125"); (11, "0.126"); (12, "0.115");
     (13, "0.098"); (14, "0.070"); (15, "0.047"); (16, "0.027"); (17, "
    0.014"); (18, "0.005")]
```

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4 Loi hypergéométrique ?...

5 Loi de Poisson

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```
(* renvoie 1 avec une probabilité p et 0 sinon *)
let bernoulli (p: float) : unit -> int  =
  fun () -> if (Random.float 1. < p) then 1 else 0;;
```

```
# bernoulli 0.2;;
- : unit -> int = <fun>
# bernoulli 0.2 ();;
- : int = 1

#  simul_dic (bernoulli 0.2) 100_000;;
- : (int * float) list = [(0, 0.80121); (1, 0.19879)]
```

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- : (int * float) list = [(0, 0.80121); (1, 0.19879)]
```

```
(* X ~ B(n,p) comme somme de var de Bernoulli B(1,p) *)
let binomial  (n: int) (p: float) :  unit -> int  =
  fun () -> som (simul_exp (bernoulli p) n);;
```

5 jetons tirés successivement dans une urne avec 5 blancs et 9 noirs. X : nombre de blancs tirés.

```
# pretty_dic "%.3f" (simul_dic (binomial 5 (5. /. 14.)) 100_000);;
- : (int * string) list =
[(0, "0.110"); (1, "0.304"); (2, "0.339"); (3, "0.188"); (4, "0.052"); (5,
      "0.006")]
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```
(* on utilise C(n,p) = (n-p+1)/p * C(n,p-1) *)
let binom = fun n p ->
   if p > n || n < 0 || p < 0 then
     0
   else
     let rec aux = fun acc num den ->
       if den <= p then
         aux ((acc * num) / den) (num - 1) (den + 1)
       else
         acc
     in aux 1 n 1;;
```

```
# ( ** );;
- : float -> float -> float = <fun>
# 2. ** 10. ;;
- : float = 1024.
```

```
(* calcul de Px(k) quand X ~ B(n,p) *)
let proba_bin (n: int) (p: float) (k: int) : float =
  (float_of_int (binom n k)) *. (p ** (float_of_int k)) *. ((1. -. p)**(
      float_of_int (n - k)));;
```

```
# ( ** );;
- : float -> float -> float = <fun>
# 2. ** 10. ;;
- : float = 1024.
```

```
(* calcul de Px(k) quand X ~ B(n,p) *)
let proba_bin (n: int) (p: float) (k: int) : float =
  (float_of_int (binom n k)) *. (p ** (float_of_int k)) *. ((1. -. p)**(
      float_of_int (n - k)));;
```

```
(* renvoie le dictionnaire des (k, Px(k)) quand X ~ B(n,p) *)
let dic_bin (n: int) (p: float) : (int * float) list =
  let rec aux = fun i acc ->
    if i > n then
      List.sort compare acc
    else
      let pr = proba_bin n p i in
      aux (i + 1) ((i,pr) :: acc)
  in aux 0 [];;
```

```
# pretty_dic "%.3f" (simul_dic (binomial 5 (5. /. 14.)) 100_000);;
- : (int * string) list =
[(0, "0.108"); (1, "0.305"); (2, "0.340"); (3, "0.187"); (4, "0.054"); (5,
    "0.006")]
```

```
# pretty_dic "%.3f" (dic_bin 5 (5. /. 14.));;
- : (int * string) list =
[(0, "0.110"); (1, "0.305"); (2, "0.339"); (3, "0.188"); (4, "0.052"); (5,
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```

