`http://user42.tuxfamily.org/dragon/index.html`

Iterations of the Dragon Curve

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Abstract

Various properties of finite iterations of the Heighway/Harter dragon curve and twindragon curve, including boundary, area, convex hull, minimum rectangle, XY convex hull, centroid, inertia, complex base $i\pm1$, area tree, and some properties of the fractal limit including boundary and fixed point.

Contents

Notation

Bits of an integer written in binary are numbered starting from 0 for the least significant bit. This is the "low" end and the most significant bit(s) are the "high" end.

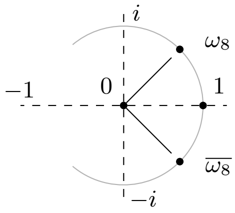
Various formulas have terms going in a repeating pattern of say 4 values according as an index $k \equiv 0 \,\text{to}\, 3 \bmod 4$. They are written like

$$[5,\ 8,\ -5,\ 9] \qquad \text{values according as } k \bmod 4$$

meaning 5 when $k \equiv 0 \bmod 4$, or 8 when $k \equiv 1 \bmod 4$, etc. Likewise periodic patterns of other lengths, usually at most 8.

Periodic patterns like this can be expressed by powers of -1 or i (or other roots of unity), but except in simple cases that tends to be less clear than the values.

A complex eighth root of unity is used variously for rotations by 45°, or its conjugate for −45°.

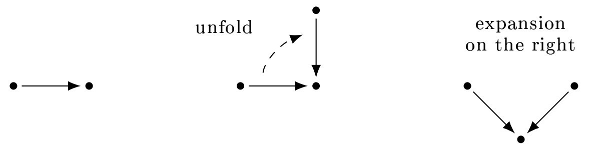


$$\omega_8 = e^{\frac{2\pi i}{8}} = \tfrac{1}{\sqrt{2}}(1{+}i) \qquad \text{eighth root of unity at } +45^\circ$$

$$\overline{\omega_8} = e^{\frac{-2\pi i}{8}} = \tfrac{1}{\sqrt{2}}(1{-}i) \qquad \text{conjugate at } -45^\circ$$

1 Dragon Curve

The dragon curve by Heighway and Harter is defined as repeated unfolding of a copy of itself beginning from a unit line segment. Or equivalently an expansion of each line segment to a pair of segments (per Bruce A. Banks).



The unfold makes the new second copy a reversal of the previous level. This is shown above by the arrow on the new copy directed towards the common middle.

The segment expansion is on the right side of each segment and the new segments are then one forward one reverse. Each expands again on the right for the next level. Every expansion gives pairs one forward, one reverse so the rule to expand on the right is equivalent to even numbered segments expand on the right and odd numbered segments expand on the left. The first segment is number 0 and so is an even segment.

Various sequences and measurements can be reached in two ways, either by considering the unfolding of the curve or by the expansion of each segment.

When the unfolding or expansion is 90°, the curve touches itself in level k=4 onwards. In the following diagram, corners are chamfered off to better see the path taken.

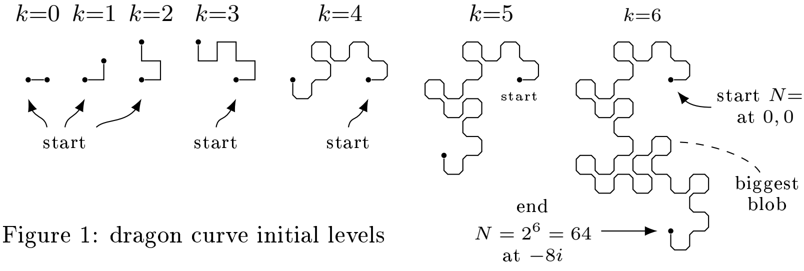
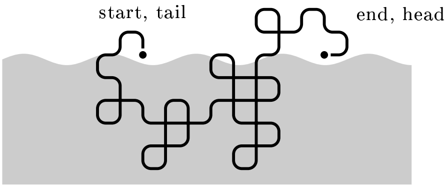


Figure 1: dragon curve initial levels

In sample images, the start of the curve can be distinguished from the end by noticing the sizes of the various “blobs” (see section 12) of touching unit squares. In k=6, the blob are 1 unit square, 3 squares, then the biggest 6 squares, before a drop to 1 unit square. This size drop, skipping the second biggest, identifies the curve end.

The supposed resemblance to a dragon (a sea dragon) is described by Gardner [18]. The start of the curve is the tail and the end is the head.



k=6 sea dragon after Gardner
(rotated so end is horizontal)

It's possible to unfold or expand on the left instead of the right. Doing so gives the same curve in mirror image. Expanding on the right has the attraction that with the initial segment in a fixed direction the curve spirals around anticlockwise, the usual mathematical direction.

1.1 Plane Filling

Davis and Knuth [12] show the dragon curve is non-crossing and plane filling, by a mixture of number representations and geometry.

Theorem 1 (Davis and Knuth). *The dragon curve touches at vertices but does not cross itself and does not overlap itself.*

Proof. Consider an infinite square grid with line segments connecting the points. The direction of each segment alternates so that they point away from an "even" point $x+y$ even, and towards an "odd" point $x+y$ odd.

Each line segment expands on the right as follows. The corners of the new lines are chamfered off to show how they touch the expansion of their opposite right-side line, and do not cross.

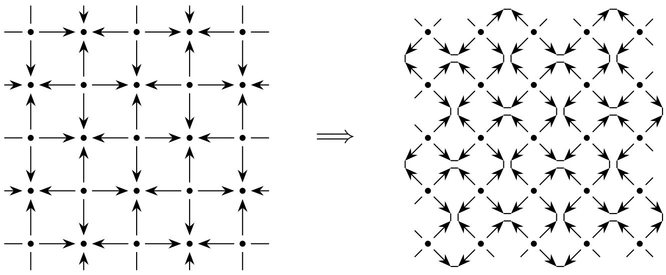


Figure 2: grid of segments and their expansions

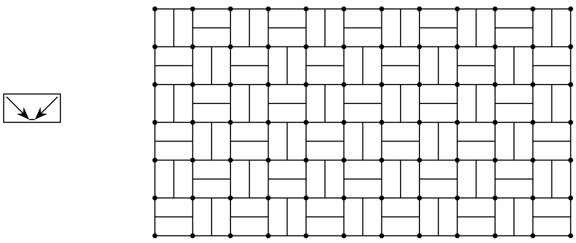
The expanded segments are the same grid pattern rotated by 45°. The pattern of arrow directions is the same.

Any subset of the full grid expands to a new bigger set with any crossings unchanged. The dragon curve begins as a single line segment which is such a subset with no crossings, and so on repeated expansions has no crossings. □

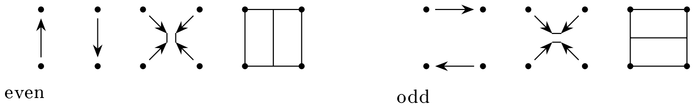
Edgar[16] with summer program contributors gives a similar proof with the expansion side going by alternating black and white squares. The curve always turns 90° so does not cross itself unless some segment is repeated (an overlap). A repeated segment can only occur when two segments at 90° expand onto each other. But a vertical always expands into its black adjacent square and a horizontal always expands into its white adjacent square, so they do not overlap.

Davis and Knuth make the argument on "generalized" dragon curves. A generalized curve expands to the right side or the left side of the directed segment according to the expansion level. The plain dragon curve is always on the right (of each directed segment), or for example the alternate paperfolding curve is alternately left side and right side. In figure 2, the effect is still opposing sides expanding together, just the whole grid taken with first a transpose (mirror image) across a diagonal.

Each pair of expanded line segments can be placed in a 2×1 rectangle. This is a classical tiling pattern[51].

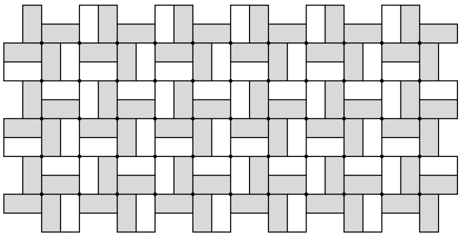


The squares are divided alternately vertically or horizontally. This is since the line segment directions alternate in each square so that the right side segment expansion is either vertical or horizontal.



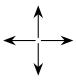
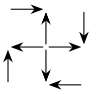
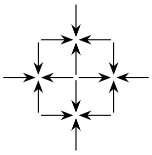
Another way to view the tiling is as a pinwheel of rectangles at each point. Points $x+y$ even have the pinwheel directed clockwise. Points $x+y$ odd have the pinwheel directed anti-clockwise, which is mirror image.

The even and odd pinwheels overlap since every line segment which becomes a 2×1 rectangle goes from an even point to an odd point. Taking just the even pinwheels gives a pattern like the following. Alternate pinwheels are shaded for contrast.

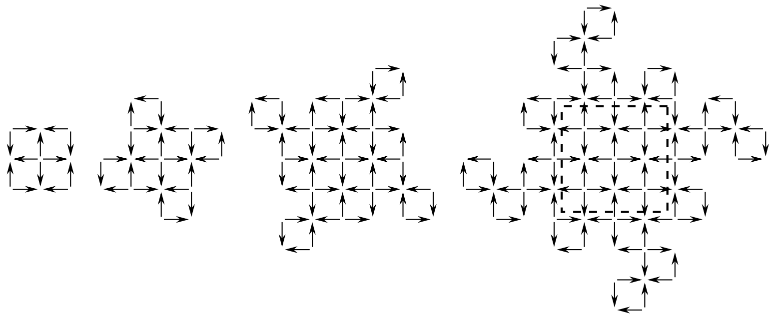


Theorem 2 (Davis and Knuth)**.** *Four copies of the dragon curve arranged at right angles fill the plane.*

Proof. The initial cross expands twice to

Take the central 2×2 block. With four expansions it grows



The dashed square is a 4×4 block at the origin. So each distinct 2×2 grows to at least 4×4. By repeated expansion, the blocks grow to an arbitrarily large square at the origin. ☐

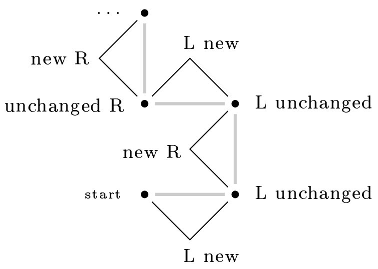
Dekking[14] generalizes this type of plane filling to a carousel theorem with conditions for four-arm plane filling by folding curves on a square grid.

1.2 Turn

Davis and Knuth[12, equation 3.3] show the dragon curve turn sequence is, in the form +1 left and −1 right, with first turn at $n=1$,

$$\begin{aligned} \mathit{turn}(2n) &= \mathit{turn}(n) && \text{even repeat} && (1)\\ \mathit{turn}(2n{+}1) &= (-1)^n && \text{odd alternately L, R} && (2)\\ &= {+}{+}{-}{+}{+}{-}{-}{+}{+}{+}{-}{-}{+}{-}{-}{+}\dots && n \geq 1 && \text{A034947} \end{aligned}$$

Per the segment expansion, the next curve level has a new turn inserted between each existing one. These expansions are alternately on the right and left so odd turns are alternately left and right which is (2). The alternate side expansion also means existing turns are unchanged, so (1). (See section 15.4 on how this pattern falls at locations in the plane.)



turns from
segment expansion,
new alternating,
existing unchanged

A generating function for *turn* follows from the recurrence. The odd terms (2) are periodic $0, 1, 0, -1$ which is generating function $(x-x^3)/(1-x^4) = x/(1+x^2)$. This is the k=0 term in (3). Further $turn(2n)$ are by substituting x^2 for the same there, etc.

$$gturn(x) = \sum_{k=0}^{\infty} \frac{x^{2^k}}{1+x^{2^{k+1}}} \tag{3}$$

turn can be calculated from n in binary, again for $n \geq 1$. High 0s are reckoned above n so *BitAboveLowestOne* always has a bit above.

$$turn(n) = \begin{cases} +1 \text{ (left)} & \text{if } BitAboveLowestOne(n) = 0 \\ -1 \text{ (right)} & \text{if } BitAboveLowestOne(n) = 1 \end{cases} \qquad n \geq 1 \tag{4}$$

$$BitAboveLowestOne(n) = 0, 0, 1, 0, 0, 1, 1, 0, 0, 0, 1, 1, \ldots \quad n \geq 1 \qquad \text{A038189}$$

The next turn, ie. the turn at point n+1, after segment n, is given similarly but above the lowest 0-bit. High 0s are reckoned above n so *BitAboveLowestZero* always has a 0 and a bit above.

$$turn(n{+}1) = \begin{cases} +1 \text{ (left)} & \text{if } BitAboveLowestZero(n) = 0 \\ -1 \text{ (right)} & \text{if } BitAboveLowestZero(n) = 1 \end{cases} \qquad n \geq 0 \tag{5}$$

$turn(n)$ and $turn(n{+}1)$ are related simply by n+1 changing low “0111” to “1000” so

$$BitAboveLowestZero(n) = BitAboveLowestOne(n{+}1) \tag{6}$$

n	$\cdots$	t	0	$1 \cdots 1$	bits, turn per t
n+1	$\cdots$	t	1	$0 \cdots 0$	

The effect of *BitAboveLowestOne* is to number turns with the first at n=1. This has the attraction of numbering points 0 to 2^n inclusive in an expansion level.

The effect of *BitAboveLowestZero* is to number turns with the first at n=0. n=0 is treated as entirely zeros so $BitAboveLowestZero(0) = 0$.

For computer calculation in a single machine word, the bit extractions can be done by some usual bit-twiddling. For example,

$$\begin{aligned} MaskAboveLowestOne(n) &= \mathrm{BITXOR}(n,\ n{-}1)\ +1 \qquad n \geq 1 \qquad (7) \\ &= 2, 4, 2, 8, 2, 4, 2, 16, 2, 4, 2, 8, 2, 4, 2, 32, \ldots \qquad \text{A171977} \end{aligned}$$

$$\mathit{BitAboveLowestOne}(n) = \begin{cases} 0 & \text{if BITAND}(n, \mathit{MaskAboveLowestOne}(n)) = 0 \\ 1 & \text{if BITAND}(n, \mathit{MaskAboveLowestOne}(n)) \neq 0 \end{cases}$$

$n{-}1$ changes low zeros "...1000" to "...0111" and XORing the two gives "0001111" which is a mask up to and including the lowest 1-bit. +1 gives "0010000" which is a mask for the single bit above the lowest 1.

Arndt [3] gives similar in C with a negation (twos-complement negation) $\mathit{MaskAboveLowestOne}(n)$ = `(n&-n)<<1` for the same result.

This bit-twiddling uses carry propagation in the CPU adder to locate the lowest 1-bit. It's common for the adder on a single machine word to be faster than a count-trailing-zeros and test-jth-bit.

Per for example Jeremy Gardiner in OEIS A034947, the turn sequence is also Kronecker symbol $\left(\frac{-1}{n}\right)$. That symbol identifies quadratic residues. For -1, its value is ± 1 according to bit above lowest 1-bit of n, which is the same as the turn sequence.

$$\left(\frac{-1}{2^k m}\right) = (-1)^{\frac{m-1}{2}} \qquad \text{Kronecker symbol, } m \text{ odd} \geq 1$$

$$\mathit{turn}(n) = \left(\frac{-1}{n}\right)$$

As noted by Davis and Knuth [12, equation 3.4], $\mathit{turn}(n)$ is multiplicative (like the Kronecker symbol is too).

$$\mathit{turn}(m.n) = \mathit{turn}(m) \,.\, \mathit{turn}(n) \qquad \text{multiplicative any } m, n$$

This follows from recurrence (1), or bits (4). Trailing 0-bits on m and n add in the product. The odd parts are then $\equiv 1 \text{ or } 3 \bmod 4$ which is $\pm 1 \bmod 4$ and so m and n multiply the same as their turn ± 1. For Kronecker symbol $(-1/n)$, any factors of 2 in n are ignored since it has $(-1/2) = +1$.

A predicate for right turns is *BitAboveLowestOne*, and left turns its opposite.

$$\begin{aligned} \mathit{TurnLpred}(n) &= n{\geq}1 \text{ and } 1 - \mathit{BitAboveLowestOne}(n) \\ &= 1,\ 1,0,\ 1,1,0,0,\ 1,1,1,0,0,1,0,0,\ldots \qquad \text{A014577} \\ \mathit{TurnRpred}(n) &= n{\geq}1 \text{ and } \quad \mathit{BitAboveLowestOne}(n) \end{aligned}$$

The "regular paperfolding sequence" is the lefts *TurnLpred*,

$$\mathit{Paper}(n) = \mathit{TurnLpred}(n) \qquad n \geq 1$$

Mendès France and van der Poorten [36] form "paperfolding numbers" in a given radix by writing the paperfolding sequence as digits below a radix point. For base 2 (ie. a binary fraction), this is the paperfolding constant. They show paperfolding numbers are transcendental in any algebraic radix.

$$\begin{aligned} \mathit{PaperConst} &= \sum_{n=1}^{\infty} \frac{\mathit{Paper}(n)}{2^n} \qquad (8) \\ &= .110110011100100\ldots \quad \text{binary} \qquad \text{A014577} \\ &= .850736188\ldots \quad \text{decimal} \qquad \text{A143347} \end{aligned}$$

A generating function for *TurnLpred* is formed by considering those n with k many low 0 bits,

$$\boxed{\cdots \text{ any } h \cdots}\boxed{0}\boxed{1}\underbrace{\boxed{0\cdots0}}_{k} \qquad n = 2^k + h.4.2^k \tag{9}$$

$$gTurnLpred(x) = \sum_{k=0}^{\infty} \frac{x^{2^k}}{1 - x^{4.2^k}}$$

$1/(1-x^{4.2^k})$ is a generating function with 1 at multiples of 4.2^k. Multiply x^{2^k} to add 2^k giving 1 at all $n = 2^k + h.4.2^k$. If a generating function for an initial part of the sequence is required then stopping at k suffices for $n < 2^{k+1}$ where the next term would begin (k+1 and h=0).

PaperConst is then *gTurnLpred* evaluated at $\frac{1}{2}$. This is the form given by Finch[17],

$$PaperConst = gTurnLpred\left(\tfrac{1}{2}\right) = \sum_{k=0}^{\infty} \frac{1}{2^{2^k}} \cdot \frac{1}{1 - \dfrac{1}{2^{2^{k+2}}}}$$

Sum of just $1/2^{2^k}$ would be the Kempner-Mahler number ahead at (23). The further factor rapidly approaches 1 but makes a much different limit. Paper-folding numbers in other radices are other $0 < |x| < 1$ in *gTurnLpred*.

A generating function for *TurnRpred* is formed in the same away, but bits "11" in (9), so factor 3 in the exponent

$$gTurnRpred(x) = gBitAboveLowestOne(x) = \sum_{k=0}^{\infty} \frac{x^{3.2^k}}{1 - x^{4.2^k}} \tag{10}$$

This is a generating function for *BitAboveLowestOne* too. The constant term is 0 which reckons $BitAboveLowestOne(0) = 0$ where n=0. This is a reasonable definition since n=0 certainly has no 1-bit to be the result bit. Again if a generating function for an initial part of the sequence is required then stopping at k suffices for $n < 3.2^{k+1}$ where the next term would begin.

Another way to think of these generating functions is from low to high by what happens for a new low bit on n. When 0 is appended, *BitAboveLowestOne* is unchanged. When 1 is appended, that bit is the lowest 1 so the second lowest bit is the result.

$$\begin{aligned} &BitAboveLowestOne(2n) = BitAboveLowestOne(n) \\ &BitAboveLowestOne(4n+1) = 0 \\ &BitAboveLowestOne(4n+3) = 1 \end{aligned}$$

So start from $g_0(x)$=0 as *BitAboveLowestOne* for n < 3, then spread to $2n$ by substituting x^2 and add $x^3/(1-x^4)$ which is 1s at all $4m$+3. This substitution is like *gturn* (3).

$$g_{j+1}(x) = g_j(x^2) + \frac{x^3}{1 - x^4}$$

g_{j+1} is good for twice as many n as g_j. Expanding repeatedly this way gives g_j as sum (10) up to $k = j-1$ inclusive and good for $n < 6.2^k$.

gturn at (3) is difference $gTurnLpred - gTurnRpred$ after cancelling common factor $1-x^{2.2^k}$ from numerator and denominator.

$$gturn(x) \;= gTurnLpred(x) - gTurnLpred(x) \;= \sum_{k=0}^{\infty} \frac{x^{2^k} - x^{3.2^k}}{1-x^{4.2^k}}$$

Another sum can be made using *turn* as signs in a harmonic series $1/n$,

$$\sum_{n=1}^{\infty} \frac{turn(n)}{n} \;= \frac{\pi}{2} \tag{11}$$

Odd turns $n=2m{+}1$ alternate ± 1 per (2) so give Gregory's series for $\pi/4$, and which is the Taylor series for $\arctan(1)$ per Leibniz,

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \;= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots \;= \arctan(1) = \frac{\pi}{4}$$

Terms with exactly k low 0-bits are these same terms scaled by $/2^k$ so $(\pi/4)/2^k$ and total $(\pi/4).(1+\frac{1}{2}+\frac{1}{4}+\cdots) = \pi/2$.

Adding terms for each k into the sum is a convergent series inserted at regular intervals into another convergent series. Or an explicit convergence calculation can be made by differences from the subseries limits. Each subseries alternates, so each difference is bounded by the magnitude of the previous term. At $n = 2^{d+1}{-}1$, previous terms $0 \le k \le d$ are each $\le 1/2^d$ and those k not yet seen are at most $1/2^k$ each so a simple total bound $(d{+}1)/2^d + \sum_{k>d} 1/2^k = (d{+}2)/2^d \to 0$.

An integral of $gturn(x)$ down-shifted is denominators n the same as (11),

$$\int_0^1 \frac{1}{x}\, gturn(x)\; dx \;= \frac{\pi}{2}$$

1.2.1 Turn State Machine

A state machine for *BitAboveLowestOne* (= *TurnRpred*) on bits of n from low to high can look for the lowest 1 and then the next bit is the result. Albers [2, figure 7.3 page 106, results L,R] gives the following form. This is simple but allows usual state machine manipulations to mechanically answer questions about intersections, consecutive terms, etc.

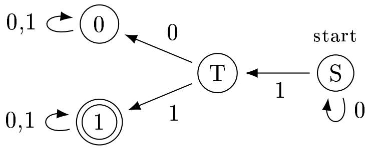


Figure 3:
BitAboveLowestOne
by bits of n
low to high

$n{=}0$ has no 1-bits so remains in state S. It can be taken as result 0 in the way noted after (10). State T is result 0 by reckoning high 0s above n.

To instead take bits of n high to low, the state must remember the result bit so far, and the lowest bit (the bit just seen). The lowest bit becomes the new result bit if the next bit of n is 1. The following is equivalent to Albers [2, figure 7.5 page 107].

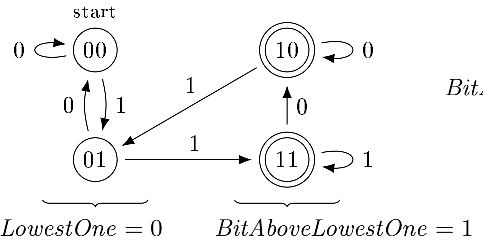


Figure 4: *BitAboveLowestOne* by bits of n high to low

These states can be numbered or labelled arbitrarily. The form above is result bit and least significant bit, so two bits picked from n.

$$\text{state} = \begin{array}{|c|c|}\hline \mathit{BitAboveLowestOne} & n \bmod 2 \\ \hline \end{array} \tag{12}$$

Each 0 transition goes to a top row $x0$, and each 1 transition goes to a bottom row $x1$.

Carpi [8, lemma 3.2] forms a family of bi-infinite words based on n written in a base p^{m+1}. Term a_n is the two digits from n above any low 0 digits, plus $p^{2(m+1)}$ if there are one or more such 0 digits. n=0 is entirely 0 digits and Carpi defines $a_0 = p^{2(m+1)} + 1$. For the case p=2 and m=0, this is the lowest 1-bit of n and the bit above it, and +4 if n even. This is equivalent to (12) suitably re-numbered ((14) below).

$$a_n = \begin{cases} 5 & \text{if } n = 0 \\ n \bmod 4 & \text{if } n \text{ odd} \\ 4 + (\mathit{OddPart}(n) \bmod 4) & \text{if } n \neq 0 \text{ even} \end{cases} \tag{13}$$

$$= 5, 1, 5, 3, 5, 1, 7, 3, 5, 1, 5, 3, 7, 1, 7, 3, \ldots \quad n \geq 0 \qquad \text{A122002}$$

$$\mathit{OddPart}(n) = n / 2^{\mathit{CountLowZeros}(n)}$$

$$= 1, 1, 3, 1, 5, 3, 7, 1, 9, 5, 11, 3, 13, 7, 15, \ldots \quad n \geq 1 \qquad \text{A000265}$$

At (13), mod 4 is understood as remainder 0 to 3. So odd n is $a_n = 1$ or 3 alternating, and even n is $a_n = 5$ or 7 according to *BitAboveLowestOne* (and it taken as 0 when n=0),

$$\begin{aligned} a_n &= 2\,\mathit{BitAboveLowestOne}(n) + 1 + (4 \text{ if } n \text{ even}) \\ &= \text{bits } \begin{array}{|c|c|c|}\hline n{+}1 \bmod 2 & \mathit{BitAboveLowestOne} & 1 \\ \hline \end{array} \\ &= 5, 1, 7, 3 \text{ according as state } 00, 01, 10, 11 \text{ in figure 4} \end{aligned} \tag{14}$$

Carpi gives a morphism h which for p=2, m=0 is as follows and is equivalent to the high to low state machine figure 4 with states labelled per (14).

$$1 \to 5, 3 \qquad 3 \to 7, 3 \qquad 5 \to 5, 1 \qquad 7 \to 7, 1 \qquad \text{starting from 5}$$

Carpi shows subsequences formed by taking every k'th term are square-free (no repeat block E E for E of any length) when k not a multiple of p, so for p=2 here k odd. This includes k=1 so the whole sequence is an alternating parity square-free infinite word, where parity means odd or even state machine labels or in Carpi's form is alternately $a_n < 4$ (at odd n) versus $a_n > 4$ (at even n). Every odd k'th term for $k > 1$ alternates this same way.

Baker, McNulty and Taylor [4] have an equivalent among their family of words $\Omega_{m,z}$ too. Case $\Omega_{2,1}$ (OEIS A112658) is a morphism with values 0, 1, 2, 3 for states 01, 00, 11, 10 in figure 4 (flip the low bit). They too show this is a square-free word.

Kao et al [27] show that all generalized paperfolding sequences with a change of "parity" applied to every second term (so making 4 values) are odd k'th term square-free, again including k=1 for the whole sequences. Their form of Carpi's construction uses values 1, 2, 3, 4 for states 00, 01, 11, 10 in figure 4, so numbered anti-clockwise around (OEIS A125047).

1.2.2 Turn Run Lengths

turn recurrence (1) gives lengths of turn runs. The odd turns L, R are parts of runs of turns. The turn between each is the same as one or the other, so forming runs of lengths 1, 2 or 3, as shown by Bates, Bunder and Tognetti [5]. All these lengths occur in curve k=4 and repeat by unfolding so that all occur infinitely.

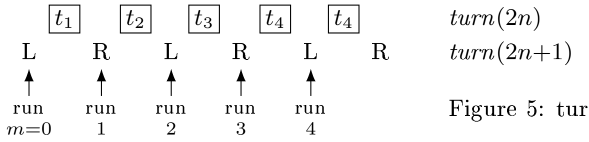


Figure 5: turn runs

Counting the first run as m=0, the run lengths of the curve continued infinitely are, using *turn*,

$$TurnRun(m) = \begin{cases} 2 & \text{if } m{=}0 \text{ (lefts)} \\ 2 - \tfrac{1}{2}\big(turn(m) + turn(m{+}1)\big) & \text{if } m \text{ even } {\geq}2 \text{ (lefts)} \\ 2 + \tfrac{1}{2}\big(turn(m) + turn(m{+}1)\big) & \text{if } m \text{ odd (rights)} \end{cases} \tag{15}$$

$$= 2, 1, 2, 2, 3, 2, 1, 2, 3, 1, 2, 3, 2, 2, \dots \qquad \text{A088431}$$

$$gTurnRun(x) = -\tfrac{1}{2} + \frac{2}{1{-}x} + \tfrac{1}{2}\,(1{-}\frac{1}{x})\; gturn(-x)$$

$$= -1 + \frac{2}{1-x} + \frac{1-x}{1-x^4} + (1{-}x)\sum_{k=1}^{\infty} \frac{x^{3.2^k-1}}{1-x^{4.2^k}} \tag{16}$$

For a curve of finite k, the run lengths instead end with a final 1 which is an unfold of the initial 1.