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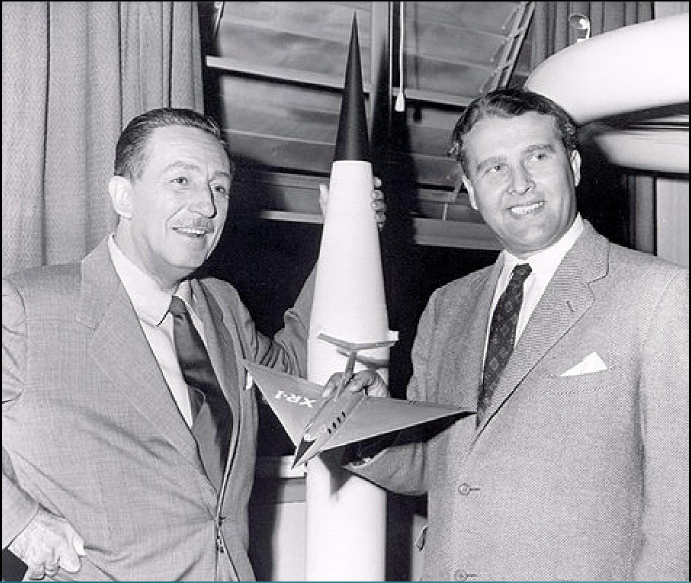
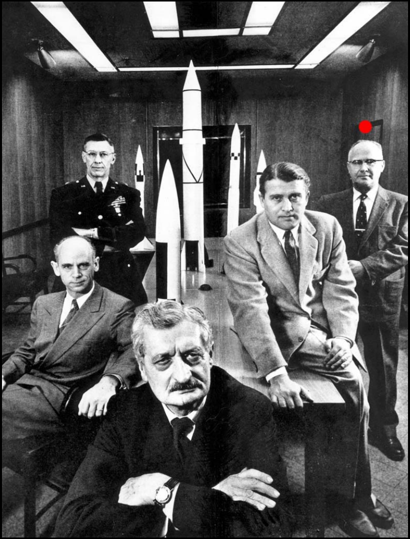
WWII

Von Braun et ses vieux amis

Von Braun et son nouvel ami



Lusser, père du V1 et Von Braun aux États-Unis...



Le V1

Le V1 de passage à Londres

No. of flying bombs per square	0	1	2	3	4	5 and over
No. of squares	229	211	93	35	7	1

Source :
http ://www.actuaries.org.uk/sites/all/files/documents/pdf/0481.pdf

576 carrés de $0,25$ km^2.

537 impacts de bombe.

$\lambda = 537/576$

No. of flying bombs per square	0	1	2	3	4	5 and over
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537 impacts de bombe.
$\lambda = 537/576$

$$\mathbb{P}(\{X = k+1\}) = \frac{\lambda}{k+1}\mathbb{P}(\{X = k\})$$

```
let poisson (lambda: float) (n: int) : float =
  let rec aux = fun k acc ->
    if k = n then acc
    else aux (k + 1) ((lambda /. (float_of_int (k + 1))) *. acc)
  in aux 0 (exp (-. lambda));;
```

$$\mathbb{P}(\{X = k+1\}) = \frac{\lambda}{k+1}\mathbb{P}(\{X = k\})$$

```
let poisson (lambda: float) (n: int) : float =
  let rec aux = fun k acc ->
    if k = n then acc
    else aux (k + 1) ((lambda /. (float_of_int (k + 1))) *. acc)
  in aux 0 (exp (-. lambda));;
```

```
let lambda = 537. /. 576. ;;

# List.map (fun k -> 576. *. (poisson lambda k)) [0;1;2;3;4];;
- : float list =
[226.742722583239527; 211.390350741666; 98.5387312050995092;
 30.6222793154736301; 7.13722395503877571]
```

No. of flying bombs per square	0	1	2	3	4	5 and over
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let lambda = 537. /. 576. ;;

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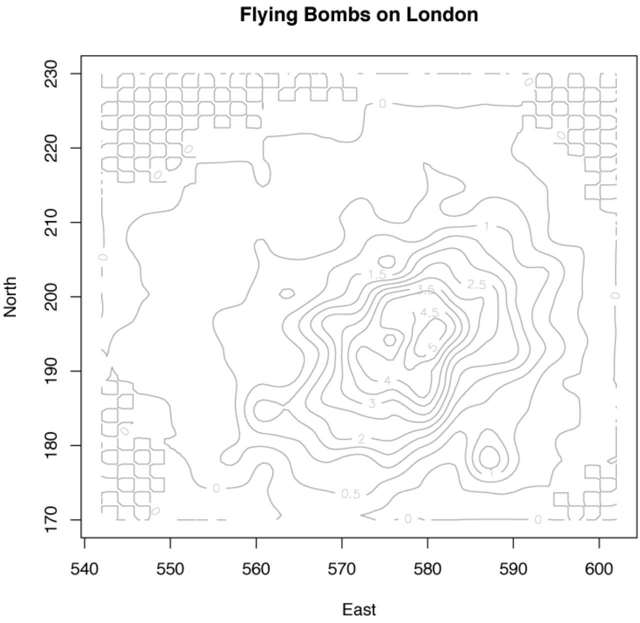
Flying Bombs on London
North
230
220
210
200
190
180
170
East
540
550
560
570
580
590
600
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5