The generating function sum (16) starts k=1 to take term $/(1{-}x^4)$ out of sum (3). The alternating $gturn(-x)$ negates that term since it has numerator x^3. In further terms, powers are all even so are unchanged by $-x$.

The pairs of consecutive *turns* in (15) can be written

$$TurnRun(m) = \begin{cases} 2 & \text{if } m{=}0 \\ 2 + (-1)^m\,\tfrac{1}{2} sturn(m) & \text{if } m \geq 1 \end{cases}$$

$$sturn(n) = turn(n) + turn(n{+}1) \tag{17}$$

$$\begin{aligned} &= \mathit{turn}(\lceil n/2\rceil) + (-1)^{\lfloor n/2\rfloor} && (18)\\ &= 2,0,0,2,0,-2,0,2,2,0,-2,0,0,-2,0,2,\ldots \quad n{\ge}1 \end{aligned}$$

For (18), a pair of integers n and $n{+}1$ is one odd and one even. *turn* for the odd one alternates $1,-1$ and the even one is the t turn in between as in figure 5. Floor and ceil in (18) combine them. Pairs of consecutive turns like this occur in the midpoint curve too, ahead in section 10.

The equivalent to *sturn* in terms of bits is sum of the two "aboves",

$$\mathit{sBitAboveLowest}(n) = \mathit{BitAboveLowestOne}(n) + \mathit{BitAboveLowestZero}(n) \qquad (19)$$

$$\begin{aligned} &= \mathit{BitAboveLowestOne}(n) + \mathit{BitAboveLowestOne}(n{+}1)\\ &= \mathit{BitAboveLowestOne}\Big(\Big\lceil \frac{n}{2}\Big\rceil\Big) + \mathit{SecondLowestBit}(n)\\ &= 0,1,1,0,1,2,1,0,0,1,2,1,1,2,1,0,0,1,1,0,\ldots \quad n{\ge}1 \end{aligned}$$

$$\mathit{SecondLowestBit}(n) = 0,0,1,1,0,0,1,1,\ldots \quad n{\ge}0 \qquad \text{A021913}$$

$$\mathit{gsBitAboveLowest}(x) = (1+x)\sum_{k=0}^{\infty} \frac{x^{3.2^k-1}}{1-x^{4.2^k}} \qquad (20)$$

$$\mathit{TurnRun}(m) = \begin{cases} 2 & \text{if } m{=}0\\ 2+(-1)^m\big(1-\mathit{sBitAboveLowest}(m)\big) & \text{if } m \ge 1 \end{cases} \qquad (21)$$

At (19), the lowest bit of n is either 0 or 1 so the second lowest bit is either the *Zero* or *One*, and no effect on the other. So cases,

$$\mathit{sBitAboveLowest}(n) = \begin{cases} \mathit{BitAboveLowestOne}(n) & \text{if } n \equiv 0 \bmod 4\\ \mathit{BitAboveLowestZero}(n) & \text{if } n \equiv 1 \bmod 4\\ \mathit{BitAboveLowestOne}(n)+1 & \text{if } n \equiv 2 \bmod 4\\ \mathit{BitAboveLowestZero}(n)+1 & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

or with (6) for *Zero* in terms of *One*,

$$\mathit{sBitAboveLowest}(n) = \begin{cases} \mathit{BitAboveLowestOne}(n) & \text{if } n \equiv 0 \bmod 4\\ \mathit{BitAboveLowestOne}(n{+}1) & \text{if } n \equiv 1 \bmod 4\\ \mathit{BitAboveLowestOne}(n)+1 & \text{if } n \equiv 2 \bmod 4\\ \mathit{BitAboveLowestOne}(n{+}1)+1 & \text{if } n \equiv 3 \bmod 4 \end{cases} \qquad (22)$$

$$= \mathit{BitAboveLowestOne}\big(n+\mathit{LowestBit}(n)\big) + \mathit{SecondLowestBit}(n)$$

The effect of (21) and these various bit combinations is that $\mathit{TurnRun}(m)$ is determined by 3 bits extracted from m,

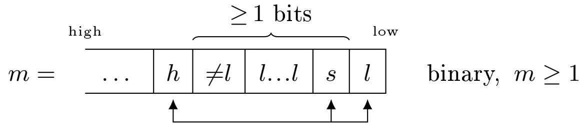


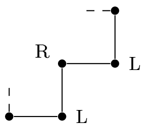
Figure 6

$$TurnRun(m) = \begin{cases} 2 & \text{if } m{=}0 \\ 3,1,2,2,2,2,1,3 & \text{for } hsl = 000, 001, 010, \dots, 111 \\ & \text{respectively (as 3-bit numbers)} \end{cases}$$

In figure 6, l is the lowest bit, s is the second lowest, and then h is above the lowest bit $\neq l$. If $s \neq l$ then h is immediately above s (the third lowest bit), otherwise bits $= l$ are skipped until finding $\neq l$.

Bits of $m{+}1$ can be considered in a similar way, with pattern $h10\dots0s'l'$ and h is above the lowest 1-bit excluding low l' but including s'. This suits if indexing *TurnRun* so that the first run is at index 1 rather than $m{=}0$ as here (and the general case starting from $m{+}1 = 2$). In this form, h is the same position and value, $l' = 1-l$ is opposite, and $s' = s$ or $1-s$ according as $l = 0{,}1$.

Bates, Bunder and Tognetti[5] show that runs of length 1 are located at $n \equiv 3 \bmod 16$ for a single R and $n \equiv 13 \bmod 16$ for a single L. These are turn sequences LRL or RLR respectively. These 3 and 13 mod 16 are not consecutive so the longest stair-step in the curve is just 4 segments.



R run length 1
sequence LRL,
$n \equiv 3 \bmod 16$

Prodinger and Urbanek[43] consider *BitAboveLowestOne* (= *TurnRpred*) as an infinite word of 0,1 symbols. They show it has bounded repetition so subword repeats ww do not occur for w of length ≥ 6, nor of length 4. Their argument can be adapted in a simple way to show no length 2 too, since ww positioned anywhere has one w containing an $n \equiv 1 \bmod 4$ and the other w containing an $n \equiv 3 \bmod 4$, but those $BitAboveLowestOne = 0, 1$ respectively, so different.

Repeats ww with w length 1 occur, being simply consecutive turns (each $TurnRun \geq 2$). A length 3 repeat can be seen at the start of $n = 1 \text{ to } 7$ and a length 5 within $n = 1 \text{ to } 15$. Unfolding then gives infinite occurrences of these lengths.

$$\overbrace{0\ 0\ 1}^{3}\ \underbrace{\overbrace{0\ 0\ 1}^{3}\ 1\ 0}_{5}\ \underbrace{0\ 0\ 1\ 1\ 0}_{5}\ 1\ 1$$

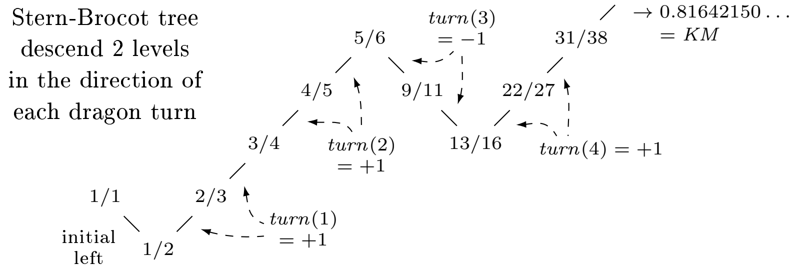
BitAboveLowestOne(1…15)
repeat subwords
lengths 3 and 5

Shallit[48] and Kmošek[30] show that taking the run lengths with each term doubled, an extra initial 1, and then used as continued fraction terms gives the Kempner-Mahler number, a sum of powers of powers of 2 of a type Kempner [28] had shown is transcendental.

$$KM\,cfrac_j = \begin{cases} 0 & \text{if } j{=}0 \ \ \text{(the integer term)} \\ 1 & \text{if } j{=}1 \\ 2\,TurnRun(j{-}2) & \text{if } j \geq 2 \end{cases}$$
$$= 0;\ 1, 4, 2, 4, 4, 6, 4, 2, 4, 6, 2, 4, 6, 4, 4, 2, 4, \dots \qquad \text{A007400}$$

$$\begin{aligned} 0+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\cdots}}}} \quad = KM &= \frac{1}{2}+\frac{1}{4}+\frac{1}{16}+\frac{1}{256}+\cdots+\frac{1}{2^{2^k}}\cdots && (23) \\ &= 0.81642150\ldots && \text{A007404} \\ &= 0.11010001\ldots \ \text{binary} && \text{A036987} \end{aligned}$$

Continued fraction terms $t_0, t_1, \ldots$ are descents down the Stern-Brocot tree of rationals by t_0 many levels left, t_1 many right, and so on, alternating left and right. Continued fraction terms which are in fact run lengths are therefore successive descents left or right according to the original sequence, in this case the dragon curve turn sequence with each value taken twice.


Stern-Brocot tree
descend 2 levels
in the direction of
each dragon turn

The tree here is drawn across the page. A left descent is downwards and a right descent is upwards. The initial 1 in the continued fraction goes to 1/2 and that is the starting point for the turns.

At each fraction, descent is to the right (the larger child) if dragon *turn* = +1. Descent is to the left (the smaller child) if *turn* = −1. For example 1/2 has children 1/3 and 2/3. Since *turn*(1) = +1 go to 2/3. Take two such descents for each dragon turn value.

Any binary sequence can be taken as directions down the Stern-Brocot tree like this. At a given node, the values in all deeper nodes are within a wedge-shaped area. The children divide that into non-overlapping smaller wedges, so any descent sequence converges towards some constant. Per Shallit and Kmošek, the dragon turns duplicated converge on an attractive sum of powers.

A predicate for the n which is the start of a turn run is

$$\begin{aligned} \mathit{TurnRunSpred}(n) &= \begin{cases} 1 & \text{if } n{=}1 \text{ or } \mathit{turn}(n{-}1) \neq \mathit{turn}(n) \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 1 & \text{if } n{=}1 \text{ or } \mathit{turn}(\lfloor n/2 \rfloor) = (-1)^{\lceil n/2 \rceil} \\ 0 & \text{otherwise} \end{cases} && (24) \\ &= 1,0,1,1,0,1,0,1,0,0,1,0,1,1,0,1,\ldots \quad n{\ge}1 && n{\ge}2 \ \ 1{-}\text{A090678} \end{aligned}$$

Form (24) is by considering cases n odd or even. Each pair of even $2j, 2j{+}1$ has the run start as one of the two. The odd turn is $(-1)^j$ and the $(-1)^{\lceil n/2 \rceil}$ expression gives opposites at $2j$ and $2j{+}1$ to compare.

Different consecutive turns are where *sturn* = 0 so that, and the equivalent using *sBitAboveLowest*,

$$\mathit{TurnRunSpred}(n) = 1 - \left|\mathit{sturn}(n{-}1)/2\right| \qquad \text{for } n \geq 2$$

$$= \mathit{sBitAboveLowest}(n{-}1) \mod 2 \qquad \text{for } n \ge 2$$

The sequence of those n which are the start of a run, $\mathit{TurnRunSpred}(n) = 1$, follows from the odd/even cases too. Counting the first run as m=0, each odd turn is at $n = 2m{+}1$ but if preceded by the same turn then its run starts 1 earlier. This can be written as an expression (26).

$$\begin{aligned}
\mathit{TurnRunStart}(m) &= 1 + \sum_{j=0}^{m-1} \mathit{TurnRun}(j) && (25)\\
&= \begin{cases} 2m & \text{if } m{\ge}1 \text{ and } \mathit{turn}(m) = (-1)^m \\ 2m+1 & \text{otherwise} \end{cases}\\
&= \begin{cases} 1 & \text{if } m{=}0 \\ 2m + \frac{1}{2} - \frac{1}{2}(-1)^m \mathit{turn}(m) & \text{if } m \ge 1 \end{cases} && (26)\\
&= 1, 3, 4, 6, 8, 11, 13, 14, 16, 19, 20, 22, 25, \ldots
\end{aligned}$$

For the sum at (25), the *TurnRun* at (15) has *turn* in pairs with alternating signs. They cancel out leaving just $\frac{1}{2}\mathit{turn}(1) = \frac{1}{2}$ at the start and $-(-1)^m \frac{1}{2}\mathit{turn}(m)$ at the end, per (26).

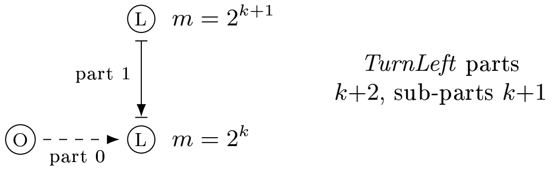
1.2.3 Turn m'th

Theorem 3. *The m'th left or right turn point n is given by mutual recurrences, with the first turn index m=0,*

$$\begin{aligned}
\mathit{TurnLeft}(m) &= \begin{cases} 1 & \textit{if } m = 0 \\ 2^{k+1} & \textit{if } m = 2^k \\ 2^{k+2} - \mathit{TurnRight}(2^{k+1}{-}1 - m) & \textit{if } m > 2^k \end{cases} && (27)\\
&\qquad \textit{where } 2^k \le m < 2^{k+1}\textit{, so } k \textit{ is high 1-bit position in } m\\
&= 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, \ldots && \text{A091072}\\
\mathit{TurnRight}(m) &= 2^{k+2} - \mathit{TurnLeft}(2^{k+1}{-}2 - m) && (28)\\
&\qquad \textit{where } 2^k{-}1 \le m < 2^{k+1}{-}1 && (29)\\
&= 3, 6, 7, 11, 12, 14, 15, 19, 22, 23, 24, 27, \ldots && \text{A091067}
\end{aligned}$$

Proof. In an expansion level k, there are 2^k segments and $2^k - 1$ turns between them. For $k \ge 1$, there are 2^{k-1} are lefts and $2^{k-1} - 1$ rights. This follows from the unfolding since the turn between the unfolding is left so lefts in the next level are *lefts* + 1 + *rights* and the rights are *lefts* + *rights*.

Expansion level k+2 unfolds as sub-parts level k+1,



The L at the unfolding is $n = 2^{k+1}$. Its m is the number of L preceding, which is 2^k when $k \ge 1$ and so that case at (27).

Bigger m are in curve part 1 shown, which by unfolding is a reverse curve. The right turns in the plain curve become left turns in the unfold, and hence *TurnRight* there, with index reversed $2^{k+1}-1-m$ so that $m = 2^k+1$ becomes the last right at $2^{k-1}-1$. And the point n back from $n = 2^{k+2}$.

Likewise *TurnRight*, but there are 2^k-1 rights preceding the unfold and hence the m range at (29). The unfold is then back from $n = 2^{k+2}$ again. ☐

Sequence $\mathit{TurnLeft}(m)-1$ is segment numbers n where the turn at the end of the segment is left, and similarly $\mathit{TurnRight}(m)-1$,

$$\begin{aligned} \mathit{TurnLeft}(m)-1 &= 0,1,3,\ 4,\ 7,\ 8,\ 9,12,\dots \\ \mathit{TurnRight}(m)-1 &= 2,5,6,10,11,13,14,18,\dots \qquad \text{A255068} \end{aligned} \tag{30}$$

Or $\mathit{TurnLeft}(m)+1$ is points n where the previous turn is left, similarly right,

$$\begin{aligned} \mathit{TurnLeft}(m)+1 &= 2,3,5,\ 6,\ 9,10,11,14,\dots \\ \mathit{TurnRight}(m)+1 &= 4,7,8,12,13,15,16,20,\dots \qquad \text{A060833} \end{aligned}$$

Theorem 4. *$n = \mathit{TurnLeft}(m)$ can be calculated by the following bit procedure*

```
$n \leftarrow 2m$ and $t \leftarrow 1$
for each bit position high to low in $n$
    if bit $= t$ then $n \leftarrow n-1$ and $t \leftarrow 1-t$
$n \leftarrow n+1$                                                    (31)
```

And $n = \mathit{TurnRight}(m)$ can be calculated by the following bit procedure

```
$n \leftarrow 2m+2$ and $t \leftarrow 1$
for each bit position high to low in $n$
    if bit $= t$ then $n \leftarrow n+1$ and $t \leftarrow 1-t$
```

The bit tested at each bit position is in the successively modified n. t is an alternating target bit value sought.

Proof. These procedures are implicit in the recurrences of theorem 3. For *TurnLeft*, it's convenient firstly to consider its second half as left turns in a reverse curve (instead of right turns counted from the end).

$$\mathit{TurnLeft}(m) = \begin{cases} 1 & \text{if } m = 0 \\ 2^k & \text{if } m = 2^{k-1} \\ 2^k + \mathit{TurnLeftRev}_k(m - 2^{k-1} - 1) & \text{if } m > 2^{k-1} \end{cases} \tag{32}$$
$$\text{where } 2^{k-1} \le m < 2^k$$

$$\mathit{TurnLeftRev}_k(m) = \begin{cases} \mathit{TurnLeft}(m) & \text{if } m < 2^{k-2} \\ 2^{k-1} + \mathit{TurnLeftRev}_{k-1}(m - 2^{k-2}) & \text{if } m \ge 2^{k-2} \end{cases} \tag{33}$$

$$\text{for } k \ge 2 \text{ and } m \text{ in the range } 0 \le m < 2^{k-1}-1 \tag{34}$$

$$k{=}2 \qquad = 1$$

$$\begin{aligned} k{=}3 &\quad = 1,2,5 \\ k{=}4 &\quad = 1,2,4,5,9,10,13 \end{aligned}$$

In general, a given level reverse curve is not a prefix of the next level reverse curve, so *TurnLeftRev* is for a particular k. Its m range is the number of left turns there, which is the number of right turns in the forward curve. At (32), the *TurnLeftRev* case is when $2^{k-1} < m < 2^k$ which means $k \geq 2$ as required at (34).

The *TurnLeft* procedure starts from $2m$. Consider a *TurnLeft2* which takes $2m$, and a *TurnLeftRev2* which takes $2m{+}1$,

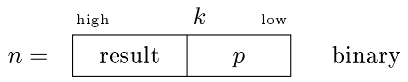
$$\begin{aligned} \mathit{TurnLeft2}(p) &= \mathit{TurnLeft}(m) && \text{where } p = 2m \\ \mathit{TurnLeftRev2}_k(p) &= \mathit{TurnLeftRev}_k(m) && \text{where } p = 2m{+}1 \end{aligned}$$

Then (32),(33) become

$$\mathit{TurnLeft2}(p) = \begin{cases} 1 & \text{if } p{=}0 \\ 2^k & \text{if } p = 2^k \\ 2^k + \mathit{TurnLeftRev2}_k(p - 2^k - 1) & \text{if } p > 2^k \end{cases} \tag{35}$$
$$\text{for } p \text{ even in the range } 2^k \leq p < 2^{k+1}$$

$$\mathit{TurnLeftRev2}_k(p) = \begin{cases} \mathit{TurnLeft2}(p - 1) & \text{if } p < 2^{k-1} \\ 2^{k-1} + \mathit{TurnLeftRev2}_{k-1}(p - 2^{k-1}) & \text{if } p \geq 2^{k-1} \end{cases} \tag{36}$$
$$\text{for } k \geq 1 \text{ and } p \text{ odd in the range } 1 \leq p \leq 2^{k+1} - 3$$

The effect of the procedure is to hold the result so far in the high bits of n, and p in the low bits.



$t{=}1$ is when within *TurnLeft2* (35). Its case $p{=}0$ is all 0-bits in p so that $t{=}1$ is not found by the procedure and there are no more changes to n. Result 1 for this case is the final $n{+}1$ at (31).

Case $p = 2^k$ is bit $t{=}1$ found, and the decrement goes to bits 011…11. The procedure continues with $t{=}0$ which is not found in those 1s so no further change to n. The final $n{+}1$ at (31) adds back the decrement so as to give 2^k as required.

Case $p > 2^k$ takes the bit from p, puts it to the result, and decrements p. Since p is not 2^k, it has at least one further 1-bit where any borrow from that subtract will stop, so the 2^k bit is unchanged in the combined n.

The *TurnLeftRev2* cases at (36) consider a p of k bits and whether its high bit 2^{k-1} is 0 or 1. A 0-bit returns to *TurnLeft2* with a decrement of p. Since p is odd, this decrement simply clears the least significant bit to 0. A 1-bit at 2^{k-1} is moved from p to the result.

TurnLeftRev2 always returns to *TurnLeft2* eventually since it is for $p \neq 2^{k+1} - 1$ so has at least one 0-bit.

The two *TurnLeft2* and *TurnLeftRev2* are thus bit $t = 1 \,\text{or}\, 0$ sought alternately and decrement of n when found.

For *TurnRight*, similar holds. Firstly *TurnRight* in terms of right turns in a reverse curve,

$$\begin{aligned}
\mathit{TurnRight}(m) &= 2^k + \mathit{TurnRightRev}_k\Big(m - (2^k - 1)\Big) \\
&\qquad \text{where } 2^k{-}1 \le m < 2^{k+1}{-}1 \\
\mathit{TurnRightRev}_k(m) &= \begin{cases} \mathit{TurnRight}(m) & \text{if } m{<}2^{k-2}{-}1 \\ 2^{k-1} & \text{if } m{=}2^{k-2}{-}1 \\ 2^{k-1}{+}\mathit{TurnRightRev}_{k-1}\Big(m{-}(2^{k-2}{-}1)\Big) & \text{if } m{>}2^{k-2}{-}1 \end{cases} \\
&\qquad \text{for } k \ge 1 \text{ and } m \text{ in the range } 0 \le m < 2^{k-1} \\
k{=}1 \quad &= 1 \\
k{=}2 \quad &= 2, 3 \\
k{=}3 \quad &= 3, 4, 6, 7
\end{aligned}$$

Then *TurnRight2* and *TurnRightRev2* on a doubled m,

$$\begin{aligned}
\mathit{TurnRight2}(p) &= \mathit{TurnRight}(m) && \text{where } p = 2m{+}2 \\
\mathit{TurnRightRev2}_k(p) &= \mathit{TurnRightRev}_k(m) && \text{where } p = 2m{+}1
\end{aligned}$$

with mutual recurrences

$$\begin{aligned}
\mathit{TurnRight2}(p) &= 2^k + \mathit{TurnRightRev2}(p - 2^k + 1) \\
&\qquad \text{for } p \text{ even in the range } 2^k \le p < 2^{k+1} \\
\mathit{TurnRightRev2}_k(p) &= \begin{cases} \mathit{TurnRight2}(p+1) & \text{if } p < 2^{k-1} - 1 \\ 2^{k-1} & \text{if } p = 2^{k-1} - 1 \\ 2^{k-1} + \mathit{TurnRightRev}_{k-1}(p{-}2^{k-1}) & \text{if } p \ge 2^{k-1} \end{cases} \\
&\qquad \text{for } k \ge 1 \text{ and } p \text{ odd in the range } 1 \le p \le 2^k
\end{aligned}$$

In the procedure, $t{=}1$ is when in *TurnRight2*. There bit 2^k moves from p to the result, p increments, and go to *TurnRightRev2*. Since p is even, this increment simply sets the lowest bit.

In *TurnRightRev2*, each high 1-bit at 2^{k-1} moves from p to the result. A high 0-bit is go back to *TurnRight* with an increment, except for case $p = 2^{k-1} - 1$ which is bits 011…11. The procedure increments that to 100…00 and seeks a 1-bit among the 0s, which there is none, so no further change and so the desired 2^{k-1}. □

Finding alternating $t = 1 \,\text{or}\, 0$ (in the successively modified n) in the procedures is almost the same as finding 01 or 10 pairs, but not quite. In *TurnLeft*, the case 2^k at (35) decrements to 011…11 which would be a further 01 and an unwanted decrement. Similarly in *TurnRight2* case $2^k{-}1$ which increments to makes a 10. Both can be made to work by checking for the increment or decrement changing bit k and stop the loop when that happens. A separate t seems easier.

Both *TurnLeft* and *TurnRight* are close to $2m$. The procedures make increments or decrements from there, or roughly speaking since there are 2^k and

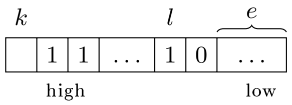
$2^k{-}1$ of each in expansion level $k{+}1$ so about half each turn. Offsets from $2m$ can be expressed

$$\begin{aligned} \mathit{TurnLeftOff}(m) &= 2m - \mathit{TurnLeft}(m) \\ &= -1, 0, 0, 1, 0, 1, 2, 1, 0, 1, 2, 2, 3, 1, 2, 1, 0, \ldots \\ \mathit{TurnRightOff}(m) &= \mathit{TurnRight}(m) - 2m \\ &= 3, 4, 3, 5, 4, 4, 3, 5, 6, 5, 4, 5, 4, 4, 3, 5, 6, \ldots \end{aligned} \tag{37}$$

Substituting into (27),(28) gives mutual recurrences

$$\begin{aligned} \mathit{TurnLeftOff}(m) &= \begin{cases} -1 & \text{if } m{=}0 \\ 0 & \text{if } e{=}0 \\ \mathit{TurnRightOff}(2^k{-}1-e)\ -2 & \text{otherwise} \end{cases} \\ &\quad \text{where } m = 2^k + e, \text{ with } 0{\le}e{<}2^k \\ \mathit{TurnRightOff}(m) &= \mathit{TurnLeftOff}(2^k{-}1-e)\ +4 \\ &\quad \text{where } m = 2^k{-}1+e, \text{ with } 0{\le}e{<}2^k \end{aligned}$$

$2^k{+}e$ takes a high bit, then the reversal is a bit flip so the descent into the opposite Left/Right finds the next 0-bit below. The recurrences can be expressed staying in left or right by taking a high run of 1s from m or from $m{+}1$ for right.



high run of 1s of m (or $m{+}1$)
bit positions 0 lowest

$$\begin{aligned} \mathit{TurnLeftOff}(m) &= \begin{cases} -1 & \text{if } m{=}0 \\ 0 & \text{if } e{=}0 \text{ and } k = l{+}1 \\ 1 & \text{if } e{=}0 \text{ and } l = 0,\ m \ge 3 \\ 2 & \text{if } e{=}0 \text{ and } l = 1,\ m \ge 6 \\ 3 & \text{if } e{=}0 \text{ other} \\ \mathit{TurnLeftOff}(e{-}1)\ +2 & \text{otherwise} \end{cases} \\ &\quad \text{where } m = 2^k - 2^l + e, \text{ with } 0 \le e < 2^{l-1} \\ \mathit{TurnRightOff}(m) &= \begin{cases} 3 & \text{if } l = 0 \\ 4 & \text{if } e = 2^{l-1} - 1 \\ \mathit{TurnRightOff}(e)\ +2 & \text{otherwise} \end{cases} \\ &\quad \text{where } m{+}1 = 2^k - 2^l + e, \text{ with } 0 \le e < 2^{l-1} \end{aligned} \tag{38}$$

The cases for left (38) are bit patterns 10…0 single bit, 1…1 all 1s, 1…10 single low 0. In both left and right, the recurrence descends with -1 on the new m. In the lefts, this is $e{-}1$. In the rights, this is since e is formed from bit run on $m{+}1$.

See ahead at (57),(58) for new highs in the offsets.

Increments between successive turns L or R are

$$\mathit{dTurnLeft}(m) = \mathit{TurnLeft}(m{+}1) - \mathit{TurnLeft}(m)$$

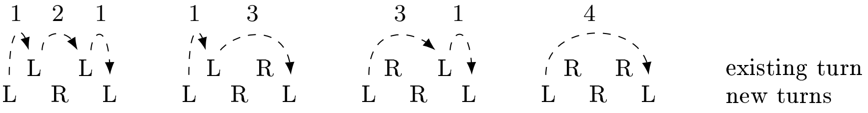
$$\begin{aligned} &= 1,2,1,3,1,1,3,3,1,1,2,1,4,1,3,3,1,\ldots && \text{A088742} \\ \mathit{dTurnRight}(m) &= \mathit{TurnRight}(m{+}1) - \mathit{TurnRight}(m) \\ &= 3,1,4,1,2,1,4,3,1,1,3,1,2,1,4,3,1,\ldots && \text{A106836} \end{aligned}$$

The expansions in figure 5 show steps are always 1, 2, 3, 4.

The *dTurnLeft* sequence follows from the turn sequence by a mapping of turn pairs, starting at the start of the turn sequence, so that the first of each pair is an odd n.

$$\text{LL} \to 1,2,1 \qquad \text{LR} \to 1,3 \qquad \text{RL} \to 3,1 \qquad \text{RR} \to 4 \tag{39}$$

This is since the turn sequence expands with alternate L or R preceding each existing turn. So a pair of turns $n, n{+}1$ where n is odd get new L,R,L as follows.



Pairs of turns expand as follows. These are the above groups of 5, without the final L of each.

LL → LLRL   LR → LLRR   RL → LRRL   RR → LRRR   starting LL

Substituting (39) into this gives a morphism for *dTurnLeft*

121 → 12131   13 → 1214   31 → 1331   4 → 134   starting from 121

This expansion is unambiguous since 2 occurs only in 121 triplets and taking those 121s leaves each 3 with an untaken 1 either before or after but not both. The expansions put each 3 in the middle of new terms, not at the start or end, so no need to consider what might be before or after an originating term.

The m'th increment can be expressed by mutual recurrences,

$$\mathit{dTurnLeft}(m) = \begin{cases} 1,2 & \text{if } m = 0,1 \\ 1 & \text{if } e = 0 \\ 3 & \text{if } e = 2^k{-}1 \\ \mathit{dTurnRight}(2^k{-}2 - e) & \text{otherwise} \end{cases}$$

$$\text{where } m = 2^k + e, \text{ with } 0{\le}e{<}2^k \text{ and } m \ge 2$$

$$\mathit{dTurnRight}(m) = \begin{cases} 3 & \text{if } m = 0 \\ 4 & \text{if } e = 2^k{-}1 \\ \mathit{dTurnLeft}(2^k{-}2 - e) & \text{otherwise} \end{cases}$$

$$\text{where } m = 2^k{-}1 + e, \text{ with } 0{\le}e{<}2^k \text{ and } m \ge 1$$

In curve unfolding, the direction reverses so the two turns which are the step swap position. This makes it necessary to descend to 1 smaller $2^k{-}2 - e$ back from the end, to stay across the same step.

In these recurrences, nothing is accumulated, just descend down m by unfolds until reaching one of the 1, 2, 3, 4 cases.

dTurnLeft can also be calculated using the *TurnLeft* procedure to find n where the step will start and then a $\mathit{dNextL}(n)$ for the step there.

Theorem 5. *The offset $\mathit{dNextL}(n)$ from any point n to the next left turn n is given by the low bits of n of the following patterns, where $\underline{1}$ means zero or more 1-bits. High 0-bits are understood on n where needed to match.*

$$\mathit{dNextL}(n) = \begin{cases} 1 & \text{if } n = \ldots 00\underline{1} \\ 2 & \text{if } n = \ldots 00\underline{1}10 \quad \text{or } \ldots 10\underline{1}11 \\ 3 & \text{if } n = \ldots 00\underline{1}101 \ \text{or } \ldots 10\underline{1}10 \\ 4 & \text{if } n = \ldots 10\underline{1}101 \end{cases}$$

And dNextR offset to the next right turn,

$$\mathit{dNextR}(n) = \begin{cases} 1 & \text{if } n = \ldots 10\underline{1} \\ 2 & \text{if } n = \ldots 001 \ \text{ or } \ldots 100 \\ 3 & \text{if } n = \ldots 000 \ \text{or } \ldots 0011 \\ 4 & \text{if } n = \ldots 00\underline{1}111 \end{cases} \tag{40}$$

$$\begin{aligned} \mathit{dNextL}(n) &= 1,1,2,1,1,3,2,1,1,1,3,2,1,3,2,1,\ldots \\ \mathit{dNextR}(n) &= 3,2,1,3,2,1,1,4,3,2,1,1,2,1,1,4,\ldots \end{aligned}$$

Proof. Bates, Bunder and Tognetti [5, section 4] show the forms of n which are the first of each run of 1, 2 or 3 consecutive L or R in the turn sequence. (They work with *TurnLpred* = 1,0.) For *dNextL*, the n immediately preceding a run of 2 rights is $\mathit{dNextL}=3$ to step past them, or the first n of such a run is $\mathit{dNextL} = 2$, and the n of the second is $\mathit{dNextL} = 1$. Similarly runs of 1 and 3, and right and left vice versa. ☐

Second Proof of Theorem 5. A mechanical approach can be made using the state machines for *TurnRpred* in figure 3 or figure 4, and their complement for *TurnLpred*. $\mathit{dNextL}(n) = 1$ is at those n where $n{+}1$ is L. This is next turn left per (5), and is a pair "any,L" at n.

$\mathit{dNextL}(n) = 2$ is a triplet "any,R,L" at n, so R at $n{+}1$ and make some state machine manipulations for a test of L at $n{+}2$, which is the bit strings of left turns all subtract 2. The intersection of R next and L second next is all $\mathit{dNextL} = 2$.

Similarly $\mathit{dNextL}(n) = 3$ is four "any,R,R,L" at n, and $\mathit{dNextL}(n) = 4$ is five "any,R,R,R,L" at n.

dNextR cases are likewise any,R, any,L,R, any,L,L,R, any,L,L,L,R. ☐

The *dTurnLeft* and *dNextL* sequences are related by inserting into *dTurnLeft* the successive steps down which are the non-left n,

$$1 \to 1 \qquad 2 \to 2{,}1 \qquad 3 \to 3{,}2{,}1 \qquad 4 \to 4{,}3{,}2{,}1$$

dNextL starts with an initial 1 for $n{=}0$ too. Similarly *dTurnRight* related to *dNextR*, with starting with initial $3,2,1$ for $n = 0,1,2$.

dTurnLeft and *dTurnRight* can use *dNextL* and *dNextR* by

$$\mathit{dTurnLeft}(m) = \mathit{dNextL}\big(\mathit{TurnLeft}(m)\big)$$

$$\mathit{dTurnRight}(m) = \mathit{dNextR}\big(\mathit{TurnRight}(m)\big)$$

These apply *dNextL* only to a left turn n, and *dNextR* only to a right turn n, which reduces their cases a little.

For computer calculation, a modest optimization over the *TurnLeft* procedure followed by *dNextL* can be made using the bits successively seen (high to low) within the *TurnLeft* procedure. This has the attraction of being one-pass so that once the bit at each position has been considered, it is not needed again. Similarly *TurnRight*. The bits seen are not the n turn as such, but they suffice to determine d.

Theorem 6. *$d = \mathit{dTurnLeft}(m)$ can be calculated by the following bit procedure*

```
$n \leftarrow 2m$ and $d \leftarrow$ 1a
for each bit position high to low in $n$
    $d \leftarrow$ transition($d$, $bit$ of $n$)
    if $d$ = s1 ... s4 then $n \leftarrow n - 1$        (41)
```

where transition(d, bit) is in the state machine of figure 7 below.

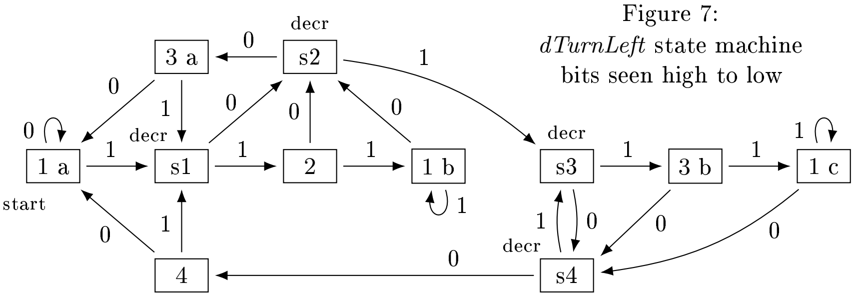


Figure 7:
dTurnLeft state machine
bits seen high to low

d is a state and the final value $1, 2, 3, 4$ is the result. The a, b, etc letters are just to distinguish states. s1 to s4 are never final states and have no result value. At (41), a transition going to any of s1 to s4 means decrement n.

Proof. The n progressed is the same as in the *TurnLeft* procedure of theorem 4. The t bit there is in the states as

t=1 states: 1a, 3a, 4, s2, s4
t=0 states: 1b, 1c, 2, 3b, s1, s3

When $bit = t$, the transition goes to a state of the opposite t, per the *Turn Left* procedure. Each state is reached either from previous states of the same t as it, or the opposite t, but never a mixture. The result is to decrement n or not according to the destination state (s1,s2,s3,s4).

The bit values seen at each bit position are not the final left turn n.

When the *TurnLeft* procedure ends with the 2^k case of *TurnLeft2*, the bits of n are 1 000 and the decrement gives 0 111. The first bit is considered before that decrement so 1 111 is seen by the state machine here.

There are 1 or more low 0s changed to 1s this way, since p in *TurnLeft2* is even. *dNextL* cases for n even and left turn (0-bit above lowest 1) are as follows.

dNextL	even n	low 1s seen
1	$\ldots01\underline{0}00$	$\ldots0\underline{1}111$
2	$\ldots0010$	$\ldots0011$
3	$\ldots1010$	$\ldots1011$

Case *dNextL* = 1 is two or more low 0s on n which together with its lowest 1 become three or more. Cases *dNextL* = 2, 3 have none of their $\underline{1}$ repeats since that would not be a left turn n. Consequently they are distinct from the 1 case. *dNextL* = 4 is never an even n.

When the *TurnLeft* procedure ends in the 0 case of *TurnLeft2*, the bits seen are of $n-1$ (which the final (31) will increment). The low bit of $n-1$ is a 0 since p is even, so n is odd. The *dNextL* cases on $n-1$ of an odd left turn n are as follows. There is no 2 since its only odd n is …11 which is not a left turn.

dNextL	odd n	$n-1$ seen
1	$\ldots001$	$\ldots000$
3	$\ldots00\underline{1}101$	$\ldots00\underline{1}100$
4	$\ldots10\underline{1}101$	$\ldots10\underline{1}100$

The state machine recognises all of the above cases. High 0-bits understood on n as necessary for the cases. In the state machine, high 0s are optional. The final state indicates the result d = *dNextL*.

s1 and s3 are not final states since they are reached only by bits 01. This could only be from low 0s changed to 1s, but there is at least one such changed bit, plus the lowest 1-bit, so always at least two low 1-bits seen.

s2 and s4 are not final states since they are reached only by bits 10 but that would be an $n-1$ and its n = …11 is not a left turn. ☐

Theorem 7. *d = dTurnRight(m) can be calculated by the following bit procedure*

```
$n \leftarrow 2m+2$ and $d \leftarrow$ 3a
for each bit position high to low in $n$
    $d \leftarrow$ transition($d$, bit of $n$)
    if $d$ = s1, s2 or s3  then $n \leftarrow n+1$          (42)
```

where transition(d, bit) is in the state machine of figure 8 below.

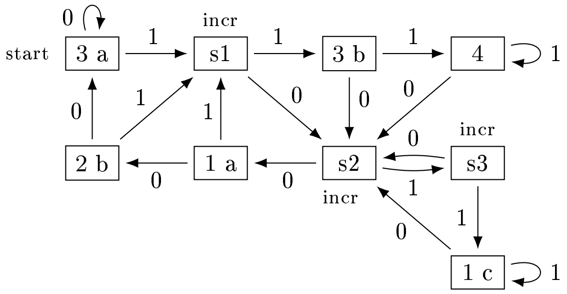


Figure 8:
dTurnRight
bits seen
high to low

d is a state and final value 1, 2, 3, 4 is the result. The a, b, etc letters are just to distinguish states. s1 to s4 are never final states and have no result value. At (42), a transition going to any of s1, s2, s3 means increment n.

Proof. The n progressed is the same as the *TurnRight* procedure in theorem 4. The target bit t there is in the states as

t=1 states:	1a, 2, 3a, s2
t=0 states:	1b, 3b, 4, s1, s3

When $bit = t$, the transition goes to a state of the opposite t, per the *Turn Right* procedure. The states are arranged so that each state is reached either from a same t or different t, but not a mixture. So increment n, or not, according to the destination state (s1,s2,s3).

State s3 exists only for the purpose of this destination increment rule. It could merge with 1c just for the result d, but a transition coming from s2 should increment n whereas the self-loop at 1c should not.

The bit values seen at each bit position are not the final right turn n.

When the *TurnRight* procedure ends with the $2^k{-}1$ case of *TurnRightRev2*, the bits of that n are 0 11…11 which increment to 1 00…00. There are one or more such low bits so that this is an even n. The first bit is considered before that increment so 0 00…00. is seen by the state machine here. The *dNextR* cases from (40) on right turn n are then

dNextR	even n	seen	*dNextR*	odd n
1	$\ldots 110$	$\ldots 100$	1	$\ldots 10\underline{1}11$
2	$\ldots 1100$	$\ldots 1000$	3	$\ldots 0011$
3	$\ldots 11\underline{0}000$	$\ldots 10\underline{0}000$	4	$\ldots 00\underline{1}111$

The even n cases are distinguished by having 2, 3, or $\geq$4, low 0-bits for $d = 1, 2, 3$ respectively, so the cases are distinct.

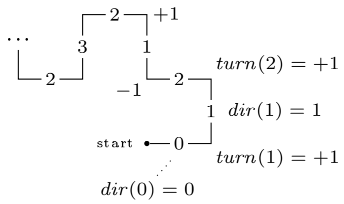
The state machine recognises all of the above. High 0-bits understood on n as necessary for the cases. In the state machine, high 0s are optional. The final state indicates the result d = *dNextL*.

s1 and s3 are not final states since they are reached only by bits 01 which is odd, so an unmodified n, but 0 above lowest 1 is not a right turn.

s2 is not a final state since it is reached only by bits 10 which is even and would occur only as an n with lowest 1 cleared to 0, but such clear is not at the least significant bit (p even as described above). ☐

1.3 Direction

Davis and Knuth [12] give segment directions in the curve as net total turn +1 or −1 up to n. Let $dir(n)$ be the direction of segment n, with first segment numbered n=0,



segment direction
$dir(n)$,
net total *turn*

$$dir(n) = \sum_{j=1}^{n} turn(j) \qquad \text{empty sum when } n{=}0 \text{ so } dir(0){=}0$$

$$= \mathit{CountBitTransitions}(n) \tag{43}$$

$$= \mathit{CountOneBits}\big(\mathit{Gray}(n)\big) \tag{44}$$

$$= 0,1,2,1,2,3,2,1,2,3,4,3,2,3,2,1,2,3,4,3,\ldots \qquad \text{A005811}$$

$$\mathit{Gray}(n) = \text{BITXOR}(n,\ \lfloor n/2 \rfloor) \qquad \text{binary reflected Gray code} \tag{45}$$

$$= 0,1,3,2,6,7,5,4,12,13,15,14,10,11,\ldots \qquad \text{A003188}$$

$$\mathit{CountOneBits}(n) = 0,1,1,2,1,2,2,3,1,2,2,3,2,3,\ldots \qquad \text{A000120}$$

CountBitTransitions is the number of 10 or 01 bits pairs (overlaps allowed). Or equivalently, the number of places a bit differs from the bit immediately above. High 0-bits are understood on n so the most significant 1-bit of n is always a transition 01.

Gray code form (44) is per George P. Davis (Knuth [31, addendum]). The shift and XOR for the Gray code gives a 1-bit at each bit transition, which can then be counted. The Gray code is a permutation of the integers in blocks $0 \le n < 2^k$ so the *dir* sequence is such a permutation of *CountBitTransitions*, and one which arranges successive steps always $+1$ or -1 (the turn sequence).

Davis and Knuth start their direction function g at $g(1) = 0$, so $g(n) = \mathit{dir}(n-1)$ and so $g(n)$ is the direction of the segment preceding point n (point $n{=}0$ as curve start still). The various relations they give have corresponding forms in *dir*. Starting $\mathit{dir}(0) = 0$ has the attraction of bitwise interpretations on n (rather than $n-1$), and the segment directions in curve k are over all n of k bits.

A generating function for *dir* is a usual factor $1/(1-x)$ for cumulative turns (per for example Ralf Stephan in A005811).

$$\mathit{gdir}(x) = \frac{1}{1-x}\,\mathit{gturn}(x) \quad = \sum_{k=0}^{\infty} \frac{x^{2^k}}{(1-x)\,(1+x^{2^{k+1}})}$$

The direct interpretation of this sum is term k coefficient 1 where the bit at k differs from the bit at $k{+}1$. This is 01 or 10 so x^{2^k} up to and not including $x^{3.2^k}$, and replicated so arbitrary bits above.

$$\begin{aligned} \frac{x^{2^k} + \cdots + x^{3.2^k-1}}{1-x^{2^{k+2}}} &= \frac{x^{2^k} - x^{3.2^k}}{(1-x)\,(1-x^{2^{k+2}})} \\ &= \frac{x^{2^k}\,(1-x^{2^{k+1}})}{(1-x)\,(1+x^{2^{k+1}})\,(1-x^{2^{k+1}})} \end{aligned} \tag{46}$$

dir is a maximum when n has a bit transition at every bit. In curve level k, this is

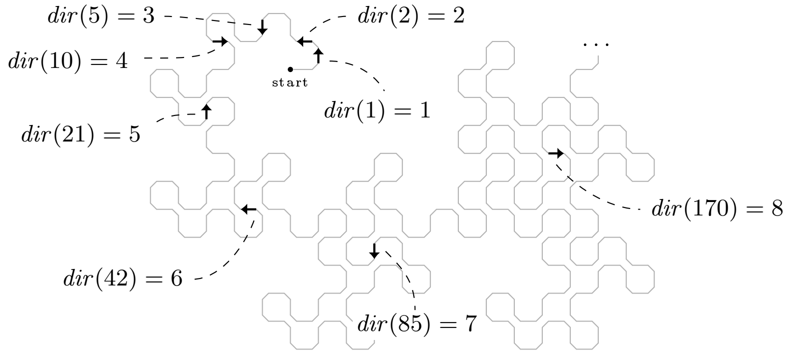
$$\max_{n=0}^{2^k-1} \mathit{dir}(n) = k \qquad \text{unique} \tag{47}$$

$$\text{at } n = \mathit{DirMaxN}_k = \left\lfloor \tfrac{2}{3}.2^k \right\rfloor \tag{48}$$

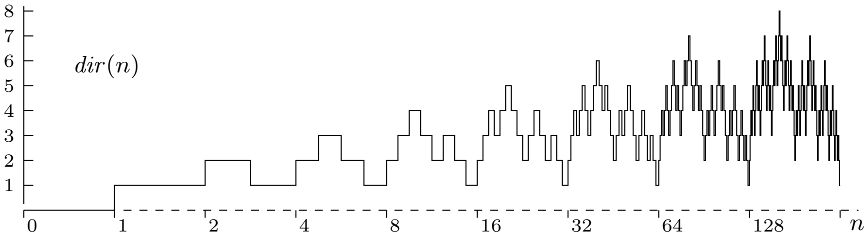
$$= 0,1,2,5,10,21,42,85,\ldots \qquad \text{A000975}$$

$$= \text{binary } 0,\ 1,\ 10,\ 101,\ 1010,\ 10101,\ 101010,\ldots \qquad \text{A056830}$$

Davis and Knuth note $DirMaxN_k$ is the first n attaining $dir(n) = k$ in the curve continued infinitely. Or in their g numbering $g(DirMaxN_k + 1) = k$, at $DirMaxN_k + 1 = \lceil \frac{2}{3} 2^k \rceil$ (A005578).



Some of the structure of dir can be illustrated in a plot. Each new high is the unique dir maximum in level k at (47). The last n of each 2^k block is the return to $dir(2^k-1) = 1$.



Blocks of $n = 2^k$ to 2.2^k-1 are shown scaled to the same width, and linear within them, in order to see successive refinements. The overall shape is preserved, just with some additional excursions up added. Arithmetically, this is since a repeat of the low bit of n is dir unchanged, or opposite of the low bit is +1. (Also ahead at (74).)

$$dir(\ldots 0) = d_1 < \begin{array}{l} dir(\ldots 00) = d_1 \\ dir(\ldots 01) = d_1 + 1 \end{array}$$

$$dir(\ldots 1) = d_2 < \begin{array}{l} dir(\ldots 10) = d_2 + 1 \\ dir(\ldots 11) = d_2 \end{array}$$

The number of left and right turns from 1 to n inclusive are

$$\begin{aligned} TurnsL(n) &= \sum_{j=1}^{n} TurnLpred(n) \\ &= 0, 1, 2, 2, 3, 4, 4, 4, 5, 6, 7, 7, 7, 8, 8, \ldots \quad n \geq 0 \\ TurnsR(n) &= \sum_{j=1}^{n} TurnRpred(n) \\ &= 0, 0, 0, 1, 1, 1, 2, 3, 3, 3, 3, 4, 5, 5, 6, \ldots \quad n \geq 0 \end{aligned}$$

A255070

$$gTurnsL(x) = \frac{1}{1-x}\, gTurnLpred(x) \;= \sum_{k=0}^{\infty} \frac{x^{2^k}}{(1-x)\,(1-x^{4.2^k})} \tag{49}$$

$$gTurnsR(x) = \frac{1}{1-x}\, gTurnRpred(x) \;= \sum_{k=0}^{\infty} \frac{x^{3.2^k}}{(1-x)\,(1-x^{4.2^k})} \tag{50}$$

All turns are left or right so total lefts plus rights is simply n. The difference lefts minus rights is net direction dir. In the generating functions, this is seen in $gdir$ term form (46) which is difference (49) – (50).

$$TurnsL(n) + TurnsR(n) \;= n \tag{51}$$

$$TurnsL(n) - TurnsR(n) \;= dir(n) \tag{52}$$

Sum and difference of (51),(52) give

$$\begin{aligned} TurnsL(n) &= \tfrac{1}{2}\big(n + dir(n)\big) \\ TurnsR(n) &= \tfrac{1}{2}\big(n - dir(n)\big) \end{aligned} \tag{53}$$

Prodinger and Urbanek[43] show (53) by induction on an equivalent to turn sequence flip and reverse unfolding. They use this to show the limit proportion of rights and lefts is 1:1, since $dir(n) \le 1 + \log_2 n$,

$$\lim_{n\to\infty} \frac{TurnsL(n)}{n} \;= \lim_{n\to\infty} \frac{TurnsR(n)}{n} \;= \frac{1}{2}$$

They calculate $TurnsR$ also by a sum counting how many numbers of the form 1100…00 are $\le n$, since these are all the $BitAboveLowestOne(n) = 1$. The sum is over j many low 0-bits.

$$TurnsR(n) = \sum_{j\ge 0} \left\lfloor \frac{n+2^j}{2^{j+2}} \right\rfloor$$

This is similar to the $gTurnRpred$ sum at (10). The equivalent for lefts is

$$TurnsL(n) = \sum_{j\ge 0} \left\lfloor \frac{n+3.2^j}{2^{j+2}} \right\rfloor \;= \sum_{j\ge 0} \left\lceil \frac{n-(2^j-1)}{2^{j+2}} \right\rceil$$

n with next turn right shown at (30) is OEIS A255068, as noted by Antti Karttunen in that entry. That sequence is defined as A255068$(t) = n$ where n is the maximum n satisfying

$$\tfrac{1}{2}\big(n - dir(n)\big) = t$$

This is $TurnsR(n) = t$. $TurnsR$ is non-decreasing and the maximum n where it is a given t occurs when the next turn $n{+}1$ is to the right.

$TurnLeft$ from theorem 3 and $TurnsL$ are inverses in the sense that

$$TurnsL\big(TurnLeft(m)\big) = m + 1$$

$TurnLeft$ is the least inverse of $TurnsL$ since its left turn at $n = TurnLeft(m)$ increments $TurnsL$. Or equivalently $n{-}1$ is the greatest for which $TurnsL(n{-}1) = m$. Similarly $TurnRight$ and $TurnsR$.

The procedure for *TurnLeft* in theorem 4 finds, for given m, the least solution n of

$$\begin{aligned} \mathit{TurnsL}(n) &= m+1 \\ \tfrac{1}{2}\big(n + \mathit{dir}(n)\big) &= m+1 \qquad (54) \\ n &= 2m+2-\mathit{dir}(n) \qquad (55) \end{aligned}$$

The successive decrements in the procedure effectively apply the 01-transition counting of *dir* to reduce n and satisfy (55). But, roughly speaking, the correctness of the procedure depends on decrement for a given transition not upsetting higher ones already considered, and as noted after the proof of theorem 4 a little care is needed.

The procedure for *TurnRight* in theorem 4 finds in a similar way

$$n = 2m+2+\mathit{dir}(n) \qquad (56)$$

These show too *TurnLeftOff* and *TurnRightOff* from (37) are

$$\begin{aligned} \mathit{TurnLeftOff}(m) &= \mathit{dir}(n) - 2 \qquad &&\text{where } n = \mathit{TurnLeft}(m) \\ \mathit{TurnLeftRight}(m) &= \mathit{dir}(n) + 2 \qquad &&\text{where } n = \mathit{TurnRight}(m) \end{aligned}$$

New highs in *TurnLeftOff* are new highs in *dir* among left turn n. For n of $k \ge 1$ many bits, the unique $\mathit{DirMaxN}_k$ at (48) is a left turn (0 above lowest 1-bit). Its m, per (54), is

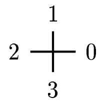
$$\begin{aligned} m &= \tfrac{1}{2}(\mathit{DirMaxN}_k + k) - 1 \qquad (57) \\ &= \left\lfloor \tfrac{1}{3}2^k + \tfrac{1}{2}k - 1 \right\rfloor \\ &= 0, 1, 3, 6, 12, 23, 45, 88, \dots \qquad k \ge 1 \qquad \text{A086445} - 1 \end{aligned}$$

New highs in *TurnRightOff* are new highs in *dir* among right turn n. To be a right turn is 1-bit above lowest 1-bit. This is a non-transition for *dir*, so in $k \ge 2$ bits maximum $\mathit{dir} = k{-}1$. So unique $n = 10\dots10\,11$ when even k, or $n = 10\dots10\,110$ when odd k, which in both cases is $n = \mathit{DirMaxN}_k + 1$. Its m then, per (56), is

$$\begin{aligned} m &= \tfrac{1}{2}\big(\mathit{DirMaxN}_k + 1 - (k{-}1)\big) - 1 \qquad (58) \\ &= \tfrac{1}{4}(3^k - 2k - 1) \\ &= 0, 1, 3, 8, 18, 39, 81, 166, \dots \qquad k \ge 2 \qquad \text{A178420} \end{aligned}$$

These offsets are the maximum number of decrements or increments made by the *TurnLeft* and *TurnRight* procedures in theorem 4. In both cases for n of k bits they make at most $k{-}1$ decrements or increments in their respective loops, and the m (or n) where that maximum occurs is unique.

$\mathit{dir}(n)$ mod 4 is a net segment direction East, North, West, or South.



direction mod 4

$$\mathit{dir}(n) \bmod 4 \equiv 0, 1, 2, 1, 2, 3, 2, 1, 2, 3, 0, 3, 2, 3, 2, 1, \dots \qquad \text{A246960}$$

$$\equiv 0 \text{ at } n = 0, 10, 18, 20, 22, 26, 34, 36, 38, 40, \dots \qquad \text{A043724}$$
$$\equiv 1 \text{ at } n = 1,\ \ 3,\ \ 7, 15, 21, 31, 37, 41, 43, 45, \dots \qquad \text{A043725}$$
$$\equiv 2 \text{ at } n = 2,\ \ 4,\ \ 6,\ \ 8, 12, 14, 16, 24, 28, 30, \dots \qquad \text{A043726}$$
$$\equiv 3 \text{ at } n = 5,\ \ 9, 11, 13, 17, 19, 23, 25, 27, 29, \dots \qquad \text{A043727}$$

Dekking[15, section 4.5] gives *dir* mod 4 (as letters a,b,c,d) in the form of an expansion (morphism), or Arndt[3, figure 1.31-G] likewise (as numbers 0 to 3),

$$0 \to 0,1 \qquad 1 \to 2,1 \qquad 2 \to 2,3 \qquad 3 \to 0,3 \qquad \text{starting } 0$$

These expansions are the segment expansions. The curve always turns left or right so even directions $0, 2$ are even segments which expand on the right, and odd directions are odd segments which expand on the left.

A morphism like this is equivalent to taking bits of n high to low. The bits go through the following state machine to find the direction mod 4 of n.

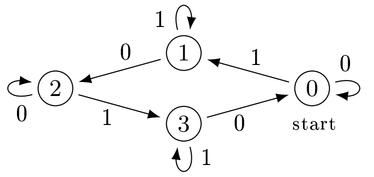
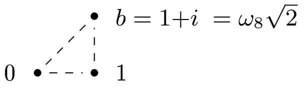


Figure 9: *dir*(n) mod 4 by bits of n high to low

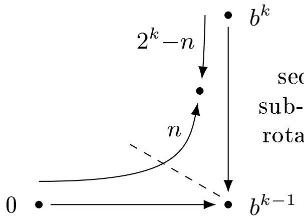
Each time a bit differs from the preceding bit, the state steps +1, thereby counting bit transitions as at (43), but with wrap-around mod 4. Similar can be done for other moduli. An even modulus is an even state for a preceding 0-bit or odd state for preceding 1-bit. An odd modulus would have separate states for 0 or 1 preceding.

1.4 Coordinates

It's convenient to number the points of the dragon curve starting n=0 at the origin and first segment directed East. Locations can then be calculated in complex numbers using powers of base $b = i{+}1$ which is the end of a single segment expansion.



Davis and Knuth[12] calculate coordinates from the curve unfolding. n in the second half of the curve is a point in that sub-curve directed back from b^k, giving a recurrence.



$$point(n) \;=\; b^k + (-i).point(2^k - n) \qquad 2^{k-1} \le n \le 2^k \tag{59}$$

They expand (59) repeatedly and take the alternating sign powers of 2 as a "folded" representation of n, and corresponding $point(n)$ form as a "revolving" representation of $x+iy$.

$$\begin{aligned} n &= 2^{k_0} + (-1).2^{k_1} + (-1)^2.2^{k_2} + \cdots + (-1)^t.2^{k_t} && \text{folded} \quad (60) \\ point(n) &= b^{k_0} + (-i).b^{k_1} + (-i)^2.b^{k_2} + \cdots + (-i)^t.b^{k_t} && \text{revolving} \quad (61) \\ & k_0 > k_1 > \cdots > k_t && \text{high to low powers} \\ &= 0,\ 1,\ 1+i,\ i,\ 2i,\ -1+2i,\ -1+i,\ -2+i,\ -2+2i, \ldots \end{aligned}$$

Re A332383, Im A332384

In (59), the midpoint $n = 2^{k-1}$ is the end of the first sub-curve and also the end of the second sub-curve. The location is the same. In folded representation (60), this is a final 2^{k-1} or $2^k - 2^{k-1}$, or negatives of those when odd number of terms above. The corresponding revolving *point* is the same since $b^{k-1} = b^k$ $+ (-i).b^{k-1}$.

For odd n, or the odd part of n, the geometric interpretation of these final terms is to arrive at the target z either from the segment before or the segment after, according as n final term $+1$ or -1 respectively.

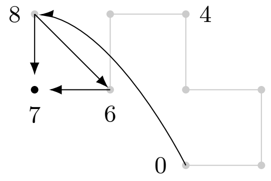


Figure 10:

n=7, $point(n)$=$-2+i$

$7 = 8 \quad - 1$

$7 = 8 - 2 + 1$

For computer calculation, if n is represented in binary then there's no need to manipulate it to form a folded representation. The sign changes for folded are at the ends of runs of 1-bits. The $+1$ folded term is one place above the bit run and the -1 folded term is at the last of the run.

	k				l			
$n =$	0	1	1	1	1	0	$\cdots$	binary
$=$	1	0	0	0	-1	0	$\cdots$	folded

$$2^{k-1} + 2^{k-2} + \cdots + 2^l \;=\; 2^k - 2^l$$

Another approach to curve unfolding is to take n in binary and for the second sub-curve calculate coordinates along a reversed curve.

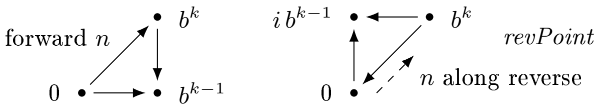


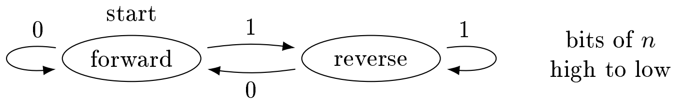
Figure 11: forward and reverse

Write n in k many bits with a the highest, either 0 or 1. Then the above expansions become

$$\text{for } n = a.2^{k-1} + n_1 \ \text{ with } n_1 < 2^{k-1}$$

$$\begin{aligned} \mathit{point}(n) &= \begin{cases} \mathit{point}(n_1) & \text{if } a=0 \\ b^{k-1} + i.\mathit{revPoint}_{k-1}(n_1) & \text{if } a=1 \end{cases} \\ \mathit{revPoint}_k(n) &= b^k - \mathit{point}(2^k{-}n) \\ &= \begin{cases} i.\mathit{point}(n_1) & \text{if } a=0 \\ i.b^{k-1} + \mathit{revPoint}_{k-1}(n_1) & \text{if } a=1 \end{cases} \end{aligned}$$

revPoint is for a particular expansion level k since in general a given reverse level is not a prefix of the next. Both forward and reverse descend to *point* or *revPoint* according as a=0 or a=1 respectively, so the bit above determines which state.



Both forward and reverse add power b at each bit a=1, but with factors of i accumulating below. A run of 1-bits has factor i from *point* and further factor i from each ib^k term in *revPoint*.

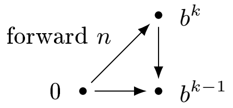
It's convenient to put a factor $-i$ through *revPoint* so its b^k is without further factor, just negations on switching forward/reverse.

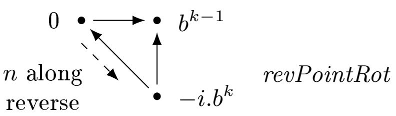
$$\mathit{revPointRot}(n) = -i.\mathit{revPoint}(n)$$

$$\mathit{point}(n) = \begin{cases} \mathit{point}(n_1) & \text{if } a=0 \\ b^{k-1} - \mathit{revPointRot}(n_1) & \text{if } a=1 \end{cases} \tag{62}$$

$$\mathit{revPointRot}(n) = \begin{cases} \mathit{point}(n_1) & \text{if } a=0 \\ b^{k-1} + \mathit{revPointRot}(n_1) & \text{if } a=1 \end{cases} \tag{63}$$

Geometrically, this $-i$ is taking the reverse curve at $-90°$. The first halves of both forward and reverse are forward curves to b^{k-1}, hence $\mathit{point}(n_1)$ at (62), (63). The second half of forward n has the reverse part there directed up to $i.b^{k-1}$ whereas the reverse curve is to end at $-i.b^{k-1}$, hence negation to rotate in (62).





The $-\mathit{revPoint}$ in (62) is at the highest 1-bit of n and at any later 01 bit pair. n can be considered to have 0s above its highest 1-bit, so it is an 01 pair too.

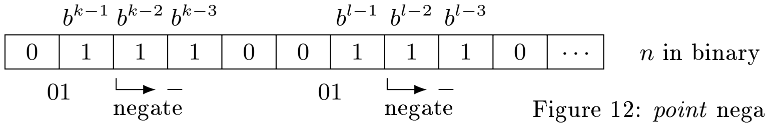


Figure 12: *point* negations

$$\begin{aligned} \mathit{point}(n) &= b^{k-1} - b^{k-2} - \cdots && \text{power for each 1-bit of } n, \\ &\quad - b^{l-1} + b^{l-2} + \cdots && \text{sign change below each 01 in } n \end{aligned} \tag{64}$$

$$
\begin{aligned}
&+ b^{m-1} - b^{m-2} - \cdots \\
&- \cdots
\end{aligned}
$$

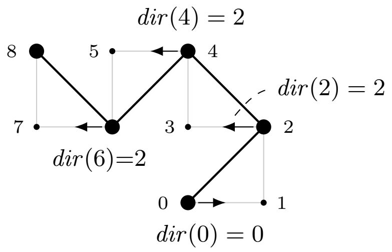
This is equivalent to the revolving representation (61) since sum of a run

$$b^{k-1} - b^{k-2} - \cdots - b^{l} \quad = b^{k-1} - \frac{b^{k-1} - b^{l}}{b-1} \quad = b^{k} + (-i).b^{l} \quad \text{revolving}$$

The expansion of each individual segment also gives a coordinate formula for a new low bit.

$$\mathit{point}(2n_1 + a) \;=\; \mathit{point}(n_1).b \;+\; i^{\,\mathit{dir}(2n_1)}.a \tag{65}$$

a is the low bit and n_1 all bits above it. $\mathit{point}(n_1).b$ is existing points expanded and if a=1 then the curve direction at $2n_1$ (so without low a bit) is the direction to go to the new point in between.



Each direction is horizontal $\mathit{dir} = 0\,\text{or}\,2$ since the curve turns ±90 at each point so an even numbered segment is horizontal, ±1.

$$
\begin{aligned}
\mathit{point}(2n{+}1) - \mathit{point}(2n) \quad &= i^{\,\mathit{dir}(2n)} \qquad \text{even segment direction} \\
&= +1, -1, -1, -1, -1, +1, -1, -1, -1, +1, +1, +1, -1, \ldots \qquad {\scriptstyle 0,1\ \mathrm{A268411}}
\end{aligned}
$$

Expanding (65) repeatedly is powers of b

$$
\begin{aligned}
n &= 2^{k-1}a_{k-1} + 2^{k-2}a_{k-2} + \cdots + 2^{1}a_1 + a_0 \\
&= \text{binary } a_{k-1}a_{k-2}\ldots a_1a_0
\end{aligned}
$$

$$
\begin{aligned}
\mathit{point}(n) = {} & b^{k-1}\; a_{k-1} && \text{high bit} \\
& + b^{k-2}\; a_{k-2}\; i^{\,\mathit{dir}(a_{k-1}0)} \\
& + b^{k-3}\; a_{k-3}\; i^{\,\mathit{dir}(a_{k-1}a_{k-2}0)} \\
& + \cdots \\
& + b^{1}\quad a_1 \quad i^{\,\mathit{dir}(a_{k-1}a_{k-2}\cdots a_2 0)} \\
& + b^{0}\quad a_0 \quad i^{\,\mathit{dir}(a_{k-1}a_{k-2}\cdots a_2 a_1 0)} && \text{low bit}
\end{aligned}
\tag{66}
$$

These powers of i are the same as the forward/reverse signs (64). $\mathit{dir}(2n_1)$ changes only when a bit pair 01 introduces a new run of 1-bits, increasing the bit transitions by 2 for dir.

All of the above coordinate formulas are expressed with i or ±1 factor determined by bits of n from high to low. For computer calculation, the formulas can be applied low to high instead by assuming lowest b^0 term has factor 1 and proceeding upwards from there. The factors on all the powers are then correct relative to each other and if the high b^{k-1} turns out not to have factor 1 then divide through by that factor (a rotation) to adjust all.

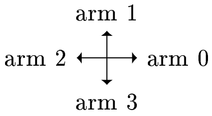
1.5 Coordinates to N

point(n) can be inverted low to high to calculate n at a given location z. Davis and Knuth do this from the revolving representation of z. It can also be done from the forward/reverse b powers (64).

Suppose $z = point(n)$ and that in (64) the total sign changes would be sign s on terms below the lowest (so how many 01 bit pairs). Then

```
unpoint($z$, $s$)            $z$ = Gaussian integer, $s = \pm 1$
  loop until $z = 0$ or $z = s.i$
    if $z \equiv 1 \bmod b^2$  then $s \leftarrow -s$
    $bit$ 0 or 1 $= z \bmod b$              bits of $n$ low to high
    if $bit$=1  then $z \leftarrow z - s$          step to multiple of $b$
    $z \leftarrow z/b$                        divide out $b$
  end loop
  if $z$=0 and $s$=1      then    $n$      in arm 0
  if $z$=$i$ and $s$=1      then  $2^k-n$  in arm 1
  if $z$=0 and $s$=$-1$     then    $n$      in arm 2
  if $z$=$-i$ and $s$=$-1$  then  $2^k-n$  in arm 3
  where $k$ is the number of bits of $n$ generated
```

Arm means a copy of the curve rotated by i^{arm}. The plain curve is arm 0.



In an unrotated copy of the curve, the two $s = \pm 1$ are directions $d = 0, 2$ (horizontal) at an even point and $d = 3, 1$ (vertical) at an odd point, respectively.

$$s = i^{d+(1 \text{ if } z \text{ odd})}$$

This is $s = i^{dir(2n)}$ which would be the next sign factor in (64). z odd is when n odd so that $dir(2n)$ is an extra bit transition.

Every point $z \neq 0$ is visited exactly twice among the 4 arms, per theorem 2, so s=+1 and s=-1 both lead to n values.

$z \equiv 1 \bmod b^2$ is when the lowest two bits of n are 01 and so a sign change for all powers below. The sign below is s so change to $-s$ for the present term and above.

For computer calculation, everything can be done in Cartesian coordinates $x+iy$ without full complex number arithmetic. $bit \equiv z \bmod b$ is simply $x+y \equiv 0, 1 \bmod 2$ and division z/b is $(x,y) \leftarrow (\frac{x+y}{2}, \frac{y-x}{2})$. The test for $z \equiv 1 \bmod b^2$ is equivalent to $x \equiv 1 \bmod 2$ and $y \equiv 0 \bmod 2$ since $z \bmod b^2$ goes in a 2×2 repeating pattern.

The loop reduces z by dividing b each time, except for the s subtraction. Considering just magnitudes, $|z|$ decreases when

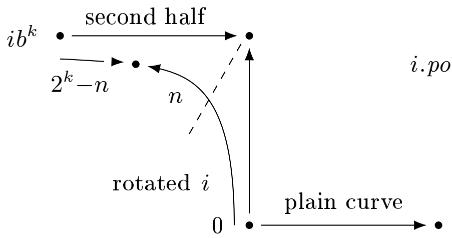
$$\begin{aligned} |z| - \left|\frac{z-s}{b}\right| &\ge |z| - \frac{|z|+1}{\sqrt{2}} = \left(1-\tfrac{1}{2}\sqrt{2}\right)|z| - \tfrac{1}{2}\sqrt{2} \\ &> 0 \quad \text{when} \ |z| > 1{+}\sqrt{2} \end{aligned}$$

So $|z|$ decreases until $|z| \le 1{+}\sqrt{2}$ and for points there it can be verified explicitly that all integer z and $s{=}\pm1$ reach one of the loop ends.

The arm 0 case ending $z{=}0, s{=}1$ is a point in the plain unrotated curve. If this arm is all that's of interest then note that it's not possible to loop until $z{=}0$ because the $z{=}s.i$ cases do not reach 0 but repeat forever generating 1-bits. Those cases occur at single-visited points of arm 0.

The arm 2 case is $z = -\mathit{point}(n)$. That negation does not change the bits generated since $z \equiv -z \bmod b^2$. The negation is on all terms so the high b^k ends with sign $s{=}{-1}$.

The arm 1 case works since the second half of arm 1 is the same direction as first half of a plain unrotated arm 0, offset by ib^k.



$$\begin{aligned} i.\mathit{point}(n) &= ib^k + \mathit{point}(2^k{-}n) \\ 2^k &\ge n > 2^{k-1} \\ 2^k{-}n &< 2^{k-1} \end{aligned}$$

This ib^k does not change any of the $z \bmod b^2$ etc calculations up to bit 2^{k-2} which is the high bit of $2^k{-}n < 2^{k-1}$. It leaves $z{=}i.b, s{=}1$ which becomes $z{=}i, s{=}1, a{=}1$ repeating forever so high 1-bits of $2^k{-}n$ for ever larger k.

Similarly the arm 3 case which is $-i.\mathit{point}(n)$ as second half of $-\mathit{point}$.

For a given n, let $\mathit{other}(n)$ be the point number which is the other visit to that location. This can be found from n without calculating the location as such. Davis and Knuth give a procedure based on transforming the revolving representation of n. Their folded n gives revolving z which after transformation is folded $\mathit{other}(n)$.

Another approach can be made using the bits of n written in binary.

high		flip		flip			low
... any	$\ne t$	100...000 011...111	t	100...000 011...111	t	1	0...0
		repeat $\ge$0 times					

Figure 13: $\mathit{other}(n)$ bit fields

```
other(n)        $n \ne 0$
    skip low 0-bits and lowest 1-bit of $n$
    $t$ = take next bit
    loop
```

```
        $n \leftarrow$ flip $0\leftrightarrow1$ run of bits plus the opposite bit above them
        if $n$ bits end before flip run complete
        then end loop, result $2^k - n$  arm $-1$ if $t$=0
                                         arm $+1$ if $t$=1
                where $k$ is the number of bits of $n$
        if take next bit $\neq t$
        then end loop, result $n$
    end loop
```

$$\begin{aligned} \mathit{other}(n) &= 0,\ \ 3,\ \ 6,\ 1,\ 12,\ 23,\ 2,\ 11,\ 24,\ 27,\ 46,\ 7,\ 4,\ 47,\ 22,\dots \\ \text{arm} &= 0,-1,-1,\ 1,-1,-1,\ 1,\ \ 0,-1,-1,-1,\ 0,\ 1,-1,\ \ 0,\dots \end{aligned} \tag{67}$$

n in binary is divided into the fields shown in figure 13. t is the bit above the lowest 1-bit. This is the turn at n as from (4).

Above t is a bit run of one or more either 0-bits or 1-bits plus one further bit which is the opposite and ends the run. Then a further t bit and another such run plus one, etc. The pattern stops when the next bit is $\neq t$. Each run plus one are bit flipped $0\leftrightarrow1$ to give $\mathit{other}(n)$.

High 0-bits are understood on n as necessary to make the fields shown. Flipping a high zero 011…11 → 100…00 gives $\mathit{other}(n)$ with more bits than n. This happens when a location within level k has its second visit in the next level k+1. (This is a join on unfolding, per section 4.2.)

If a flipped run continues infinitely into the high 0-bits on n then $\mathit{other}(n)$ is in an adjacent arm. If t=0 then it is the preceding arm ($-90°$) since t=0 is a left turn so the other arm is on the right. If t=1 then it is the next arm ($+90°$) since t=1 is a right turn so the other arm is on the left. In both these cases, $\mathit{other}(n) = 2^k - \text{flipped } n$.

The arm at (67) is therefore $\mathit{turn}(n)$ when at a single-visited point (the other visit being in that other arm). So with the single-visit predicate ahead in section 5.1,

$$\mathit{otherArm}(n) = -\mathit{turn}(n)\,.\,\mathit{Spred}_\infty(n)$$

For computer calculation in a single machine word, the runs for $\mathit{other}(n)$ can be located and bit flips applied by some bit twiddling.

This *other* bit flip procedure is found by taking bits of n and the forward/reverse powers they imply (64) then apply an *unpoint* to those powers with opposite sign.

```
other($n$)       $n \neq 0$
    $s = 1$           sign on $n$
    $h = -1$          sign on $\mathit{other}(n)$, starting opposite
    $\delta = 0$
    loop
        $a_0$ = low bit of $n$,   $a_1$ = second lowest bit of $n$
        if $a_1, a_0 = 0,1$   then $s \leftarrow -s$
        $z = b.a_1 + a_0 + \delta$
        $c_0 = 0$ or $1 \equiv z \bmod b$          $\mathit{other}(n)$ bits, low to high
        if $z \equiv 1 \bmod b^2$   then $h \leftarrow -h$
        $\delta \leftarrow (\delta + a_0.s - c_0.h)/b$
        drop lowest bit of $n$
```

```
end loop
```

s is the sign below the last bit of n. If n bits are 01 then it changes to $-s$ for the present term of n and above. h is the sign below the bits of the $\mathit{other}(n)$ being calculated. Taking only the low bits of n and $\mathit{other}(n)$ does not in general give the same location. δ is the offset from location n to $\mathit{other}(n)$. It changes when the b powers in n and $\mathit{other}(n)$ are not the same (different sign, or zero and not zero).

Following up through possible bits of n gives combinations of $s, h, \delta, \mathit{bit}$ as states of a finite state machine. The state incorporates a "current" bit since two bits are required at each step. The next higher bit is taken as input and the output is a bit of $\mathit{other}(n)$ at the "current" position. The higher bit goes into the new state. The initial state is $s{=}1, h{=}{-1}, \delta{=}0$ and bit = low of n.

The states and outputs simplify to the bit flips above. δ takes seven possible values 0, ±1, $\pm i$, $\pm(1{-}i)$.

The turn bit t above lowest 1 is unchanged by this *other* going up by states when within the same arm. This is a complicated way to see *turn* at first and second visit are the same (see section 15.4).

A run 01...11 is flipped by adding 1. A run 10...00 is flipped by subtracting 1. Differences $n - \mathit{other}(n)$ occurring within an arm are therefore ±1 at bit positions with 2 zeros below, per the fields in figure 13.

$$\text{binary}\quad \boxed{\pm1}\underset{\geq 2\text{ zeros}}{\boxed{0\ldots0}}\overbrace{\boxed{\pm1}\underset{\geq 2\text{ zeros}}{\boxed{0\ldots0}}}^{\text{repeat}} \qquad \begin{array}{c} d = n - \mathit{other}(n) \\ \text{(in same arm)} \end{array}$$

$$\mathit{Opred}(d) = \begin{cases} 1 & \text{if } d = \sum \pm 2^{p_j} \text{ where} \\ & \quad \text{each } p_{j+1} \geq p_j + 3 \text{ and } p_0 \geq 2 \\ 0 & \text{otherwise} \end{cases}$$
$$= 0,0,0,0,\ 1,0,0,0,\ 1,0,0,0,\ 0,0,0,0,\ 1,\ldots$$
$$=1 \text{ at } |d| = 4,8,16,28,32,36,56,60,64,68,72,\ldots$$

Differences can also be considered by BITXOR(n, $\mathit{other}(n)$) which is 1-bits in the flipped runs and so length $\geq$2 with one 0-bit between each, and least two 0-bits at the low end.

$$\text{binary}\quad \overbrace{\underset{\geq 2}{\boxed{11\ldots11}}\boxed{0}}^{\text{repeat 1 or more}}\underset{\geq 1}{\boxed{0\cdots0}} \qquad \mathit{OXpred}$$

$$\mathit{OXpred}(c) = \begin{cases} 1 & \text{if } c = \text{BITXOR}(n, \mathit{other}(n)) \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$$
$$=1 \text{ at } c = 12,24,28,48,56,60,96,108,112,120,124,\ldots$$
$$= \text{binary } 1100, 11000, 11100, 110000, 111000, \ldots$$

1.6 Direction Cumulative

Blecksmith and Laud [6] count the number of bit blocks in n written in binary (their b_n). A bit block is a run of consecutive 1s or consecutive 0s (but not the high 0 bits). The top of a block is at a bit different from the bit immediately above (with high 0s understood) so block count is the dragon curve $\mathit{dir}(n)$.

They consider a total dir summed from 0 to n inclusive, but here it's convenient to instead sum 0 to $n{-}1$, so n many terms,

$$\begin{aligned} \mathit{DirCumul}(n) &= \sum_{j=0}^{n-1} \mathit{dir}(j) && (68) \\ &= 0, 0, 1, 3, 4, 6, 9, 11, 12, 14, 17, \ldots && \text{A173318} \end{aligned}$$

They use chained probability matrices to calculate the sum by matrix multiplication and give an example to 2^k inclusive, which for $\mathit{DirCumul}$ here is

$$\begin{aligned} \mathit{DirCumul}(2^k{+}1) &= \begin{cases} 1 & \text{if } k{=}0 \\ k.2^{k-1} + 2 & \text{if } k \geq 1 \end{cases} \\ &= 1, 3, 6, 14, 34, 82, \ldots && k{\geq}2 \;\; 2{\times}\text{A131056} \end{aligned}$$

This is dir of all segments of dragon curve k plus 1 more segment. The extra is $\mathit{dir}(2^k) = 2$. Within level k, the *Gray* form at (44) for dir is a permutation of all n of k bits, so $\mathit{DirCumul}$ is total 1-bits among all n of k many bits. Each bit is 0 or 1 in all combinations so half of $k.2^k$,

$$\begin{aligned} \mathit{DirCumul}(2^k) &= k.2^{k-1} && (69) \\ &= 0, 1, 4, 12, 32, 80, \ldots && \text{A001787} \end{aligned}$$

Theorem 8. *For* $n = a_k \ldots a_0$ *in binary,*

$$\begin{aligned} \mathit{DirCumul}(n) &= \sum_{j=0}^{k} 2^j.a_j.\Big(\frac{j}{2} + \mathit{dir}\big(2\Big\lfloor \frac{n}{2^{j+1}} \Big\rfloor\big)\Big) && (70) \\ &= \quad 2^k \quad .a_k \quad . \quad \tfrac{k}{2} && (71) \\ &\quad + 2^{k-1}.a_{k-1}.\big(\tfrac{k-1}{2} + \mathit{dir}(a_k 0 \qquad)\big) \\ &\quad + 2^{k-2}.a_{k-2}.\big(\tfrac{k-2}{2} + \mathit{dir}(a_k a_{k-1} 0)\big) \\ &\quad \ldots \\ &\quad + 2^1 \quad .a_1 \quad .\big(\; \tfrac{1}{2} \;\; + \mathit{dir}(a_k \ldots a_2 0 \quad)\big) && (72) \\ &\quad + 2^0 \quad .a_0 \quad .\big(\; \tfrac{0}{2} \;\; + \mathit{dir}(a_k \ldots a_2 a_1 0)\big) && (73) \end{aligned}$$

Proof. A recurrence for $\mathit{dir}(n)$ removing the low bit of n can be formed in a similar way to Knuth [31] (on $g(n) = \mathit{dir}(n{-}1)$)

$$\mathit{dir}(n) = \mathit{dir}\Big(\Big\lfloor \frac{n}{2} \Big\rfloor\Big) + [0, 1, 1, 0] \qquad (74)$$

This follows since if the lowest bit of n differs from the second lowest then it is an extra bit transition for $\mathit{dir}(n)$ over what $\mathit{dir}(\lfloor n/2 \rfloor)$ has, hence +1 when

$n \equiv 1, 2 \bmod 4$.

Applying this in *DirCumul* sum (68), successive $j, j{+}1$ give two copies of $\lfloor j/2 \rfloor$, which is two *DirCumul* of $\lfloor n/2 \rfloor$, plus one extra *dir* if n odd. Term $[0, 1, 1, 0]$ in (74) is 1 on every second term so goes as $n/2$, with adjustment for where n falls mod 4. So recurrence

$$\begin{aligned} \mathit{DirCumul}(n) &= 2\,\mathit{DirCumul}\big(\Big\lfloor \frac{n}{2} \Big\rfloor\big) + \mathit{dir}\big(\Big\lfloor \frac{n}{2} \Big\rfloor\big).[0,1] + \Big\lfloor \frac{n}{2} \Big\rfloor + [0,0,0,1] \\ &= 2\,\mathit{DirCumul}\big(\Big\lfloor \frac{n}{2} \Big\rfloor\big) + \mathit{dir}\big(2\Big\lfloor \frac{n}{2} \Big\rfloor\big).[0,1] + \Big\lfloor \frac{n}{2} \Big\rfloor \qquad (75) \end{aligned}$$

Form (75) incorporates $[0, 0, 0, 1]$ into *dir* by doubling its argument, giving n with lowest bit zeroed out. (That term is only for odd n so it could instead be written $\mathit{dir}(n{-}1)$ if preferred.)

Repeatedly expanding (75) is the theorem. $2\,\mathit{DirCumul}$ in (75) puts terms of successive expansions at successive higher powers of 2. The first expansion is at 2^0, being *dir* of n with low bit zeroed out. Factor $[0, 1]$ means that term is only when n odd, which in (73) is expressed by factor bit a_0.

$\lfloor n/2 \rfloor$ in (75) adds copies of bits of n at each power of 2. The first expansion is $\lfloor n/2 \rfloor = a_k \ldots a_1$. Bit a_1 is thus $\frac{1}{2}$ in its 2^1 term at (72). Then the third lowest bit a_2 gets 2 copies, and so on successively up to k copies of the high bit a_k at (71). ☐

Each term of (70) has bit a_j as a factor so the effect is at each 1-bit to put $j/2$, plus *dir* of bits above with a_j cleared to 0. For example,

high							low		
1	1	1	0	0	1	1	1	n in binary	
7	6	5			2	1	0	$\times\frac{1}{2} = \frac{1}{2}j$	(76)
0	2	2			2	4	4	$\mathit{dir}(a_k \ldots a_{j+1}0)$	(77)
2^7							2^0		

The $\frac{1}{2}j$ terms at (76) alone are

$$\mathit{PosPowers}(n) = \tfrac{1}{2}\sum_{j=0}^{k-1} a_j.j.2^j \;= 0, 0, 1, 1, 4, 4, 5, 5, 12, 12, \ldots \qquad \text{A136013}$$

This is OEIS A136013. That sequence is conceived as repeated floors and factor 2

$$\mathit{PosPowers}(n) = 2\,\mathit{PosPowers}\Big(\Big\lfloor \frac{n}{2} \Big\rfloor\Big) + \Big\lfloor \frac{n}{2} \Big\rfloor$$

which is the expansion in *DirCumul*. Thinking of it as $\frac{1}{2}j$ at each 1-bit position j, in the manner of Marc LeBrun's comment there, is perhaps easier.

Factor $\frac{1}{2}$ in *PosPower* can also be treated by shifting the terms down 1 bit position, so 1, 2, etc at one place below each 1-bit of n, and nothing for the lowest bit of n.

The *dir* terms of (70) can also be considered by counts of bit runs. At each 1-bit, put 2× how many runs of 1-bits are above that bit, and not including that 1-bit itself. Hence the example at (77) has 0 at the high bit of the top run and 2 at the high bit of the second run.

In (71), *dir* of bits above is similar to *point* form (66), but at (66) terms are powers of b, and *dir* is just for $1, i, -1, -i$, not its magnitude as here.

DirCumul can be treated by curve unfolding with the help of a sum of end terms in level k, being n many values from $\mathit{dir}(2^k{-}1)$ downwards,

$$\mathit{revDirCumul}_k(n) = \sum_{j=0}^{n-1} \mathit{dir}(2^k{-}1 - j)$$

Then, like *point*, for an n in the range $2^{k-1} \le n < 2^k$, the unfold means *DirCumul* first half 2^{k-1} per (69) and second half remaining *revDirCumul*.

$$\begin{gathered}\mathit{DirCumul}(n) = (k{-}1).2^{k-1} + r + \mathit{revDirCumul}_{k-1}(r) \qquad (78)\\ \text{for } n = 2^{k-1} + r \text{ with } 0 \le r < 2^{k-1}\end{gathered}$$

The second half is turned +90° from the curve starting East, so each segment there is $\mathit{dir} + 1$. This is $+r$ at (78).

revDirCumul is for a particular k since in general the curve end of one level is not a prefix of the next higher level. *revDirCumul* on unfolding is

$$\mathit{revDirCumul}_k(n) = \begin{cases} n + \mathit{DirCumul}(n) & n{<}2^{k-1} \\ (k{-}1).2^{k-2} + 2^{k-1} + \mathit{revDirCumul}_{k-1}(r) & n{\ge}2^{k-1} \end{cases}$$
$$\text{where } n = 2^{k-1} + r \text{ for the } n{\ge}2^{k-1} \text{ case}$$

As back in figure 11, the first half of the curve is turned +90° so each $\mathit{dir} + 1$ there which is n for $n < 2^{k-1}$, or is 2^{k-1} for $n \ge 2^{k-1}$.

The $(k{-}1).2^{k-2}$ terms here are the j of sum (71). The $r, n, 2^{k-1}$ additions give the *dir* terms.

The folded binary representation of Davis and Knuth at (60) locates bit transitions. The representation with an even number of terms, which is t odd and lowest term sign -1, gives *dir* as

$$\mathit{dir}(n) = t + [1, 0]_n \qquad \text{folded } t \text{ odd}$$

The representation with an odd number of k powers, so t even, effectively arrives at a point from the far end of the curve (figure 10) so gives direction of the segment preceding the point

$$\mathit{dir}(n{-}1) = t + [1, 0]_n \qquad \text{folded } t \text{ even}$$

For *DirCumul*, working through the powers and directions of folded terms k_j gives, for either folded representation of n,

$$\mathit{DirCumul}(n) = \sum_{j=0}^{t} (-1)^j \, (\tfrac{1}{2}k_j + j) \, 2^{k_j} \qquad \text{folded } n$$

Blecksmith and Laud [6] also consider a ratio of *dir* over bit length (their $\delta_j = b_j/k_j$). Bit length is the maximum *dir* (47), so *DirRatio* ranges up to 1.

$$\mathit{DirRatio}(n) = \frac{\mathit{dir}(n)}{\mathit{bitlength}(n)} \qquad \text{for } n \ge 1$$

$$= 1,\ \ 1, \tfrac{1}{2},\ \ \tfrac{2}{3}, 1, \tfrac{2}{3}, \tfrac{1}{3},\ \ \tfrac{1}{2}, \tfrac{3}{4}, 1, \tfrac{3}{4}, \tfrac{1}{2}, \ldots$$

$$\mathit{bitlength}(n) = \lceil \log_2 (n{+}1) \rceil \qquad \text{length of } n \text{ written in binary}$$

$$= 0,\ 1,\ 2, 2,\ 3, 3, 3, 3,\ 4, 4, \ldots \qquad \text{A029837}$$

They consider then a cumulative total of this ratio

$$\mathit{DirRatioCumul}(n) = \sum_{j=1}^{n} \mathit{DirRatio}(j) \qquad \text{for } n \geq 1$$

$$= 1,\ \ 2, \tfrac{5}{2},\ \ \tfrac{19}{6}, \tfrac{25}{6}, \tfrac{29}{6}, \tfrac{31}{6},\ \ \tfrac{17}{3}, \tfrac{77}{12}, \tfrac{89}{12}, \ldots$$

and conjecture the mean ratio converges

$$\frac{\mathit{DirRatioCumul}(n)}{n} = 1,\ 1,\ \tfrac{5}{6},\ \tfrac{19}{24},\ \tfrac{5}{6},\ \tfrac{29}{36},\ \tfrac{31}{42},\ \tfrac{17}{24},\ \tfrac{77}{108}, \ldots$$

$$\to \tfrac{1}{2} \qquad \text{as } n \to \infty$$

Theorem 9.

$$\lim_{n\to\infty} \frac{\mathit{DirRatioCumul}(n)}{n} \to \frac{1}{2}$$

Proof. Let $k = \mathit{bitlength}(n)$. $\mathit{DirRatioCumul}(n)$ first sums over complete blocks of j of each bit length $p < k$. A block of p is 2^{p-1} many terms is

$$\text{all } j = 1xxxx \text{ of length } p, \text{ ie. all } \mathit{bitlength}(j) = p$$

$$\begin{aligned} \mathit{block}_p &= \sum_{j=2^{p-1}}^{2^p-1} \mathit{DirRatio}(j) \\ &= \frac{\mathit{DirCumul}(2^p) - \mathit{DirCumul}(2^{p-1})}{p} \\ &= \big(1 + \tfrac{1}{p}\big).\tfrac{1}{2}.2^{p-1} \qquad (79) \\ &= 1,\ \tfrac{3}{2},\ \tfrac{8}{3},\ 5,\ \tfrac{48}{5},\ \tfrac{56}{3},\ \tfrac{256}{7},\ 72,\ \tfrac{1280}{9},\ \tfrac{1408}{5}, \ldots \qquad \tfrac{1}{p}\times\text{A001792} \end{aligned}$$

(79) follows from *DirCumul* form (69). Or directly, high 1 bit of j is always a transition for *dir*, then $p{-}1$ many bits below it, each of which can be a transition or not, so $1 + \frac{1}{2}(p{-}1)$ each, and divide p for ratios.

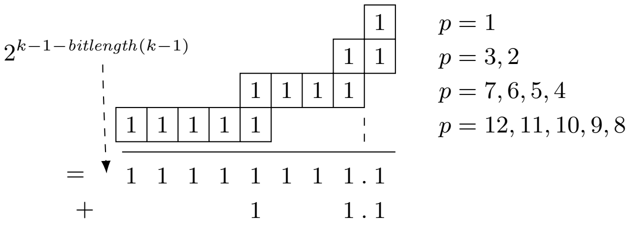
$\mathit{DirRatioCumul}(2^{k-1}{-}1)$ is all blocks up to $k{-}1$ inclusive,

$$\begin{aligned} \mathit{DirRatioCumul}(2^{k-1}{-}1) &= \sum_{p=1}^{k-1} \mathit{block}_p \ = 1,\ \tfrac{5}{2},\ \tfrac{31}{6},\ \tfrac{61}{6},\ \tfrac{593}{30},\ \tfrac{1153}{30}, \ldots \\ &= \tfrac{1}{2}(2^{k-1}{-}1)\ + E \qquad (80) \\ \text{where } E &= \sum_{p=1}^{k-1} \frac{2^{p-2}}{p} \ = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{5}{3},\ \tfrac{8}{3},\ \tfrac{64}{15}, \ldots \end{aligned}$$

(80) is $\frac{1}{2}$ for each of its $2^{k-1}{-}1$ many terms, plus E. An upper bound on E can be formed by decreasing each p denominator to its high bit alone,

$$\frac{2^{p-2}}{p} \le \frac{2^{p-2}}{2^{bitlength(p)-1}} = 2^{p-1-bitlength(p)}$$

This is successive power-of-2 bits, with an overlap of one place where *bit length* increases and so drops the bit position. For example, up to $k{-}1 = 12$,



The row of 1s sum to < the bit position immediately above the high $p = k{-}1$ term, and the duplicates are at most the same too. So bound

$$DirRatioCumul(2^{k-1}{-}1) \;\le\; \tfrac{1}{2}(2^{k-1}{-}1) \;+ 2^{k-bitlength(k)}$$

This, and $E \ge 0$ for mean always $\ge \frac{1}{2}$, suffices for the theorem just at $2^{k-1}{-}1$.

$$\begin{aligned} \frac{DirRatioCumul(2^{k-1}{-}1)}{2^{k-1}{-}1} &\le \tfrac{1}{2} + \frac{2^{k-bitlength(k)}}{2^{k-1}-1} \\ &< \tfrac{1}{2} + \frac{4}{2^{bitlength(k)}} \end{aligned}$$

n has various *DirRatioCumul* terms with j of bit length k too. They can be treated in sub-blocks. For $n = 1a_2a_3\ldots a_k$ binary bits,

$$\begin{aligned} top(n) &= \sum_{j=2^{k-1}}^{n} dir(j) \qquad \text{all } bitlength(j) = k && (81) \\ &= DirCumul(n{+}1) - DirCumul(2^{k-1}) \\ &= dir(n) + \sum_{q=2}^{k} a_q.2^{k-q}.\Big(\tfrac{1}{2}(k{-}q) + dir(1a_2...a_{q-1}0)\Big) && (82) \\ &= \tfrac{1}{2}k\,(n{-}2^{k-1}{+}1) \;+ F && (83) \end{aligned}$$

$$\text{where } F = dir(n){-}\tfrac{1}{2}k + \sum_{q=2}^{k} a_q.2^{k-q}.\Big(dir(1a_2...a_{q-1}0) - \tfrac{1}{2}q\Big)$$

(82) is where a 1-bit $a_q = 1$ of n means sum a sub-block of $dir(j)$ with j of the form

$$j = 1a_2\ldots a_{q-1}0xxxx \qquad k \text{ bits total}$$

There are $k{-}q$ many bits "$xxxx$" so 2^{k-q} many j. Each of them has the same bit transitions down to the q position, which is *dir* to there. Then $xxxx$ is all combinations of transition or not so $\frac{1}{2}(k{-}q)$.

The powers 2^{k-q} are all bits of n except the highest. (83) takes them and $\frac{1}{2}k$ out from the sum. Factor $n{-}2^{k-1}{+}1$ is the total number of terms in sum (81). For ratios, all are the same bit length k.

$$\frac{top(n)}{k} = \tfrac{1}{2}(n - 2^{k-1} + 1) \; + \frac{F}{k}$$

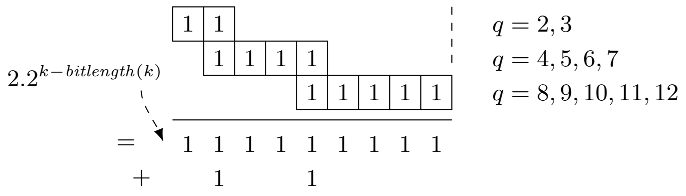
An upper bound on F/k can be formed by firstly $dir \le bitlength$ per (47) so at most $q - \frac{1}{2}q$ each term. Then take all terms irrespective of the bits of n, plus the separate $dir(n)$ term, so

$$F/k \;\le\; \tfrac{1}{2} + \sum_{q=2}^{k} \tfrac{1}{2}\,\frac{q}{k}\,2^{k-q}$$

Increase $1/k$ by decreasing k to its high bit alone so $1/2^{bitlength(k)-1}$. Increase q to the bit above its length so $2^{bitlength(q)}$.

$$F/k \;\le\; \tfrac{1}{2} + \sum_{q=2}^{k} 2^{k-bitlength(k)\;-q\;+bitlength(q)}$$

This is bits at $2^{k-bitlength(k)}$ and successively down with q, but an overlap when $bitlength(q)$ increases. The first term q=2 has $q - bitlength(q) = 0$.



Like E, the row of 1s sum to at most a 1 bit above them, and the duplicates at most the same, so

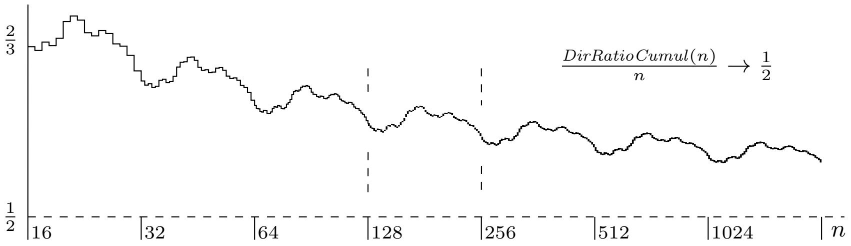
$$F/k \;\le\; \tfrac{1}{2} + 4.2^{k-bitlength(k)}$$

Total upper bounds on E and F/k give

$$\begin{aligned}\frac{DirRatioCumul(n)}{n} &\le \frac{\tfrac{1}{2}n + \tfrac{1}{2} + 5.2^{k-bitlength(k)}}{n} \\ &\le \tfrac{1}{2} + \frac{1}{2n} + \frac{10}{2^{bitlength(k)}} \quad \to \tfrac{1}{2}\end{aligned}$$

A lower bound for F/k is formed in the same way as the upper bound using $dir \ge 0$ so have the same terms in k and q but negative. It and (80) for the initial blocks total always each term $\ge \frac{1}{2}$ is then the same limit $\frac{1}{2}$ from below. ☐

$2^{bitlength(k)}$ etc are used to bring terms to a common form where a bound on their sum is easily made. With $2^{bitlength(k)} > k = \log_2\lceil n{+}1\rceil$, the bound decreases slowly, as $1/\log n$. Actual descent of the mean can be illustrated in a plot. Blocks of $n = 2^k$ to $2.2^k{-}1$ are scaled to the same width (and linear within them) in order to show their similarity.



Blecksmith and Laud conjecture a similar limit $(b-1)/b$ for mean ratios counting digit blocks in any base b. That should follow from an argument similar to above. A same digit is 1 non-transition, so new blocks $b-1$ out of b.

1.7 Segments in Direction

Theorem 10. *The number of segments in direction* $d = 0,1,2,3 \bmod 4$ *of dragon curve level* k *are*

$$\begin{aligned}
S(k,d) &= 1,0,0,0 \quad \textit{for } d \equiv 0 \,\text{to}\, 3 && \textit{for } k=0 \\
&= \tfrac{1}{4}\Big(2^k \,+\, b^k(-i)^d + \overline{b^k(-i)^d}\Big) && \textit{for } k \ge 1 \qquad (84) \\
&= \tfrac{1}{4}\,\Big(|b^k + i^d|^2 \,-1\Big) \\
&= \tfrac{1}{4}\Big(2^k \,+\, s(k{-}2d)\,.\,2^{\lfloor k/2 \rfloor}\Big) && (85) \\
s(m) &= 2,2,0,-2,-2,-2,0,2 && 2{\times}\text{A046980} \\
&\qquad \textit{according to } m \equiv 0 \,\text{to}\, 7 \mod 8
\end{aligned}$$

$$\begin{aligned}
S(k,0) &= 1,1,1,1,2,\;\; 6,16,36,72,136,256,\ldots && \text{A038503} \\
S(k,1) &= 0,1,2,3,4,\;\; 6,12,28,64,136,272,\ldots && \text{A038504} \\
S(k,2) &= 0,0,1,3,6,10,16,28,56,120,256,\ldots && \text{A038505} \\
S(k,3) &= 0,0,0,1,4,10,20,36,64,120,240,\ldots && \text{A000749}
\end{aligned}$$

Proof. The segments in direction $d{=}0$ are those n which have $dir(n) \equiv 0 \bmod 4$. So from (44), a count 0, 4, 8, 12, etc many 1-bits among the k bits of $Gray(n)$. $Gray(n)$ is a permutation of the integers 0 to $2^k{-}1$ so this is also those counts on just n. The possible positions for the 1-bits are a binomial coefficient.

$$\begin{aligned}
S(k,0) &= \tbinom{k}{0} + \tbinom{k}{4} + \tbinom{k}{8} + \tbinom{k}{12} + \cdots \\
S(k,1) &= \tbinom{k}{1} + \tbinom{k}{5} + \tbinom{k}{9} + \tbinom{k}{13} + \cdots \\
S(k,2) &= \tbinom{k}{2} + \tbinom{k}{6} + \tbinom{k}{10} + \tbinom{k}{14} + \cdots \\
S(k,3) &= \tbinom{k}{3} + \tbinom{k}{7} + \tbinom{k}{11} + \tbinom{k}{15} + \cdots \\
S(k,d) &= \sum_{j=d,d+4,\ldots} \tbinom{k}{j} \qquad (86)
\end{aligned}$$

These sums are from Cournot[11] (and Ramus[44]). The $\sin\frac{\pi}{4}$ and $\cos\frac{\pi}{4}$ they give are written here in (84) using b. Form (85) is a half power to emphasise the result is always an integer. Factor $s(m)$ is

$$s(m) = \frac{b^m + \bar{b}^m}{2^{\lfloor m/2 \rfloor}} \qquad \square$$

$\frac{1}{4}2^k$ in (85) shows $S(k,d)$ has $\frac{1}{4}$ of the segments in each direction, plus or minus the half power term. When k is even there are two d which are exactly $\frac{1}{4}2^k$ due to $s(2) = s(6) = 0$. In the sample values above, the last column has two counts of $256 = \frac{1}{4}2^{10}$.

The maximum difference between these counts, as binomial sums, was a Mathematics Magazine problem [41]. In the s form (85), and for any k, there is always $s(m)$ factors 2 and -2 so that

$$\left(\max_{d=0}^{3} s(k{-}2d)\right) - \left(\min_{d=0}^{3} s(k{-}2d)\right) = 4 \qquad \text{any } k$$

and hence

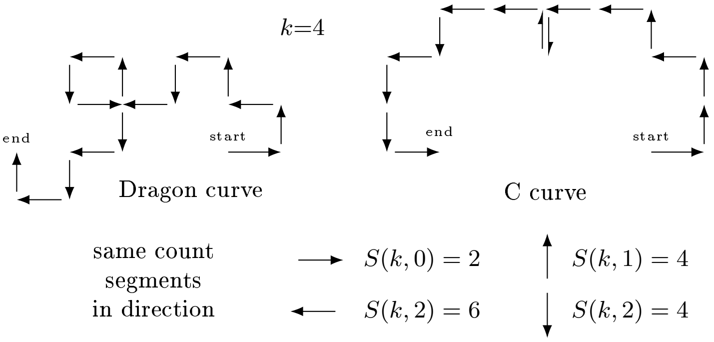
$$\left(\max_{d=0}^{3} S(k,d)\right) - \left(\min_{d=0}^{3} S(k,d)\right) = 2^{\lfloor k/2 \rfloor}$$

The total segments is simply 2^k. Since the curve always turns ±90°, the number of verticals and horizontals are the same for $k \geq 1$.

$$\begin{aligned}
\text{total} \quad & S(k,0) + S(k,1) + S(k,2) + S(k,3) = 2^k \\
\text{horizontals} \quad & S(k,0) + S(k,2) = \tfrac{1}{2}2^k \qquad k \geq 1 \\
\text{verticals} \quad & S(k,1) + S(k,3) = \tfrac{1}{2}2^k \qquad k \geq 1
\end{aligned}$$

These counts are for segment directions along the curve. Various diagrams here such as figure 2 have arrows in direction of expansion, so each segment expands on the right. Counts of those directions are obtained simply by swapping the two verticals $S(k,1)$ and $S(k,3)$. The verticals are the odd numbered segments since the curve always turns ±90°. They expand on the left so swapping gives counts for expanding on the right. This arises in section 8 for the curve centroid.

The segment counts are the same as in the Lévy C curve. Its *dir* direction is *CountOneBits*(n) without the *Gray* of (44), so the same binomial sums (86). The endpoint is consequently the same, but the sequence of segment directions and the shape are not the same, for $k \geq 2$.



Theorem 11. *Among the first n segments of the dragon curve, the number in direction $d \bmod 4$ is*

$$SN(n,d) = \tfrac{1}{4}\Big(n + 2\operatorname{Re}(-i)^d \mathit{point}(n) \ + \big((-1)^d \textit{ if } n \textit{ odd}\big)\Big) \tag{87}$$

$$\begin{aligned}
SN(n,0) &= 0,1,1,1,1,1,1,1,1,1,1,2,2,2,2,\ldots\\
SN(n,1) &= 0,0,1,1,2,2,2,2,3,3,3,3,3,3,3,\ldots\\
SN(n,2) &= 0,0,0,1,1,2,2,3,3,4,4,4,4,5,5,\ldots\\
SN(n,3) &= 0,0,0,0,0,0,1,1,1,1,2,2,3,3,4,\ldots
\end{aligned}$$

Proof. Segments alternate horizontal and vertical so total horizontals are $\lceil n/2 \rceil$ which is SN directions 0 plus 2. The difference of directions 0 and 2 is the net horizontal position $\operatorname{Re} \mathit{point}$,

$$SN(n,0) + SN(n,2) = \lceil n/2 \rceil \tag{88}$$

$$SN(n,0) - SN(n,2) = \operatorname{Re} \mathit{point}(n) \tag{89}$$

(88)+(89) and (88)−(89) give

$$\begin{aligned}
SN(n,0) \ &= \tfrac{1}{2}\big(\lceil n/2 \rceil + \operatorname{Re} \mathit{point}(n)\big)\\
SN(n,2) \ &= \tfrac{1}{2}\big(\lceil n/2 \rceil - \operatorname{Re} \mathit{point}(n)\big)
\end{aligned}$$

Similarly for the verticals

$$\begin{aligned}
SN(n,1) + SN(n,3) &= \lfloor n/2 \rfloor\\
SN(n,1) - SN(n,3) &= \operatorname{Im} \mathit{point}(n)
\end{aligned}$$

$$\begin{aligned}
SN(n,1) \ &= \tfrac{1}{2}\big(\lfloor n/2 \rfloor + \operatorname{Im} \mathit{point}(n)\big)\\
SN(n,3) \ &= \tfrac{1}{2}\big(\lfloor n/2 \rfloor - \operatorname{Im} \mathit{point}(n)\big)
\end{aligned}$$

The $\pm \operatorname{Re}, \operatorname{Im}$ parts here are selected in (87) by $\operatorname{Re}(-i)^d \mathit{point}$, and the floor or ceil $n/2$ by the $(-1)^d$ offset part. □

Second Proof of Theorem 11. These counts are a cumulative sum of a direction predicate

$$\mathit{DirPred}(n,d) = \begin{cases} 0 & \text{if } \mathit{dir}(n) \equiv d \bmod 4 \\ 1 & \text{if not} \end{cases}$$

$$\begin{aligned}
\mathit{DirPred}(n,0) &= 1,0,0,0,0,0,0,0,0,1,0,0,\ldots\\
\mathit{DirPred}(n,1) &= 0,1,0,1,0,0,0,1,0,0,0,0,0,\ldots\\
\mathit{DirPred}(n,2) &= 0,0,1,0,1,0,1,0,1,0,0,0,1,\ldots\\
\mathit{DirPred}(n,3) &= 0,0,0,0,0,1,0,0,0,1,0,1,0,\ldots
\end{aligned}$$

A segment step $z = 1, i, -1, -i$ can be expressed as such a predicate by

$$\tfrac{1}{4}\Big(2 + 2\operatorname{Re}(-i)^d z \ - 2\left|\operatorname{Im}(-i)^d z\right|\Big) = \begin{cases} 1 & \text{if } z \text{ in direction } d \\ 0 & \text{if not} \end{cases}$$

Then with steps $\mathit{dpoint}(n) = \mathit{point}(n{+}1) - \mathit{point}(n) \ = i^{\mathit{dir}(n)}$

$$\begin{aligned} SN(n,d) &= \sum_{j=0}^{n-1} \mathit{DirPred}(n,d) \\ &= \tfrac{1}{4}\Bigg(\Big(\sum_{j=0}^{n-1} 2\Big) + \Big(2\,\mathrm{Re}\,(-i)^d \sum_{j=0}^{n-1} \mathit{dpoint}(j)\Big) \qquad (90) \\ &\qquad\qquad - \Big(2\sum_{j=0}^{n-1} \big|\mathrm{Im}(-i)^d \mathit{dpoint}(j)\,\big|\,\Big)\Bigg) \qquad (91) \end{aligned}$$

The Re part (90) cumulative *dpoint* is *point*(n). The |Im| part (91) is sum of terms 0 or 1 according as *dpoint* is horizontal or vertical, after rotation by d. The curve always turns left or right so segments are alternately horizontal and vertical so half each giving

$$2\sum_{j=0}^{n-1} \big|\mathrm{Im}\,(-i)^d \mathit{dpoint}(j)\big| \;= n - \big((-1)^d \text{ if } n \text{ odd}\big)$$

which subtracted from $\sum 2 = 2n$ is per (87). ☐

A complete level k is 2^k segments $SN(2^k,d) = S(k,d)$. Its end $\mathit{point}(2^k) = b^k$ is the conjugate b^k parts of (84).

2 Cubic Recurrence

Various formulas in the sections below have a cubic recurrence of the form

$$a_k = a_{k-1} + 2a_{k-3}$$

which has characteristic polynomial

$$x^3 - x^2 - 2 \qquad (92)$$

so a_k can be written as powers of the roots r, r_2, r_3 of that polynomial

$$a_k \;= X.r^k + Y.r_2^k + Z.r_3^k \qquad (93)$$

The roots r, r_2, r_3 are found by the usual cubic formula (Ferro/Tartaglia/Cardano). The real root r in decimal is per Daykin and Tucker[13].

$$C = \sqrt[3]{28 + \sqrt{29 \times 27}} \quad = 3.825455\ldots$$

$$r = \frac{1}{3}\left(1 + C + \frac{1}{C}\right) \quad = 1.695620\ldots \qquad \text{A289265}$$

continued fraction 1,1,2,3,1,1,59, 1,1,1,2,3,1,1,2,1,41, . . .

$$r_2 = \frac{1}{6}\left(2 - C - \frac{1}{C}\right) + \frac{\sqrt{3}}{6}\left(C - \frac{1}{C}\right) i$$

$$r_3 = \overline{r_2}$$

$$r_2, r_3 = -0.347810\ldots \pm 1.028852\ldots i$$

r_2 and r_3 are complex conjugates. The second highest polynomial coefficient is Vieta's formula for roots one at a time $r + r_2 + r_3 = -(-1)$ so real parts

$$\operatorname{Re} r_2 = \operatorname{Re} r_3 \ = \tfrac{1}{2}(1 - r) \tag{94}$$

The low polynomial coefficient is product roots for Vieta's formula all roots $r.r_2.r_3 = -(-2)$, so magnitudes

$$|r_2| = |r_3| = \sqrt{\frac{2}{r}} \ = \sqrt{r^2 - r} \ = 1.086052\ldots \tag{95}$$

Chang and Zhang [9] show the Hausdorff dimension of the boundary of the dragon fractal follows from the root r

$$dimH = \frac{\log r}{\log \sqrt{2}} \ = 2 \log_2 r \ = 1.523627\ldots \qquad \text{A272031}$$

In the power form (93), the imaginary parts of r_2 and r_3 cancel out. The r^k term dominates eventually since r_2 and r_3 are smaller in magnitude. Magnitudes are seen in the decimals, or $C > \sqrt[3]{27} = 3$ so $r > 1$ and so at (95) $\sqrt{r^2 - r} \ < r$.

$|r_2| = |r_3| > 1$ is again seen in the decimals, or their imaginary part alone is greater than 1 by reducing

$$\begin{gathered} C > \sqrt[3]{28 - \tfrac{145}{64} \ + \sqrt{27{\times}27}} \ = \tfrac{15}{4} \\ (\operatorname{Im} r_2)^2 = (\operatorname{Im} r_3)^2 \ > \tfrac{3}{36}(\tfrac{15}{4} - \tfrac{4}{15})^2 \ = 1 + \tfrac{481}{43200} \ > 1 \end{gathered} \tag{96}$$

This means the contribution of r_2, r_3 to the powers (93) is unbounded and it's not enough to take just $X.r^k$ and round to an integer or similar (the way for example Fibonacci numbers from a single power of ϕ rounded).

A ratio of recurrence values approaches power of r which is their difference in index. For example ratio $k{+}1$ over k approaches r

$$\frac{a_{k+1}}{a_k} = \frac{X.r^{k+1} + Y.r_2^{k+1} + Z.r_3^{k+1}}{X.r^k + Y.r_2^k + Z.r_3^k} \to \frac{r^{k+1}}{r^k} = r \qquad \text{as } k \to \infty$$

Various formulas such as area A_k (section 4) include a 2^k term. That power dominates r eventually since $r < 2$.

$$\begin{gathered} 3 = \sqrt[3]{0 + \sqrt{27 \times 27}} \ < C < \sqrt[3]{35 + \sqrt{29 \times 29}} = 4 \\ r \ < \tfrac{1}{3}(1 + 4 + \tfrac{1}{3}) = \tfrac{16}{9} \ < 2 \end{gathered} \tag{97}$$

Exactly when some 2^k growth exceeds an r^k depends on their respective factors. For example theorem 33 is when double-visited points exceed single-visited points.

Another simple relation is $r > \sqrt{2}$ using bounds (97) again. This shows r^k exceeds a half power $\sqrt{2}^k$ or $2^{\lfloor k/2 \rfloor}$.

$$r^2 > \left(\tfrac{1}{3}(1 + 3 + \tfrac{1}{4})\right)^2 \ = 2 + \tfrac{1}{144} \tag{98}$$

Coefficients X, Y, Z for the powers depend on the initial values of the recurrence. They can be found by some linear algebra to make initial powers sum to the initial values. A simple example is starting $0, 0, 1$ which is dJA_k

(section 4.3). Other initial values can be constructed from sums of these.

$$\begin{pmatrix} 1 & 1 & 1 \\ r & r_2 & r_3 \\ r^2 & r_2^2 & r_3^2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} X &= \frac{1}{(r-r_2)(r-r_3)} &&= \frac{3}{C^2+1+\dfrac{1}{C^2}} &&= 0.191053\ldots \\ Y &= \frac{1}{(r-r_2)(r_3-r_2)} &&= \frac{-6}{\left(C-\dfrac{1}{C}\right)^2 + \sqrt{3}\left(C^2-\dfrac{1}{C^2}\right)i} \\ &= -0.095526\ldots + 0.189727\ldots i \\ Z &= \overline{Y} \qquad \text{conjugate} \\ dJA_k &= X.r^k + Y.r_2^k + Z.r_3^k \end{aligned}$$

X is real since r_2 and r_3 are conjugates. The real part of Y is $-\frac{1}{2}X$ since $r_3 - r_2$ is purely imaginary

$$\frac{Y}{X} = \frac{r-r_3}{r_3-r_2} = -\frac{1}{2} - \frac{3r+1}{4\,\mathrm{Im}\,r_2}\,i \qquad \text{so } \mathrm{Re}\,Y = -\tfrac{1}{2}X$$

X can be simplified a little by the polynomial second highest coefficient which is Vieta's formula roots two at a time $r.r_2 + r.r_3 + r_2.r_3 = 0$, and with $r_2.r_3 = 2/r$ from (95)

$$X = \frac{1}{r^2+2r_2r_3} = \frac{1}{r^2+4/r} = \frac{1}{3r^2-2r} \tag{99}$$

$\mathbb{Q}[r]$, being rationals and r, is a number field in the usual way, closed under addition, subtraction, multiplication and division. Powers can be rolled up or down with identity $r^3 = r^2 + 2$ to reach a desired basis, such as $1, r, r^2$ in the denominator at (99).

Sometimes a form with one or two powers can be found. A computer search helps that, then choose an attractive form. For example (94),(95) and then rolling the resulting powers of r gives an $(\mathrm{Im}\,r_2)^2$ form which is another way to see $|\mathrm{Im}\,r_2| > 1$ as from (96).

$$(\mathrm{Im}\,r_2)^2 = (\mathrm{Im}\,r_3)^2 = \frac{2}{r} - \left(\tfrac{1}{2}(1-r)\right)^2 = 1 + \frac{4}{r^8}$$

Division in $\mathbb{Q}[r]$ can be done by a little linear algebra in the basis to ask what quotient multiplied by divisor gives the dividend. For example X from (99) would seek α, β, γ satisfying $(\alpha+\beta r+\gamma r^2)\,(3r^2-2r) = 1$, which has solution $X = -\frac{3}{58} + \frac{5}{29}r - \frac{1}{58}r^2$. Sometimes coefficients of the reciprocal are simpler, as is the case here for X.

Generating functions for the recurrence have a denominator which is the characteristic polynomial with powers reversed in the usual way.

$$\frac{1}{1-x-2x^3}$$

It's helpful to write the generating functions in partial fractions so such a cubic part has numerator terms $1, x, x^2$. The coefficients in the numerator can then show which formulas differ only by a factor etc. Sometimes there is a geometric interpretation, or it could be merely numerical.

A generating function for a_{k-1} is $x.ga(x)$, or for a_{k+1} is $\frac{1}{x}.(ga(x)-a_0)$ in the usual way. These forms too can be written in partial fractions with numerator terms $1, x, x^2$. Again the coefficients can identify relations between different quantities with an index offset. Usually a little computer search is needed as several different sets of coefficients and possible offsets arise.

Any cubic linear recurrence values can be expressed as a linear combination of at most 3 others (of the same kind of recurrence) by some linear algebra to find factors which make their initial values terms sum to the desired target, or the same on generating function numerator coefficients. This is similar to the number field division above and is helpful to express one sequence in terms of another when preparing to divide. This occurs in several limits and some of the centroid calculations.

3 Boundary

3.1 Left Boundary

Theorem 12. *The dragon curve left boundary length is*

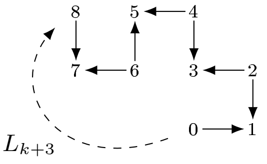
$$L_k = \begin{cases} 1, 2, 4, 8 & \textit{for } k = 0 \textit{ to } 3 \\ L_{k-1} + 2L_{k-3} & \textit{for } k \geq 4 \end{cases} \qquad \textit{left boundary} \tag{100}$$

$$= 1, 2, 4, 8, 12, 20, 36, 60, 100, 172, 292, 492, 836, \ldots \qquad {\scriptstyle k\geq 1 \quad 2\times\text{A203175}}$$

$$\textit{Generating function} \quad gL(x) = -1 + 2\frac{1 + x^2}{1 - x - 2x^3} \tag{101}$$

Proof. L_0 through L_3 are seen from the initial shapes in figure 1.

A level k+3 curve comprises eight level k sub-curves. They expand to make k+4 as follows



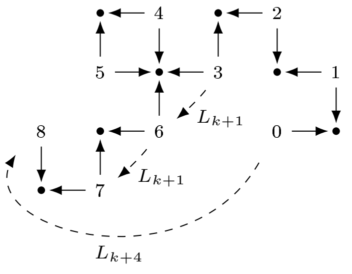


Figure 14: left boundary recurrence

Points $0, 3, 6, 7$ are endpoints of level k curves meeting at 90° angles. Endpoints are not re-visited and the perfect meshing of curves at right angles means no other curve section touches them, so those points are on the left boundary.

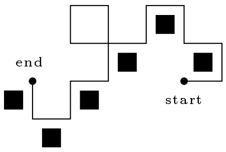
The L_{k+4} boundary sections 3 to 6 and 6 to 7 are L_{k+1}, ie. curve k+1. The remaining parts 0 to 3 and 7 to 8 are the shape and sub-curve directions of L_{k+3}. So a direct geometric form of the recurrence

$$L_{k+4} = L_{k+3} + 2L_{k+1} \qquad \text{for } k \geq 0 \tag{102}$$

□

Point 6 in figure 14 is the midpoint between curve start and end. That midpoint is on the left boundary for $k \geq 2$, since in k=2 it is single-visited and the absent sub-curves remain absent on subsequent expansion.

Daykin and Tucker[13] consider the number of unit squares on the boundary, as in the following example. Each boundary square may have 1, 2 or 3 curve segments on its sides and is counted just once each.



k=4
left boundary squares
$LQ_4 = 6$

Theorem 13 (Daykin and Tucker). *The number of unit squares on the left boundary of dragon curve k is*

$$LQ_k = \begin{cases} 1, 1, 2 & \textit{for } k = 0 \textit{ to } 3 \\ LQ_{k-1} + 2LQ_{k-3} & \textit{for } k \geq 4 \end{cases} \qquad \textit{left squares} \qquad (103)$$

$$= 1, 1, 2, 4, 6, 10, 18, 30, 50, 86, 146, 246, 418, \ldots \qquad \text{A203175}$$

$$\textit{Generating function} \quad gLQ(x) = \frac{1 + x^2}{1 - x - 2x^3}$$

Proof. In figure 14, the sub-curves which meet at points 3, 6, 7 do so at 90° angles so they have no boundary squares in common. The same recurrence (102) for the left boundary segments applies

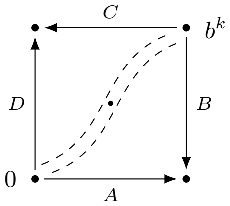
$$LQ_{k+4} = LQ_{k+3} + 2\, LQ_{k+1} \qquad \text{for } k \geq 0$$

except it can begin from $LQ_{0,1,2} = 1, 1, 2$. □

Theorem 14 (Daykin and Tucker). *The left boundary squares are symmetric in* 180° *rotation about a point* $\frac{1}{2}b^{\max(1,k)}$ *where* $b = 1{+}i$. *For* $k \geq 1$ *this is half way between curve start and end.*

Proof. For k=0 there is a single left side boundary square so it is symmetric in 180° rotation about its centre $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}b$.

For $k \geq 1$, arrange four dragon curves $k-1$ in a square. This begins as a unit square which is a subset of the grid in theorem 1 so remains non-overlapping.



The sides A and B are the expansion of a curve level k going across the diagonal from 0 to b^k. Likewise sides C and D by symmetry in 180° rotation.

Copies of the dragon curve traverse all segments in the plane without crossing so this square arrangement traverses all segments within the square. And so the boundary squares of the two diagonal curves must coincide and so are symmetric in 180° rotation about the middle $\frac{1}{2}b^k$. ☐

Further nature of the two left halves is ahead in theorem 18 as two right sides, and theorem 30 as four joins.

A consequence of this symmetry is that the number of left boundary segments and left boundary squares are related

$$L_k = 2\, LQ_k \qquad \text{for } k \ge 1 \tag{104}$$

In the half of the boundary up to midpoint $\frac{1}{2}b^k$, each boundary square has 1, 2 or 3 boundary segments. Then in the second half they are in 180° rotation so the same but 3, 2 or 1 segments, giving net 2 segments per square.

In the L and LQ recurrences, this is seen from the starting values $L_{1,2,3} = 2\,LQ_{1,2,3}$ and then same recurrence for each maintains that factor.

Likewise a factor of 2 between the generating functions, but with a constant which adjusts for initial $L_0 = LQ_0 = 1$ not doubled.

$$gL(x) \;=\; 2\,gLQ(x) \;-\; 1$$

The squares each touch 1, 2 or 3 boundary segments. The number of squares with each number of sides can be counted separately. The total is LQ.

$$LQ_k = LQ1_k \;+\; LQ2_k \;+\; LQ3_k$$

Theorem 15. *The number of left boundary squares on dragon curve k which touch 1, 2 or 3 boundary segments respectively are*

$$\begin{aligned}
LQ1_k &= 1,0,1,2 && \textit{for } k = 0 \textit{ to } 3 \\
LQ2_k &= 0,1,0,0 \\
LQ3_k &= 0,0,1,2 \\
LQs_k &= LQs_{k-1} + 2\,LQs_{k-3} && \textit{each } s = 1,2,3, \textit{ for } k \ge 4
\end{aligned} \tag{105}$$

$$\begin{aligned}
LQ1_k &= 1,0,1,2,2,4,8,12,20,36,60,100,172,\ldots && k{\ge}3 \quad 2{\times}\text{A203175} \\
LQ2_k &= 0,1,0,0,2,2,2,\;\; 6,10,14,26,\;\; 46,\;\; 74,\ldots && k{\ge}1 \quad \text{A052537} \\
LQ3_k &= 0,0,1,2,2,4,8,12,20,36,60,100,172,\ldots && k{\ge}3 \quad 2{\times}\text{A203175}
\end{aligned}$$

$$\begin{aligned}
\textit{Generating functions} \quad gLQ1(x) &= \tfrac{1}{2} + \tfrac{1}{2}\frac{1 - x + 2x^2}{1 - x - 2x^3} \\
gLQ2(x) &= \frac{x - x^2}{1 - x - 2x^3} \\
gLQ3(x) &= gLQ1(x) \;-\; 1
\end{aligned}$$

Proof. In figure 14, the sub-curves which meet at points 3, 6, 7 do so at 90° angles. No new sides touch the squares on each side of those points, so the same recurrence (103) as for LQ applies to each separate count, with the different initial values. ☐

Boundary segments L_k is sum of boundary squares and how many sides they have,

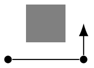
$$L_k = LQ1_k + 2\,LQ2_k + 3\,LQ3_k$$

The symmetry of theorem 14 gives 1-side and 3-side equal,

$$LQ1_k = LQ3_k \qquad \text{for } k \geq 1$$

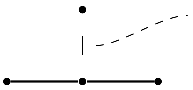
In the recurrences (105), these two have identical starting values at $k = 1, 2, 3$ and the recurrence is the same so they continue to be equal. This equality can also be seen geometrically from the turn at each square.

The segment after a square does not turn left or it would be a further side.



no turn left or would be an extra side
on this square

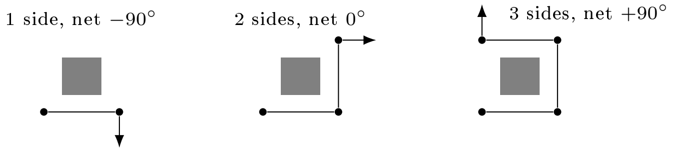
The segment after a square does not go straight ahead because there are no straight ahead segments on the boundary. If there was then the segment perpendicular could not be traversed without crossing or repeating when four curves fill the plane.



if boundary went straight ahead then could
not traverse the perpendicular segment

Figure 15: boundary not straight ahead

So the next boundary segment after a square is to the right. Taken relative to the first segment of the square, this is −90°, 0° or +90° for 1, 2 or 3 sides respectively.



1 side, net −90°

2 sides, net 0°

3 sides, net +90°

Figure 16: boundary square turns

For $k \geq 1$, the last segment of the curve is North $dir(2^k-1) = 1$ since it is a 90° unfold of the initial East segment, or by dir and 2^k-1 as all 1-bits.

dir of a segment is the sum of all turns. For a boundary segment, dir is also sum of just turns along the boundary because there are no crossings in the non-boundary parts of the curve.

The last segment has a 1-side square since the first turn in the curve is to the left which unfolded is a preceding right turn at the end. The boundary turns after the squares up to but not including the last square must be total net

$$-1.\big(LQ1_k - 1\big) + 0.LQ2_k + 1.LQ3_k = dir(2^k-1) = 1 \qquad \text{for } k \geq 1$$
$$\text{so } LQ1_k = LQ3_k$$

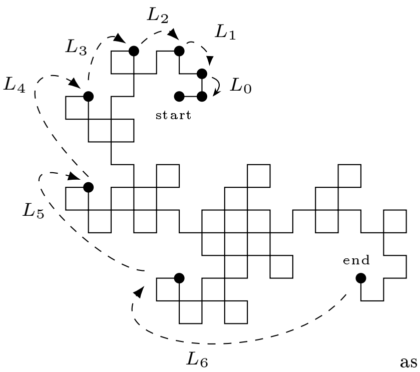
$LQ1, LQ3$ can be written in terms of LQ by noticing $LQ1_{3,4,5} = 2, 2, 4$ is 2× the start $LQ_{0,1,2} = 1, 1, 2$, and the same recurrence thereafter. The remaining squares are $LQ2$.

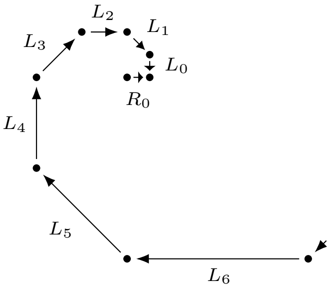
$$\begin{aligned} LQ1_k = LQ3_k &= 2LQ_{k-3} && k \ge 3 \\ LQ2_k = &\quad LQ_k - 4LQ_{k-3} \end{aligned}$$

Since LQ_k grows as a power r^k from section 2, the proportions of $1, 2, 3$-side squares among the total left squares are, with equality $LQ1_k = LQ3_k$ for $k \ge 1$,

$$\begin{aligned} \frac{LQ1_k}{LQ_k} = \frac{LQ3_k}{LQ_k} &\to \frac{2}{r^3} &= 0.410245\ldots && \text{as } k \to \infty \qquad (106) \\ \frac{LQ2_k}{LQ_k} &\to 1 - \frac{4}{r^3} &= 0.179509\ldots \end{aligned}$$

3.2 Right Boundary





Right boundary
as sum of left boundaries
$R_7 = 1 + L_0 + L_1 + \cdots + L_6$

Theorem 16. *The dragon curve k right boundary length is*

$$R_k = 1 + \sum_{j=0}^{k-1} L_j \qquad \textit{right boundary} \tag{107}$$

$$= \begin{cases} 1, 2, 4, 8, 16 & \textit{for } k = 0 \text{ to } 4 \\ 2R_{k-1} - R_{k-2} + 2R_{k-3} - 2R_{k-4} & \textit{for } k \ge 5 \end{cases} \tag{108}$$

$$= 1, 2, 4, 8, 16, 28, 48, 84, 144, 244, 416, 708, 1200, 2036, \ldots \qquad k \ge 1 \quad \text{A227036}$$

$$\textit{Generating function} \quad gR(x) = 1 - \frac{2}{1-x} + \frac{2\,(1 + x + x^2)}{1 - x - 2x^3} \tag{109}$$

When k=0 the sum in (107) is taken as empty so $R_0 = 1$.

Proof. For k=0 the curve is a single line segment $R_0 = 1$. Thereafter, per Daykin and Tucker, the curve k unfolds to become k+1

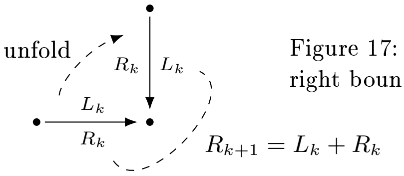


Figure 17:
right boundary unfold

The midpoint is two dragon curve endpoints meeting at 90° so there is no change to the boundary on the outside of the two curves at that point.

$$\begin{aligned} R_{k+1} &= L_k + R_k \\ &= L_k + L_{k-1} + \cdots + L_0 + R_0 \end{aligned} \tag{110}$$

Usual ways to sum recurrence L_k (100) give the R_k recurrence (108). Or substitute $L_k = R_{k+1} - R_k$ into (100),

$$(R_{k+1} - R_k) \;=\; (R_k - R_{k-1}) \,+\, 2\,(R_{k-2} - R_{k-3}) \qquad \text{for } k \ge 4$$

The generating function $gR(x)$ follows from the recurrence or derive it from $gL(x)$. Multiply $\frac{1}{1-x}$ to sum terms, multiply x to shift that to begin at term k=1 and add all-ones $\frac{1}{1-x}$ for $R_0 = 1$ in each term.

$$gR(x) \;=\; \frac{1}{1-x} + x\,\frac{1}{1-x}\,gL(x) \qquad \square$$

Theorem 17 (Daykin and Tucker). *The number of boundary squares on the right side of dragon curve k is*

$$RQ_k = 1 + \sum_{j=0}^{k-1} LQ_j \qquad \textit{right boundary squares} \tag{111}$$

$$= \begin{cases} 1, 2, 3 & \textit{for } k = 0 \textit{ to } 2 \\ RQ_{k-1} + 2RQ_{k-3} & \textit{for } k \ge 3 \end{cases} \tag{112}$$

$$= 1, 2, 3, 5, 9, 15, 25, 43, 73, 123, 209, 355, 601, \ldots \qquad \text{A003476}$$

$$\textit{Generating function} \quad gRQ(x) = \frac{1 + x + x^2}{1 - x - 2x^3}$$

Proof. The same sum of left sides applies for squares as is does for segments since the segments at the end of each LQ unfold are at right angles so squares on them do not overlap, hence (111).

Recurrence (112) follows from some recurrence or generating function manipulations. It also has a direct geometric interpretation,

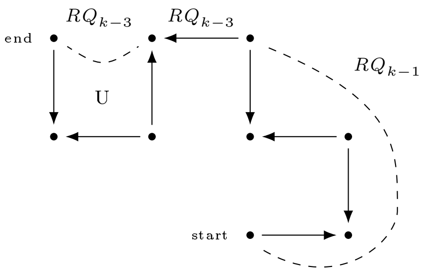
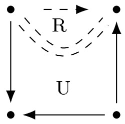


Figure 18: right boundary squares RQ_k recurrence parts $k \geq 3$

The boundary squares in the U shaped part are RQ_{k-3} since they are also boundary squares on the right of the absent sub-curve at the top.



A square of curves like this encloses the area within, and since all enclosed unit squares have all 4 sides traversed, the squares on the boundary of the U coincide with the squares on the boundary of the R. (The same holds for a left side LQ squares, when it is that side which is absent.) □

The same recurrence as R holds for RQ, with different initial values. The initial values of RQ allow it to simplify to 3 terms instead of 4.

The two RQ_{k-3} in figure 18 together are the unfolded left side of $k{-}2$, and its symmetric halves per theorem 14.

Theorem 18 (Daykin and Tucker). *The left boundary squares are related to right boundary squares by*

$$LQ_k = 2\,RQ_{k-2} \qquad k \geq 2 \qquad (113)$$

and similarly for boundary segments

$$L_k = 2R_{k-2} + 2 \qquad k \geq 3 \qquad (114)$$

Proof. Both relations follow numerically from the generating functions or the recurrences. LQ and RQ are as simple as initial values $LQ_{2,3,4} = 2\,LQ_{0,1,2}$ and the same recurrence.

The geometric interpretation is

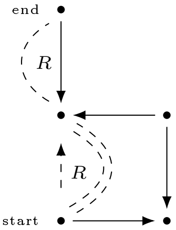


Figure 19: left boundary as two right boundaries $k{-}2$

The U shape boundary squares are RQ the same as in theorem 17. The left side here is the second half of the curve there. □

L and R both grow as powers of the cubic root r as from section 2. The limit for ratio L over R follows from the unfolding difference $L_k = R_{k+1} - R_k$. The same limit holds for boundary squares LQ over RQ.

$$\frac{L_k}{R_k} = \frac{R_{k+1} - R_k}{R_k} \to r - 1 \;\; = 0.695620\ldots \qquad \text{as } k{\to}\infty \tag{115}$$

Inverse R over L is $\frac{1}{r-1} = \frac{1}{2}r^2$ from r as root of polynomial (92) divided through by r^2 so $r{-}1 = 2/r^2$. Or using (114) for R in terms of L, for $k{\ge}1$,

$$\frac{R_k}{L_k} = \frac{\frac{1}{2}L_{k+2} - 2}{L_k} \to \tfrac{1}{2}r^2 \;\; = 1.437564\ldots \qquad \text{as } k{\to}\infty$$

Unfolding also gives counts of 1, 2, 3 side right boundary squares as sums of corresponding lefts from theorem 15.

$$\begin{aligned}
\mathit{RQ1}_k = 1 + \sum_{j=0}^{k-1} \mathit{LQ1}_j \;&= \begin{cases} 1,2,2 & \text{if } k = 0 \text{ to } 2 \\ 2RQ_{k-3}+1 & \text{if } k \ge 3 \end{cases} \\
&= \begin{cases} 1 & \text{if } k{=}0 \\ LQ_{k-1}+1 & \text{if } k{\ge}3 \end{cases} \\
= 1, 2, 2, 3, 5, 7, 11, 19, 31, 51, 87, 147, \ldots&
\end{aligned} \tag{116}$$

$$\begin{aligned}
\mathit{RQ2}_k = \sum_{j=0}^{k-1} \mathit{LQ2}_j \qquad &= dJA_k \;\text{ ahead at (163)} \\
= 0, 0, 1, 1, 1, 3, 5, 7, 13, 23, 37, 63, \ldots& \qquad\qquad k{\ge}2 \text{ A077949}
\end{aligned} \tag{117}$$

$$\begin{aligned}
\mathit{RQ3}_k = \sum_{j=0}^{k-1} \mathit{LQ3}_j \;&= \begin{cases} 1 & \text{if } k{=}0 \\ \mathit{RQ1}_k - 2 & \text{if } k{\ge}1 \end{cases} \\
= 0, 0, 0, 1, 3, 5, 9, 17, 29, 49, 85, 145, \ldots&
\end{aligned} \tag{118}$$

$$\begin{aligned}
g\mathit{RQ1}(x) &= \frac{(1+x)(1-x-x^3)}{(1-x)(1-x-2x^3)} &&= -\tfrac{1}{2} + \frac{1}{1-x} + \tfrac{1}{2}\frac{1+x}{1-x-2x^3} \\
g\mathit{RQ2}(x) &= \frac{x^2}{1-x-2x^3} \\
g\mathit{RQ3}(x) &= \frac{(1+x)\,x^3}{(1-x)(1-x-2x^3)} &&= \tfrac{1}{2} - \frac{1}{1-x} + \tfrac{1}{2}\frac{1+x}{1-x-2x^3}
\end{aligned}$$

The sub-parts of figure 18 give recurrences for these squares too. In the reversed end RQ_{k-3} part, sides flip $1{\leftrightarrow}3$. 2-side squares are unchanged so for them a direct recurrence the same as full RQ.

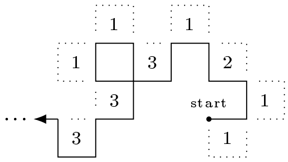
$$\begin{aligned}
\mathit{RQ1}_k &= \mathit{RQ1}_{k-1} + \mathit{RQ1}_{k-3} + \mathit{RQ3}_{k-3} \\
\mathit{RQ2}_k &= \mathit{RQ2}_{k-1} + 2\,\mathit{RQ2}_{k-3} \\
\mathit{RQ3}_k &= \mathit{RQ3}_{k-1} + \mathit{RQ3}_{k-3} + \mathit{RQ1}_{k-3}
\end{aligned} \tag{119}$$

The sub-parts of figure 18 also give a top-down recurrence for how many sides a particular boundary square has. Number right boundary squares starting $m{=}0$ for the first, then the $1{\leftrightarrow}3$ flip in the second RQ_{k-3} becomes cases

$$RQsides(m) = \begin{cases} 1,1,2 & \text{if } m = 0 \text{ to } 2 \\ RQsides\big(m - RQ_{k-1}\big) & \text{if } m - RQ_{k-1} < RQ_{k-3} \\ 4 - RQsides\big(RQ_k - 1 - m\big) & \text{if } m - RQ_{k-1} \ge RQ_{k-3} \end{cases} \tag{120}$$

$$\text{where } RQ_{k-1} \le m < RQ_k \ \text{ for } k \ge 3$$

$$= 1,1,2,1,3,1,1,3,3,1,1,2,2,3,3,1,1,2,1,3,\ldots$$



RQsides

The sides sequence can be used to draw the boundary. The boundary turns right around each boundary square, for its number of sides, and then left between squares, like shown for the left side in figure 16. See section 18.6 for an L-system, and see *Rturn* ahead in theorem 22 for turns individually.

3.3 Total Boundary

Theorem 19. *The dragon curve total boundary length is left plus right,*

$$B_k = R_k + L_k \ = R_{k+1} \tag{121}$$
$$= 2,4,8,16,28,48,84,144,244,416,708,1200,\ldots \qquad \text{A227036}$$

$$\textit{Generating function} \quad gB(x) = \tfrac{1}{x}\,(gR(x) - 1)$$
$$= \frac{2+2x^2}{(1-x)(1-x-2x^3)} \tag{122}$$
$$= 2\left(\frac{-1}{1-x} + \frac{2+x+2x^2}{1-x-2x^3}\right)$$

Proof. $B_k = R_{k+1}$ follows from (110), being one more unfolding. The first term is dropped from the generating function $gR(x)$ in (109) and that simplifies the terms for $gB(x)$. ☐

All the boundary lengths start by doubling with k but when the curve begins to touch and enclose area then the boundary is less than double. For B this happens at $B_4 = 28$ onwards.

Total boundary squares are

$$BQ_k = RQ_k + LQ_k \ = RQ_{k+1} \tag{123}$$
$$= 2,3,5,9,15,25,43,73,123,209,355,601,\ldots \qquad \text{A003476}$$

$$\text{Generating function} \quad gBQ(x) = (gRQ(x) - 1)/x$$
$$= \frac{2+x+2x^2}{1-x-2x^3}$$

And similarly 1, 2 or 3 side squares

$$BQ1_k = RQ1_k + LQ1_k = RQ1_{k+1} \tag{124}$$

$$\begin{aligned} BQ2_k &= RQ2_k + LQ2_k = RQ2_{k+1} \\ BQ3_k &= RQ3_k + LQ3_k = RQ3_{k+1} \end{aligned}$$

The proportion of right boundary out of the total follows from B in terms of R from (121). B grows as a power r^k from section 2 so

$$\frac{R_k}{B_k} = \frac{B_{k-1}}{B_k} \to \frac{1}{r} = 0.589754\ldots \qquad \text{as } k{\to}\infty \tag{125}$$

The rest $1-\dfrac{1}{r}$ is left boundary, or by L in terms of R from (114), and hence B,

$$\frac{L_k}{B_k} = \frac{2B_{k-3}+4}{B_k} \to \frac{2}{r^3} = 0.410245\ldots \qquad \text{as } k{\to}\infty \tag{126}$$

The total $\frac{1}{r} + \frac{2}{r^3} = 1$ is r as root of the cubic polynomial (92), divided through by r^3.

The limits are the same for boundary squares in both cases, as from (123) and (113). The L/B limit (126) is the same as $LQ2/LQ$ from (106).

3.4 Right Boundary Segment Numbers

The 1-side and 3-side boundary squares can be taken in two types each, according to their ends being odd or even and consequently how they expand. There are then 5 types of boundary squares,

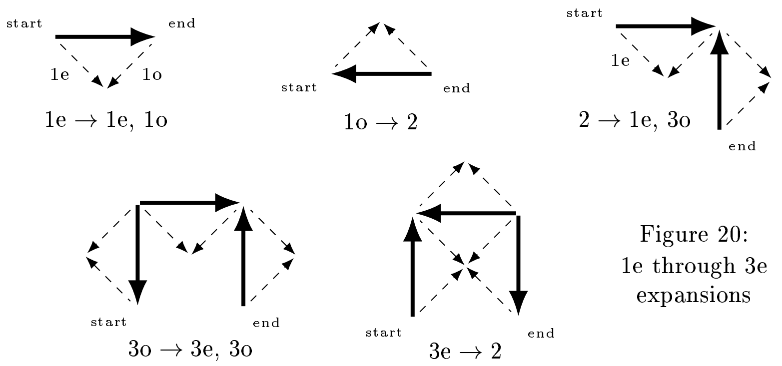


Figure 20:
1e through 3e
expansions

1e and 1o are "even" or "odd" according as their start location is even or odd ($x{+}y$ even or odd). Arrows are shown forward and reverse so expansion is on the right.

3o and 3e are similarly odd or even according as their start location. The "e" of 3e can also be thought of as "enclosing" since its expansion encloses a unit square (and leaves a 2-side).

2-side squares are a single type $RQ2$ from (117).

The expansions of figure 20 are mutual recurrences. A given type in $k{-}1$ goes to 1 or 2 types in k. For example, 1e in k is reached from 1e or from 2 in $k{-}1$.

$$\begin{aligned} RQ1e_k &= RQ1e_{k-1} + RQ2_{k-1} \qquad & RQ3o_k &= RQ3o_{k-1} + RQ2_{k-1} \\ RQ1o_k &= RQ1e_{k-1} & RQ3e_k &= RQ3o_{k-1} \end{aligned}$$

$$RQ2_k = RQ1o_{k-1} + RQ3e_{k-1} \qquad \text{starting } RQ1e_0 = 1 \text{ and others } 0$$

The effect is 1e and 3o are cumulative 2, starting at $RQ1e_0 = 1$ and $RQ3o_0 = 0$. 1o and 3e are one later 1e and 3o respectively.

$$\begin{aligned}
RQ1e_k &= 1 + \sum_{j=0}^{k-1} RQ2_j \quad = JA_k + 1 \;\; \text{ahead in theorem 25} \\
&= 1, 1, 1, 2, 3, 4, 7, 12, 19, 32, 55, 92, \dots \\
RQ3o_k &= \sum_{j=0}^{k-1} RQ2_j \quad = JA_k \;\; \text{ahead in theorem 25} \\
&= 0, 0, 0, 1, 2, 3, 6, 11, 18, 31, 54, 91, \dots \qquad \text{A003479}
\end{aligned}$$

$$\begin{aligned}
gRQ1e(x) &= \tfrac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1-x-2x^3}\right) \\
gRQ3o(x) &= \tfrac{1}{2}\left(\frac{-1}{1-x} + \frac{1}{1-x-2x^3}\right)
\end{aligned}$$

Total 1-side or 3-side are per (116),(118)

$$\begin{aligned}
RQ1e_k + RQ1o_k &= RQ1_k \\
RQ3e_k + RQ3o_k &= RQ3_k
\end{aligned}$$

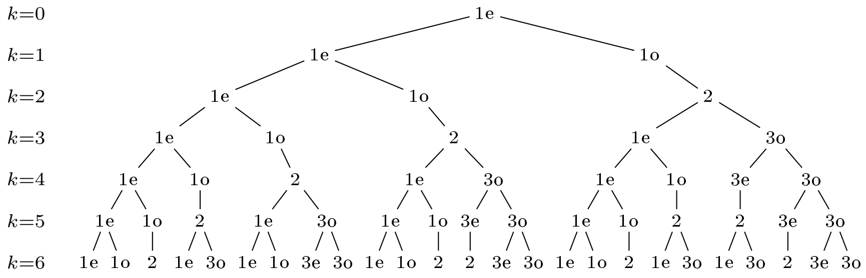
The sub-parts of figure 18 apply to the square types. In the U part, the boundary flips sides 1↔3 like (119), and also flip e↔o since reversed start becomes end.

$$\begin{aligned}
RQ1e_k &= RQ1e_{k-1} + RQ1e_{k-3} + RQ3o_{k-3} \\
RQ1o_k &= RQ1o_{k-1} + RQ1o_{k-3} + RQ3e_{k-3} \\
RQ3e_k &= RQ3e_{k-1} + RQ3e_{k-3} + RQ1o_{k-3} \\
RQ3o_k &= RQ3o_{k-1} + RQ3o_{k-3} + RQ1e_{k-3}
\end{aligned}$$

By unfolding, the whole curve boundary is 1 later. On continuing around to the left boundary, the “e” or “o” type is reckoned by going anti-clockwise. For example in k=0 single segment, the right side is 1e and the left is 1o.

$$\begin{aligned}
BQ1e_k &= RQ1e_{k+1} & BQ3e_k &= RQ3e_{k+1} \\
BQ1o_k &= RQ1o_{k+1} & BQ3o_k &= RQ3o_{k+1}
\end{aligned} \tag{127}$$

Expansions can be illustrated in a tree



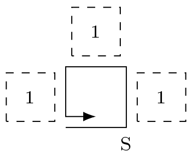
The reversals of figure 18 are seen for example where 1e and 3o sibling subtrees (underneath each 2) are mirror image and flip 1↔3, e↔o. The two right-most in row k=3 are the biggest shown.

The expansions are a bottom-up definition for *RQsides* at (120). That sequence is $1, 2, 3$ without distinguishing e or o, but the rule is

$$\mathit{RQsides}(m) = 1 = \begin{cases} 1o & \text{when second of a pair of consecutive 1,1} \\ 1e & \text{otherwise} \end{cases}$$

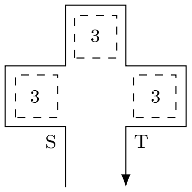
$$\mathit{RQsides}(m) = 3 = \begin{cases} 3e & \text{when first of a pair of consecutive 3,3} \\ 3o & \text{otherwise} \end{cases}$$

1o occurs solely from expansion $1e \to 1e, 1o$ so is second of a pair. Three consecutive 1s do not occur, since that 1o becomes 2 on next expansion. Or geometrically, the boundary turning away from 3 squares of 1-side would be a segment overlap, as shown in the following diagram. If S was curve start, so no preceding segment, then that would be no overlap, but the curve does not start this way.



three consecutive 1-side
right boundary squares
would be segment overlap

3e occurs solely from expansion $3o \to 3e, 3o$ so is first of a pair. Three consecutive 3s do not occur, since that would be a 1-wide gap S--T in the following diagram, into which an adjacent curve cannot enter without segment overlaps for plane filling. If S was curve start, so no segment below it, then that would permit filling, but the curve does not start this way.



three consecutive 3-side
right boundary squares
would be a hole unreachable
for plane-filling

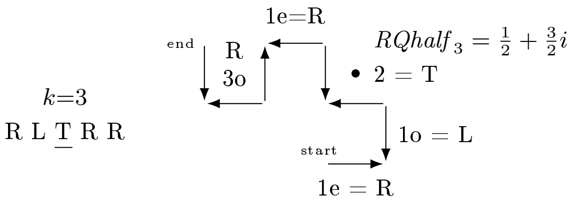
Theorem 20. *For $k \neq 1$, RQ_k is odd and the half-way RQ square has its centre located at*

$$\mathit{RQhalf}_k = \begin{cases} \frac{1}{2} - \frac{1}{2}i & \text{if } k{=}0 \\ \frac{3}{2} + \frac{1}{2}i & \text{if } k{=}2 \\ \left(\frac{3}{10} - \frac{4}{10}i\right) b^k \; + \left(\frac{3}{10} + \frac{1}{10}i\right)(-1)^k & \text{if } k{=}1 \text{ or } k \geq 3 \end{cases} \tag{128}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}i,\ 1,\ \tfrac{3}{2} + \tfrac{1}{2}i,\ \tfrac{1}{2} + \tfrac{3}{2}i,\ -\tfrac{3}{2} + \tfrac{3}{2}i,\ -\tfrac{5}{2} + \tfrac{1}{2}i,\ \dots$$

For k=1, $RQ_1 = 2$ is even and RQhalf_1 is taken as the corner where those 2 squares touch. This corresponds to the general formula $k \geq 3$.

Proof. The theorem can be verified explicitly for $k \leq 3$. Boundary squares and types (figure 20) around $k = 3$ are

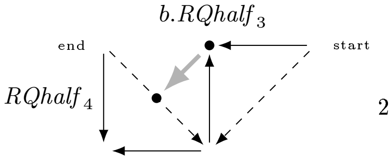


It's convenient to combine boundary square types 1e etc into R,L,T according as they grow like a right, left or two-side respectively. The expansions of figure 20 are then

$$\begin{aligned}
R &= \text{1e or 3o} && \rightarrow R, L \text{ for 1e, or } L, R \text{ for 3o}\\
L &= \text{1o or 3e} && \rightarrow T\\
T &= \ 2 && \rightarrow R, R
\end{aligned}$$

k=3 square types are R,L, $\underline{\text{T}}$, R,R. One R each side will expand to the same number of squares, which leaves L,T,R to consider on expansion.

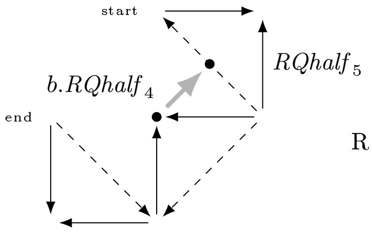
T in the orientation of k=3 expands by multiplying b and segment replacements as follows



2 = T expansion

Square types L,T,R expand to T, R,R=3o, R,L. So the new half way is the R type 3o which is second of expanded T. There is an R each side which will again expand the same way each, leaving T,R=3o,L to consider. The centre location is offset by $RQhalf_4 = b.RQhalf_3 - \frac{1}{2} - \frac{1}{2}i$.

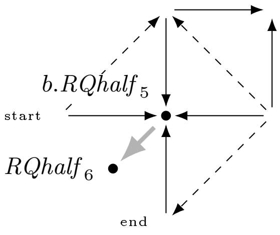
R = 3o oriented with its open side upwards expands,



R = 3o expansion

Square types T,R=3o,L expand to R,R, L,R, T. So the new half way is the L type 3e which is first from R=3o. The centre location is offset by $RQhalf_5 = b.RQhalf_4 + \frac{1}{2} + \frac{1}{2}i$. R,R, $\underline{\text{L}}$,R, T has again an R each side which expand the same subsequently, leaving R,L,T to consider.

L=3e with its open side to the left expands,



L = 3e expansion

Square types R, L=3e, T expand to R,L, T, R,R. So the new half way is the T. There is again an R each side which expand the same subsequently, leaving L,T,R to consider. The centre location is offset by $RQhalf_6 = b.RQhalf_4 - \frac{1}{2} - \frac{1}{2}i$. L,T,R is the same as k=2, but with the T oriented 180° around. So the pattern of which square types and offsets repeats

$$RQhalf_k = RQhalf_{k-1} + [-\tfrac{1}{2}-\tfrac{1}{2},\ \tfrac{1}{2}+\tfrac{1}{2}] \qquad k\geq 4 \qquad \square$$

For all boundary squares, unfolding from theorem 19 is $BQ_k = RQ_{k+1}$ so the whole curve half-way is half-way of RQ_{k+1}. Since $RQ_k > LQ_k$ for $k \geq 1$, this is before the unfolding of the left, so location

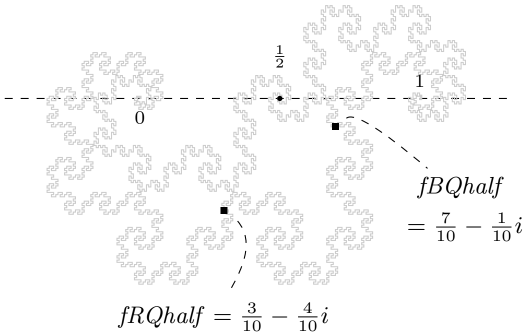
$$BQhalf_k = RQhalf_{k+1} \tag{129}$$

For k=0, $BQ_0 = 2$ is even. (129) gives curve end $BQhalf_0 = 1$. This can be reckoned as where the two squares touch when going anti-clockwise around the curve.

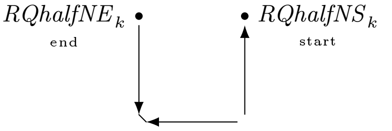
For the curve scaled to unit length, the right boundary half way limit is coefficient of b^k at (128).

$$\frac{RQhalf_k}{b^k} \to fRQhalf = \tfrac{3}{10} - \tfrac{4}{10}i \tag{130}$$

$$\frac{BQhalf_k}{b^k} \to fBQhalf = b.fRQhalf = \tfrac{7}{10} - \tfrac{1}{10}i \tag{131}$$



Each $RQhalf$ square has a first and last point number n.



$RQhalf_k$ unit square
first and last points n

The steps forward or not for each expansion in theorem 20 gives recurrences for these, and with initial values at k=3 formulas

$$\begin{aligned}
\mathit{RQhalfNS}_k &= 2\mathit{RQhalfNS}_{k-1} + [0,1,0] \qquad k \ge 3 \\
&= \begin{cases} 0 & \text{if } k{=}0 \\ 1 & \text{if } k{=}1 \\ \frac{9}{28}2^k - \frac{1}{7}[4,1,2] & \text{if } k \ge 2 \end{cases} \qquad (132) \\
&= 0,1,1,2,5,10,20,41,82,164,\ldots \\
\mathit{RQhalfNE}_k &= 2\mathit{RQhalfNE}_{k-1} - [0,0,3] \qquad k \ge 3 \\
&= \begin{cases} 1 & \text{if } k = 0 \text{ or } 1 \\ \frac{11}{28}2^k + \frac{1}{7}[6,12,3] & \text{if } k \ge 2 \end{cases} \\
&= 1,1,2,4,8,13,26,52,101,202,\ldots
\end{aligned}$$

For k=1, RQhalf_1 is the point between two squares. It is reckoned as same $n = \mathit{RQhalfNS}_1 = \mathit{RQhalfNE}_1 = 1$ start and end.

Theorem 21. *Segments n on the right boundary of the dragon curve are all and only those of the following bit pattern*

$$\begin{aligned}
\mathit{Rpred}(n) &= \begin{cases} 0 & \textit{if bits of } n \textit{ form figure 21} \\ 1 & \textit{if bits of } n \textit{ not of that form} \end{cases} \\
&{=}1 \ \textit{at } n = 0\ldots 20,\ 25\ldots 41,\ 50,51,52,\ 57,\ldots \\
&{=}0 \ \textit{at } n = 21\ldots 24,\ 42\ldots 49,\ 53\ldots 56,\ldots
\end{aligned}$$

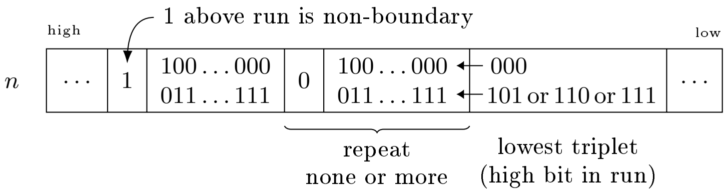


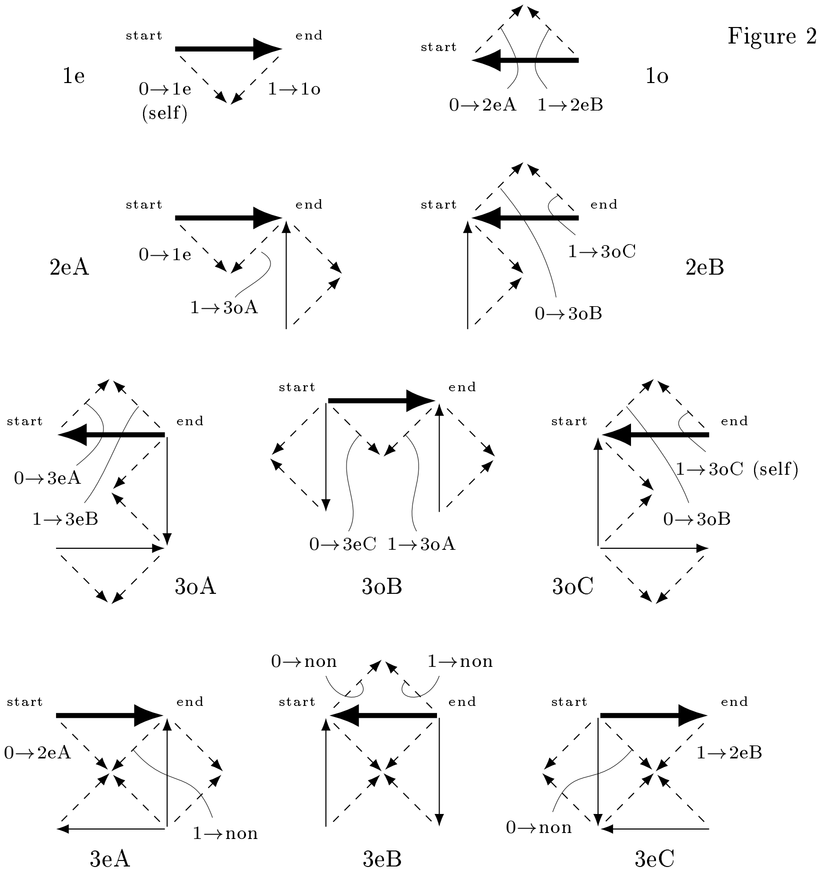
Figure 21: *NonRpred* bit fields

Triplet 000, 101, 110, or 111, is at the lowest place such a triplet occurs. A run 100…00 or 011…11 is above it. The high bit of the triplet is the low bit of the run, so determines its type. If a 0 above the run then the next run is determined just by the next higher bit. If a 1 above then that is a non-right-boundary n.

Various n are not of this form, for example no such triplet at all, or bits ending before a run completed or no 1 above a run. Such n are *Rpred*.

Proof. The following segment types in boundary square types occur. The thick horizontal is the segment. Expansion is in the manner of figure 20, with the expansion of the particular segment being of interest. The first expanded segment is a new low 0-bit and the second is a new low 1-bit. They may be enclosed by the expansion of adjacent segments.

Figure 22



The 2 and 3 side segments are shown with their further sides. The expansion of those can enclose to make non-boundary segments. In 3eA, the expanded 1-bit segment is enclosed so that it and all further expansions of it are non-boundary.

Only surrounding segments which may touch the given segment need to be considered. Anything at more than a 90° angle cannot touch since it would overlap possible further segments in between. In 1e, both the 0-bit and 1-bit cases can ignore the other segment since they are 270° away and so cannot touch.

Transitions between configurations are then

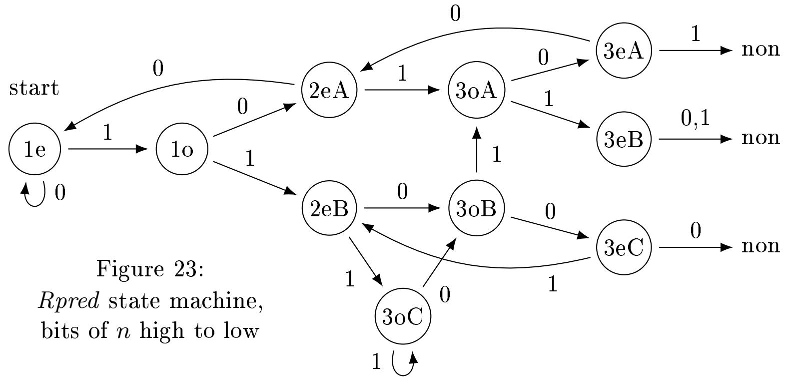
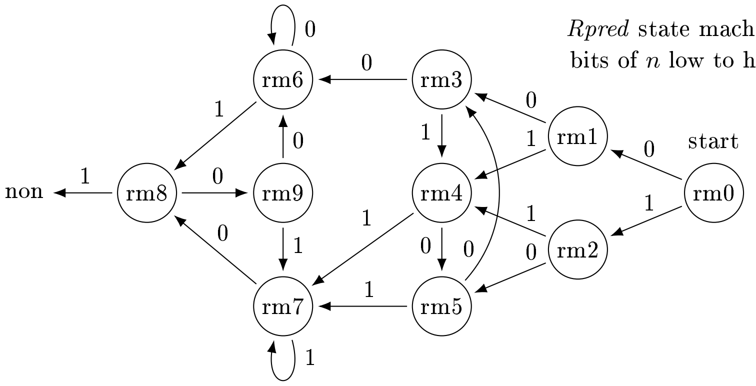


Figure 23:
Rpred state machine,
bits of n high to low

States 2eB and 3oC can be merged as a single state self-loop or to 3oB, the same as 3oC. The first side in 3oC is too far away to affect the end segment, so behaves the same as 2eB.

A given n is identified as boundary or non-boundary by following its bits high to low through the states. The initial state is 1e which is a single even-numbered segment n=0.

Usual DFA state machine manipulations can reverse to take bits of n from low to high. rm0 is the initial state, representing no bits at all.



Rpred state machine,
bits of n low to high

The initial six states rm0 to rm5 find the lowest occurrence of a triplet 000, 101, 110 or 111. Triplet 000 goes to rm6. Triplets 101, 110 or 111 go to rm7.

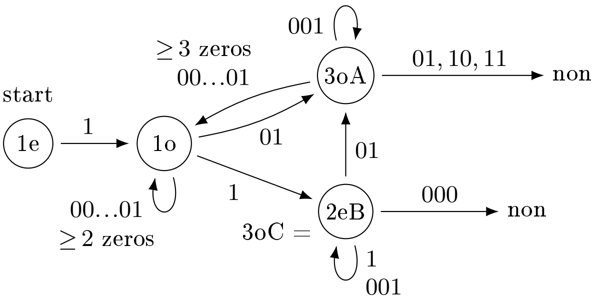
rm6 is a run 100…00 and rm7 is a run 011…11. There is at least one bit and its opposite in a run, so minimum 10 or 01 respectively. The high 0 or 1 from the triplet goes to rm6 or rm7 and is thus part of the lowest run.

Both run types end in rm8 and from there a 1 above is a non-boundary, or 0 above goes to another run.

Boundary segments are any n which does not reach a 1-bit above a run in this way. ☐

The minimum for a non-boundary is a 1-bit above a run of two bits, of which one is the high of the triplet. So for example 11000 is non-boundary with 000 triplet, or 10110 with 110 triplet.

In high to low figure 23, and with 2eB and 3oC states merged as a 2eB, it can be noticed at 1o, 2eB or 3oA bits 001 loop around to repeat that state. Transitions among those states by runs 00…001 suffice to express *Rpred*,



Rpred by
runs 0001

At 3oA, a single 1 bit remains boundary, but a further bit either 10 or 11 goes to non-boundary. In the full high to low, this is 3eB where either further bit goes non.

1o is reached by a run of $\geq$3 zeros, either from 3oA or such a run remaining in 1o. The initial 1-bit for 1e to 1o can be thought of as infinite zeros above so also a run $\geq$3 zeros. 2eB is reached by a run of no zeros. 3oA is reached by a run of 1 zero. In each case, a run of 2 zeros remains in the same state.

With *Rpred* as 1 or 0, its sum is right boundary length.

$$R_k = \sum_{n=0}^{2^k-1} \mathit{Rpred}(n)$$

This sum can be calculated from the bit patterns of figure 21. It's convenient to count the non-boundary segments and from that by difference the boundary segments. This is more complicated than theorem 12 and theorem 16 but is a combinatorial interpretation for the boundary lengths.

Runs 100...000 or 011...111 with 0 or 1 above are $\geq$3 bits of 2 types.

Triplet and bits below are also a run $\geq$3 bits and no choices in the bits below the triplet. Two arbitrary low bits reach one of rm3, rm4, or rm5. To avoid making a triplet yet the bits must stay among those states and only a single value in each does that $(1,0,0$ respectively).

The high bit of the triplet is part of the run above it and the triplet determines the run type 100...00 or 011...11. So there are 4 types of triplet and 1 type of lowest run. It's convenient to treat that as net 2 types each.

The number of ways to form such runs is a sum of binomials

$$\mathit{Rnon}_k = 2^k - R_k = \sum_{l=5}^{k} 2^{k-l} \sum_{t=1}^{l-2\,\geq\,3t} 2^{t+1} \binom{l-2-2t}{t} \tag{133}$$

There are l bits for the non-boundary 1-bit and all below it. The $k-l$ bits above it are arbitrary so 2^{k-l}.

Within the l bits there are to be $t \geq 1$ many runs plus the triplet run. As above, there is net 2 types for each so 2^{t+1}.

Each combination of runs is a composition (partition with order) $l = l_0 + l_1 + \cdots + l_t$, with each part $\geq$3 except the second last which is $\geq$2. Those minimums can be treated by starting them each with 2 bits or 1 bit respectively, and the remaining length $l-(2t+1)$ is then a composition of $t+1$ parts each $\geq$1. In the usual way for compositions, this is binomial $\binom{l-(2t+1)-1}{t}$.

Another way to think of the run compositions is that there are l positions where $t+1$ runs might end, except the minimum lengths remove $2t+1$ possible end positions, and the highest run must end at the top l position, leaving choices t ends within $l-(2t+1)-1$ positions.

Let the number of runs in the l bits from (133) be

$$\begin{aligned} \mathit{TwoRuns}(l) = {} & \text{number of compositions of } l \text{ as } {\geq}2 \text{ parts,} \\ & \quad \text{one } {\geq}2\text{, the rest } {\geq}3\text{, and each part 2 types} \\ = {} & \sum_{t=1}^{l-2-2t\,\geq\,t} 2^{t+1} \binom{l-2-2t}{t} \end{aligned}$$

$$= 0,0,0,0,0,4,8,12,24,44,72,124,216,\ldots \qquad 4{\times}\mathrm{A003479}$$

Taking the ≥ 2 part separately as length h, of 2 types, leaves one or more parts each ≥ 3.

$$\begin{aligned}
\mathit{TwoRuns}(l) &= 2\sum_{h=2}^{l-h\,\geq\,3} \mathit{OneRuns}(l{-}h) \\
\mathit{OneRuns}(m) &= \text{ number of compositions of } m \text{ as } {\geq}1 \text{ parts} \\
&\qquad \text{each part } {\geq}3 \text{ and 2 types} \\
&= \sum_{t=0}^{m-3-2t\,\geq\,t} 2^{t+1}\binom{m{-}3-2t}{t} \\
&= 0,0,0,2,2,2,6,10,14,26,46,74,126,218,366,618,\ldots \\
&\qquad\qquad 2{\times}\mathrm{A077949}, \quad n{\geq}1\ \mathrm{A052537}
\end{aligned}$$

OneRuns can be written as a recurrence sum by counting either the whole of m as a single part of 2 types, or initial part $f \geq 3$ of 2 types and the rest further *OneRuns*.

$$\mathit{OneRuns}(m) = \begin{cases} 0 & \text{if } m < 3 \\ 2 + 2\displaystyle\sum_{f=3}^{m-f\,\geq\,3} \mathit{OneRuns}(m{-}f) & \text{if } m \geq 3 \end{cases}$$

The sum is taken as empty when $m < 6$, so $\mathit{OneRuns}(3,4,5) = 2$.

Taking the $f{=}3$ highest term $2\,\mathit{OneRuns}(m{-}3)$ out of the sum leaves the terms of $\mathit{OneRuns}(m{-}1)$. This is the dragon curve recurrence,

$$\begin{aligned}
\mathit{OneRuns}(m) &= \mathit{OneRuns}(m{-}1) \;+\; 2\,\mathit{OneRuns}(m{-}3) \\
&\qquad \text{starting } \mathit{OneRuns}(0,1,2,3) = 0,0,0,2
\end{aligned}$$

OneRuns and *TwoRuns* occur variously elsewhere as dragon cubics, or counting runs for $\mathit{other}(n)$ from figure 13. *TwoRuns* values are in section 4.2 at (161) as join area $\mathit{TwoRuns}(l) = 4\,\mathit{JA}_{l-2}$. *OneRuns* values are in section 4.3 as join area increment $\mathit{OneRuns}(m) = 2\,\mathit{dJA}_{m-1}$. The connection to area is that 4 right-side non-boundary segments make a right-side enclosed unit square, so $\mathit{Rnon}_k = 4\,\mathit{AR}_k$.

The high to low states of figure 23 are how many sides on the unit square to the right of segment n.

$$\begin{aligned}
\mathit{Rsides}(n) &= 1,2,3,4 \text{ per states 1,2,3,non of figure 23} \qquad (134) \\
&= 1,1,2,2,1,3,3,3,1,1,3,3,3,3,3,3,1,1,2,2,2,4,\ldots
\end{aligned}$$

Rsides does not depend on a curve level k, since level end $\mathit{turn}(2^k) = 1$ left so the continuing curve puts no additional segment on the last boundary square.

For *Rsides*, states 2eB and 3oC are kept separate to distinguish 2 sides from 3 sides. States 3eB and "non" are separate to distinguish 3 from 4. A sides predicate testing for just certain combinations could merge one or both those state pairs if the respective distinctions are not needed. For example a predicate identifying just n with 1-side square needs neither distinction.

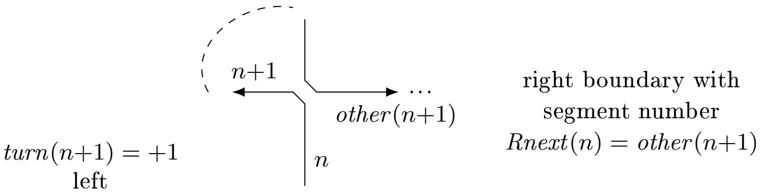
Rsides is related to *RQsides* (120), but *RQsides* is only boundary squares so without the 4-sides, and it counts each square just once however many sides it has. So *Rsides* with 4s omitted is sequence *RQsides* with each entry replicated its own number of times (2 becomes 2,2, and 3 becomes 3,3,3).

Runs of non-boundary segments are always a multiple of 4 long because they form enclosed squares with 4 sides each. But how many boundary segments are between such runs can vary.

Right boundary segment numbers can be iterated using the bit flips of $other(n)$ from figure 13.

$$Rnext(n) = \begin{cases} other(n{+}1) & \text{if } turn(n{+}1){=}1 \text{ and } other(n{+}1) \text{ same arm} \\ n{+}1 & \text{if not} \end{cases}$$

When $turn(n{+}1) = +1$ (left) at a double-visited point, the curve is turning into an enclosed area,



Stepping across to $other(n{+}1)$ is the segment where it comes out to resume the boundary. This happens first at $n{=}20$ where $Rnext(20) = other(20{+}1) = 25$.

This step across is only when $turn(n{+}1) = +1$. $turn(n{+}1) = -1$ right remains on the boundary. Some of those right turns might be double-visited, but their *other* would either skip some boundary or go to a non-boundary segment.

The previous boundary segment similarly

$$Rprev(n) = \begin{cases} other(n) - 1 & \text{if } turn(n){=}1 \text{ and } other(n) \text{ same arm} \\ n - 1 & \text{if not} \end{cases}$$

The boundary square sides *RQsides* and their expansions can be used to form a right boundary turn sequence. Squares are $RQsides - 1$ many right turns to go around the square, then a left turn between squares. A direct calculation can be made of the boundary turns as follows.

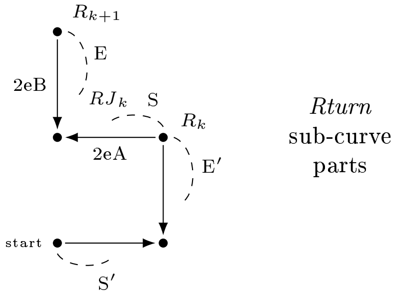
Theorem 22. *Number right boundary points of the dragon curve starting* $m{=}0$ *at the origin, so the first boundary turn is at* $m{=}1$. *The turn* $+1$ *left or* -1 *right is given by recurrence*

$$Rturn(m) = \begin{cases} 1 & \text{if } m = 1,2 \\ Rturn(m - R_k) & \text{if } m < RJ_k \\ -1 & \text{if } m = RJ_k \\ Rturn(m - L_k) & \text{if } m > RJ_k \end{cases} \tag{135}$$

$$\text{where } m \text{ is in the range } R_k \le m < R_{k+1} \text{ for } k \ge 1 \tag{136}$$

$$
\begin{aligned}
&= + + - + + - - + + + - - + - - \dots \\
RJ_k &= \tfrac{1}{2} R_{k+2} - \lceil \tfrac{2}{3} k \rceil \qquad (137) \\
&= 2, 3, 6, 12, 21, 38, 68, 117, 202, 348, \dots \\
gRJ(x) &= \tfrac{2}{3} \frac{2-x}{(1-x)^2} + \tfrac{1}{3} \frac{1+x}{1+x+x^2} + \frac{3+2x+4x^2}{1-x-2x^3}
\end{aligned}
$$

Proof. For an m in the range (136), consider the curve up to R_{k+1} and its $k{-}1$ sub-curves as follows,



Some of sub-curve 2eA is enclosed, but the portion marked S preceding where it touches 2eB is the same as curve start S′, so can reduce to $m - R_k$.

Some of sub-curve 2eB is enclosed, but the portion marked E after where it touches 2eA is the same as E′. Per the unfolding in theorem 16, difference $R_{k+1} - R_k = L_k$ so reduce to $m - L_k$.

2eA and 2eB meet at point RJ. If $k{-}1 = 1$ so that the sub-curves shown are unit segments, then it is already a right turn as shown. But in later expansion levels the curve turns left at RJ to go into enclosed area whereas the boundary turns right to skip that. Hence -1 turn at RJ in (135).

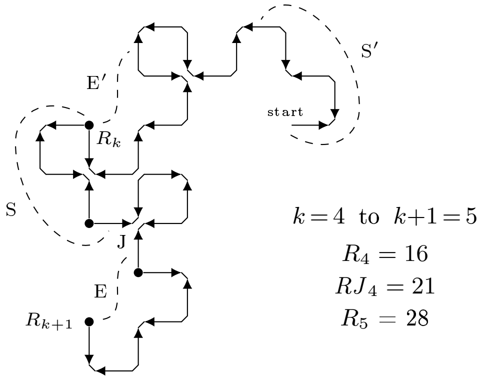
Point RJ_k along the boundary is R_k up to the 2eA, and then the number of unenclosed segments in 2eA (of level $k{-}1$). The expansions of figure 22 are mutual recurrences for 2eA length,

$$
\begin{aligned}
\mathit{R2eA}_k &= R_{k-1} + \mathit{R3oA}_{k-1} && \text{starting } \mathit{R2eA}_0 = 1 \\
\mathit{R3oA}_k &= \mathit{R3eA}_{k-1} + \mathit{R3eB}_{k-1} && \text{starting } \mathit{R3oA}_0 = 1 \\
\mathit{R3eA}_k &= \mathit{R2A}_{k-1} + \mathit{R3eB}_{k-1} && \text{starting } \mathit{R3eA}_0 = 1 \\
\mathit{R3eA}_k &= \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathit{R2eA}_k &= 1, 2, 4, 5, 10, 20, 33, 58, 104, 177, \dots \\
\mathit{R3oA}_k &= 1, 2, 1, 2, 4, 5, 10, 20, 33, 58, \dots \\
\mathit{R3eA}_k &= 1, 1, 2, 4, 5, 10, 20, 33, 58, 104, \dots \\
RJ_k &= R_k + \mathit{R2eA}_{k-1}
\end{aligned}
$$

Working through the recurrences shows RJ is per (137). That formula can define an initial RJ_0 too, though the proof here only uses $k \geq 1$. ☐

$k{=}4$ curve through to $k{+}1 = 5$ is the first where the curve at RJ turns left into enclosed area but the boundary turns right.



The right boundary is the whole curve until $RJ_4 = 21$. So $\mathit{Rturn}(m) = \mathit{turn}(m)$ for $m \le 20$, which is an alternative to continuing recurrence (135) down to $m=1,2$.

In (135), all right turns -1 occur at RJ points, suitably reduced. The effect is to identify all m where the curve turns into enclosed area and have boundary right turn there. The 2-side squares, which are right turns already, with nothing enclosed, can be thought of as prospective subsequent enclosures. RJ_k is the biggest enclosure in its R_k to R_{k+1} range, then the parts S′,E′ reduced may contain smaller ones.

A similar recurrence gives the segment number n which is m'th on the right boundary, reckoning the first as m=0. These are all the n which are $\mathit{Rpred}(n)$.

$$\mathit{Rn}(m) = \begin{cases} m & \text{if } m \le 1 \\ 2^k + \mathit{Rn}(m - R_k) & \text{if } m < RJ_k \\ 2^k + \mathit{Rn}(m - L_k) & \text{if } m \ge RJ_k \end{cases} \tag{138}$$

$$\text{where } m \text{ is in the range } R_k \le m < R_{k+1} \text{ for } k \ge 1$$

$$= 0 \ldots 20,\ 25 \ldots 41,\ 50, 51, 52,\ 57, \ldots$$

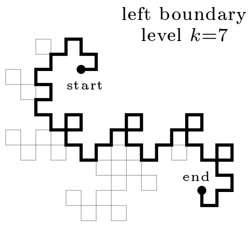
The reductions by R_k or L_k both reduce n by a block of 2^k segments. The first non-boundary segment is n=21 so that (138) can be $\mathit{Rn}(m)=m$ for $m \le 20$ instead of the recurrence all the way down to $m \le 1$ if desired.

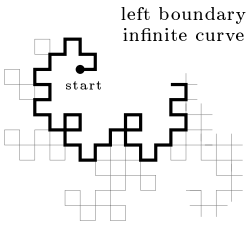
3.5 Left Boundary Segment Numbers

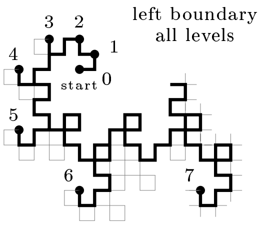
Some of the left boundary in level k is enclosed by level k+1 and so is no longer on the boundary. This is unlike the right boundary which is never enclosed so its level k boundary segment numbers are a prefix of the level k+1 boundary segment numbers.

Three forms of left boundary segment numbers are considered here

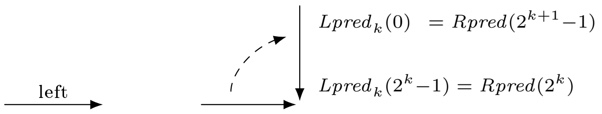
- segments on boundary for particular level k
- segments on boundary for every level, so the curve continued infinitely
- segments on boundary for some level, a union of all left boundaries







A level k left boundary unfolds to the right boundary of level k+1.

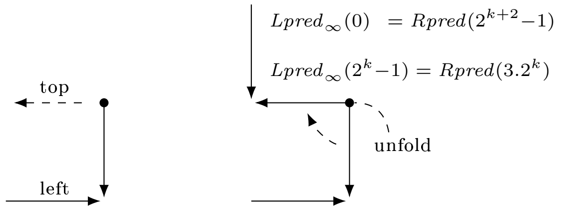


$$Lpred_k(n) = Rpred(2^{k+1}-1-n) \qquad 0 \le n < 2^k$$

This subtraction is n bit flipped 0↔1 and extra high 1-bit. That 1-bit has the effect of starting in state 1o.

$$2^{k+1}-1-n = \quad \begin{array}{|c|c|}\hline 1 & \overbrace{\;n \text{ flipped } 0\leftrightarrow1\;}^{k \text{ bits}} \\ \hline\end{array}$$

For the curve continued infinitely, it suffices to consider an extra curve at the end. The segments of $n < 2^k$ enclosed by that extra are not on the boundary continued infinitely.



The possible further curve section marked "top" does not touch the left boundary since both are left sides and so do not touch (the two opposing right sides of the square touch).

Unfolding brings the left side to the end of the right side.

$$Lpred_\infty(n) = Rpred(2^{k+2}-1-n) \qquad \text{any } k \text{ where } 2^k > n$$

$$=1 \text{ at } \; n = 0 \ldots 6,\; 11, 12, 13,\; 22 \ldots 27,\; 44 \ldots 54,\; \ldots$$

$$=0 \text{ at } \; n = 7 \ldots 10,\; 14 \ldots 21,\; 28 \ldots 43,\; 55 \ldots 90,\; \ldots$$

This subtraction is n bit flipped 0↔1 like above, but this time an extra two high 1-bits above however many bits are necessary to represent n. These two high 1-bits have the effect of starting in state 2eB.

$$2^{k+2}-1-n \quad \begin{array}{|c|c|c|}\hline 1 & 1 & \overbrace{\;n \text{ flipped } 0\leftrightarrow1\;}^{k \text{ bits, with } 2^k > n} \\ \hline\end{array}$$

For the left boundary of any level k, the minimum enclosing is when k is the smallest $2^k > n$, being the bits required to represent n with no high 0s.

$$\begin{aligned}
&\mathit{Lpred}_{\mathit{any}}(n) = \mathit{Lpred}_k(n) \qquad k \text{ smallest } 2^k > n \\
&\quad =1 \text{ at } \; n = 0\ldots7,\ 11\ldots15,\ 22\ldots31,\ 44\ldots63,\ \ldots \\
&\quad =0 \text{ at } \; n = 8,9,10,\ 16\ldots21,\ 32\ldots43,\ 64\ldots90,\ \ldots
\end{aligned}$$

Similar holds for an *Lsides* number of sides on the unit square to the left of segment n, in terms of the corresponding terms of the corresponding *Rsides* from (134).

$$\begin{aligned}
\mathit{Lsides}_k(n) &= \mathit{Rsides}(2^{k+1} - 1 - n) \\
k{=}0 \quad &= 1 \\
k{=}1 \quad &= 2, 2 \\
k{=}2 \quad &= 3, 3, 3, 1 \\
k{=}3 \quad &= 3, 3, 3, 3, 3, 3, 1, 1
\end{aligned}$$

$$\begin{aligned}
\mathit{Lsides}_\infty(n) &= \mathit{Lsides}_{k+2}(n) \qquad \text{where } k \text{ big enough } n < 2^k \\
&= 3,3,3,3,3,3,2,4,4,4,4,2,3,3,4,4,4,4,4,4,4,4,3,1,\ldots
\end{aligned}$$

Or left boundary squares numbered by an index m starting $m{=}0$, in terms of the corresponding *RQsides* from (120).

$$\begin{aligned}
\mathit{LQsides}_k(m) &= \mathit{LQsides}\big(\mathit{RQ}_{k+1} - 1 - m\big) \\
k{=}0 \quad &= 1 \\
k{=}1 \quad &= 2 \\
k{=}2 \quad &= 3, 1 \\
k{=}3 \quad &= 3, 3, 1, 1
\end{aligned}$$

$$\begin{aligned}
\mathit{LQsides}_\infty(m) &= \mathit{LQsides}_{k+2}(m) \qquad \text{where } k \text{ big enough } m < \mathit{LQ}_k \\
&= 3,3,2,3,1,3,3,1,1,3,3,2,2,1,1,3,3,2,3,1,3,\ldots
\end{aligned}$$

Left boundary segment numbers can be iterated similar to the right boundary using the bit flips of $\mathit{other}(n)$ from figure 13, stepping across a double-visited right turn.

$$\mathit{Lnext}_\infty(n) = \begin{cases} \mathit{other}(n{+}1) & \text{if } \mathit{turn}(n{+}1){=}{-1} \text{ and } \mathit{other} \text{ on same arm} \\ n{+}1 & \text{if not} \end{cases}$$

Within a level k, the $\mathit{other}(n)$ should be taken only within the same k,

$$\mathit{Lnext}_k(n) = \begin{cases} \mathit{other}(n{+}1) & \text{if } \mathit{turn}(n{+}1){=}{-1} \text{ and } \mathit{other}{<}2^k \text{ on same arm} \\ n{+}1 & \text{if not} \end{cases}$$

For the previous segment, no distinction is needed between k or continued infinitely, though within k it's necessary to check decrease $\mathit{other}(n) < n$ so as not to go to across a join into bigger k.

$$Lprev(n) = \begin{cases} other(n) - 1 & \text{if } turn(n){=}{-1} \text{ and } other{<}n \text{ on same arm} \\ n-1 & \text{if not} \end{cases}$$

4 Area

Where the dragon curve touches, it encloses unit squares. This can be on the left or right side of the curve.

Since the curve always turns ±90° the left or right side enclosed squares alternate. Left squares have an even $x{+}y$ lower left corner. Right squares have an odd $x{+}y$ lower left corner.

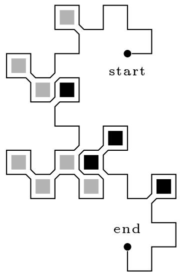


Figure 24: area $k{=}6$

$AL_6 = 7$ ▨

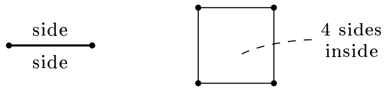
$AR_6 = 4$ ■

$A_6 = 11$ total

Lemma 1. *Consider line segments on a square grid where any enclosed unit square has segments on all 4 sides. The enclosed area A and boundary B are related to total line segments N by*

$$4A + B = 2N \tag{139}$$

Proof. Count the sides of the line segments. There are N segments so total $2N$ sides. Each side is either a boundary or is inside.



There are B outside sides on the boundary. The inside sides are all in enclosed unit squares. Each area A square has 4 inside sides, so $4A$ and total $B + 4A = 2N$. □

Theorem 23 (Daykin and Tucker). *The area enclosed by dragon curve level k is*

$$\begin{aligned} A_k &= \tfrac{1}{2}2^k - \tfrac{1}{4}B_k && \textit{area} \\ &= \begin{cases} 0,0,0,0,1 & \textit{for } k = 0 \textit{ to } 4 \\ 4A_{k-1} - 5A_{k-2} + 4A_{k-3} - 6A_{k-4} + 4A_{k-5} & \textit{for } k \ge 5 \end{cases} && (140) \\ &= 0,0,0,0,1,4,11,28,67,152,335,724,1539,\ldots && \text{A003230} \end{aligned}$$

$$\textit{Generating function}\quad gA(x) = \frac{x^4}{(1{-}x)\,(1{-}2x)\,(1{-}x{-}2x^3)}$$

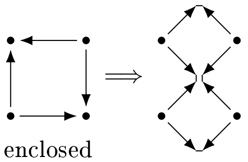
$$= \tfrac{1}{2}\Big(\frac{1}{1-x} + \frac{1}{1-2x} - \frac{2+x+2x^2}{1-x-2x^3}\Big) \qquad (141)$$

Proof. In the dragon curve, an enclosed unit square always has all four sides traversed exactly once. If there was ever a bigger area then the curve would have to cross itself to traverse the inner lines to produce the "all segments traversed" pattern when four curves fill the plane. So lemma 1 applies.

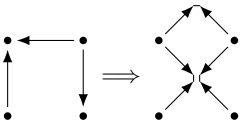
$$4A_k + B_k = 2.2^k$$

The recurrence (140) is from the usual way to work a power into an existing linear recurrence. It can also be obtained from the generating function which introduces $1/(1{-}2x)$ for 2^k to subtract from. The characteristic equation of the recurrence has a new factor $x{-}2$ for 2 as a new root. ☐

Second Proof of Theorem 23. When each segment expands, an existing enclosed unit square becomes 2 enclosed unit squares. A *BQ3e* type boundary square expands to 1 enclosed unit square. Other square types of figure 20 do not enclose new area (they do not have opposing right sides).



enclosed
unit square



BQ3e
boundary square

$$A_k = 2A_{k-1} + \mathit{BQ3e}_{k-1} \quad = \sum_{j=0}^{k-1} 2^{k-1-j}\, \mathit{BQ3e}_j \qquad (142)$$

Some recurrence or generating function manipulation then gives A_k. The sum of descending powers of 2 of *BQ3e* is factor $\frac{1}{1-2x}$ on $g\mathit{BQ3e}(x)$, then further factor x for index shift so the sum starts $k{-}1$.

$$\begin{aligned} gA(x) &= \frac{x}{1-2x}\, g\mathit{BQ3e}(x) \\ \text{where } g\mathit{BQ3e}(x) &= g\mathit{RQ3o}(x) \end{aligned}$$

☐

The limit area for the curve scaled to endpoints a unit length is the coefficient of the 2^k term,

$$\frac{A_k}{2^k} \to \frac{1}{2} \qquad (143)$$

4.1 Area Left and Right Sides

The curve does not cross itself so an enclosed unit square is either on the left or right side of the curve. The two sides can be counted separately

$$A_k = AR_k + AL_k$$

Theorem 24 (Daykin and Tucker)**.** *The area enclosed on the right side of dragon curve level k is*

$$\begin{aligned} AR_k &= \begin{cases} 0 & \textit{for } k=0 \\ A_{k-1} & \textit{for } k \ge 1 \end{cases} \qquad\qquad \textit{right area} \\ &= 0,0,0,0,0,1,4,11,28,67,152,335,724,\dots \qquad\qquad \text{A003230} \end{aligned}$$

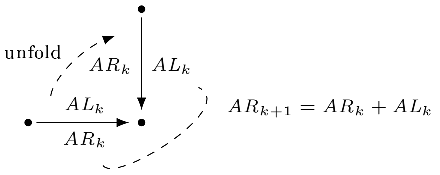
$$\begin{aligned} \textit{Generating function} \quad gAR(x) &= x\, gA(x) \\ &= \tfrac{1}{4}\left(-1 + \frac{2}{1-x} + \frac{1}{1-2x} - \frac{2\,(1+x+x^2)}{1-x-2x^3}\right) \end{aligned}$$

and on the left side

$$\begin{aligned} AL_k &= \begin{cases} 0 & \textit{for } k=0 \textit{ to } 3 \\ A_k - A_{k-1} & \textit{for } k \ge 4 \end{cases} \qquad\qquad \textit{left area} \\ &= 2^{k-2} - \tfrac{1}{4}L_k \qquad\qquad (144) \\ &= 0,0,0,0,1,3,7,17,39,85,183,389,815,\dots \qquad\qquad \text{A003478} \end{aligned}$$

$$\begin{aligned} \textit{Generating function} \quad gAL(x) &= \frac{x^4}{(1-2x)(1-x-2x^3)} \\ &= \tfrac{1}{4}\left(1 + \frac{1}{1-2x} - 2\frac{1+x^2}{1-x-2x^3}\right) \end{aligned}$$

Proof. Per Daykin and Tucker, the unfolding of figure 17 can be drawn with areas on the left and right of the curve.



At the midpoint the two curves meet at 90° so no new area is enclosed on the right, making the right area the previous total area.

$$AR_{k+1} = AR_k + AL_k = A_k \tag{145}$$

The left side follows from that as total area difference,

$$\begin{aligned} AL_k + AR_k &= A_k \\ AL_k &= A_k - AR_k \ = A_k - A_{k-1} \end{aligned} \tag{146}$$

The generating function follows as $gAL(x) = gA(x) - x\,gA(x)$.

Form (144) is lemma 1 applied only to left sides of each segment, with total 2^k such sides.

$$4AL_k + L_k = 2^k \qquad \square$$

Repeated unfolding gives all area as successive copies of the left area, so a cumulative AL,

$$A_k = \sum_{j=0}^{k} AL_j \tag{147}$$

This is also in the generating functions. Factor $\frac{1}{1-x}$ is a cumulative sum

$$gA(x) = \frac{1}{1-x}\, gAL(x)$$

Bates, Bunder and Tognetti[5] consider runs of consecutive turns left or right in the paperfolding sequence. They show the number of runs of 3 consecutive lefts or rights in curve level k are

$$\begin{aligned} \mathit{A3left}_k &= 2^{k-4} \qquad\qquad \text{for } k \ge 4 \\ \mathit{A3right}_k &= 2^{k-4} - 1 \end{aligned}$$

They show too that the n where a run of 3 lefts start or 3 rights start are respectively

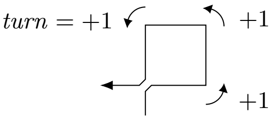
$$\begin{aligned} 8\,\mathit{TurnLeft} &= 8, 16, 32, 40, 64, \ 72, \ 80, 104, \ldots && \text{A106841} \\ 8\,\mathit{TurnRight} - 2 &= 22, 46, 54, 86, 94, 110, 118, 150, \ldots && \text{A106838} \end{aligned}$$

since expanding the curve 3 times is additional turns before each existing turn,

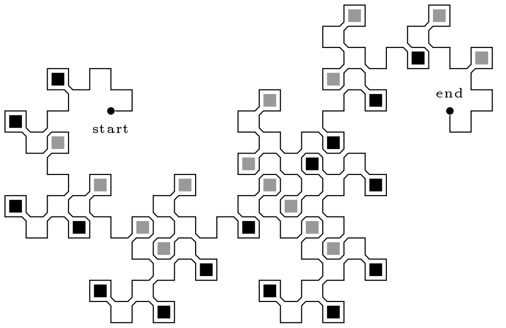
	$\boxed{t_1}$		$\boxed{t_2}$		$\ldots$		$\mathit{turn}(8n)$
LLRLLRR		LLRRLRR		LLRLLRR		$\ldots$	$\mathit{turn}(n \not\equiv 0 \bmod 8)$
before odd		before even		before odd			

The new turns before odd are the turns of $n = 1$ to 7. The new turns before even are the turns of $n = 9$ to 15. By unfolding, the latter are a reverse and L $\leftrightarrow$ R flip of 1 to 7. Unfolding puts these alternately before odd and even existing turns. The only way to make a run LLL is for an existing turn to be an L to go with the LL following it. The only way to make a run RRR is an existing turn R to go with the RR preceding it.

Runs of 3 turns left or right form an enclosed unit square on the left or right of the curve respectively.



three consecutive left turns,
left-side enclosed unit square



k=8
LLL squares black
RRR squares grey

total
$\mathit{A3left}_8 = 16$
$\mathit{A3right}_8 = 15$
$A_8 = 67$

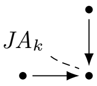
Limits for the proportion of these types of enclosed squares out of the total left or right side areas are, since AL and AR both grow as 2^{k-2},

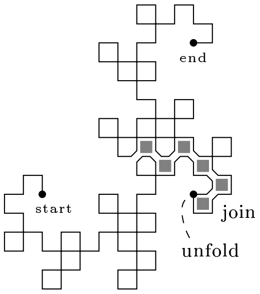
$$\frac{A3left_k}{AL_k} \to \frac{1}{4} \qquad \frac{A3right_k}{AR_k} \to \frac{1}{4}$$

No segment has a 3-turn square on both sides. That would be a turn sequence LLLRRR or RRRLLL and they do not occur. There is no such sequence in k=4 and thereafter unfolding makes a new 3-turn at the unfold, but the preceding and following turns are not then a pair of 3s.

4.2 Join Area

When the dragon curve unfolds the two copies enclose the same area each and where they meet encloses a further join area.





join area
$JA_6 = 6$
between two
k=6 curves
meeting

Theorem 25 (Daykin and Tucker). *The join area enclosed by two dragon curves level k is*

$$\begin{aligned}
JA_k &= A_{k+1} - 2A_k \qquad \textit{join area} && (148)\\
&= AL_{k+1} - A_k && (149)\\
&= \begin{cases} 0,0,0,1 & \textit{if } k = 0 \textit{ to } 3\\ 2\,JA_{k-1} - JA_{k-2} + 2\,JA_{k-3} - 2\,JA_{k-4} & \textit{if } k \geq 4 \end{cases}\\
&= 0,0,0,1,2,3,6,11,18,31,54,91,154,263,446,\ldots \qquad {\scriptstyle k\geq 3 \ \ \text{A003479}}
\end{aligned}$$

$$\textit{Generating function} \quad gJA(x) = \frac{x^3}{(1-x)(1-x-2x^3)} \qquad (150)$$

$$= -\tfrac{1}{2}\frac{1}{1-x} + \tfrac{1}{2}\frac{1}{1-x-2x^3}$$

Proof. The join area is the additional area A_{k+1} has over two copies of A_k.

This is entirely on the left side of the curve so the join is the additional area which AL_{k+1} has over $AL_k + AR_k = A_k$ which is the unfolding on that side, giving (149). ☐

(148) is the $BQ3e$ increment of A as from (142) so that

$$JA_k = BQ3e_k \qquad (151)$$

Join area can also be calculated from the shortfall of boundary B_{k+1} over two B_k. Each new enclosed unit square in the join is 4 boundary sides no longer on the boundary

$$JA_k = \tfrac{1}{4}\big(2B_k - B_{k+1}\big)$$

The join is on the left of the curve and so is the shortfall of left boundary L_{k+1} over a left and right $R_k + L_k$ which is the unfold on that side. Then with $R_k + L_k = R_{k+1}$ from (110) a left right difference

$$JA_k = \tfrac{1}{4}\big(R_{k+1} - L_{k+1}\big)$$

Join as extra over area doubling (148) also gives total area as a sum of joins

$$\begin{aligned} A_k &= JA_{k-1} + 2A_{k-1} \\ &= JA_{k-1} + 2JA_{k-2} + 4JA_{k-3} + 8JA_{k-4} + \cdots \qquad (152) \\ &= \sum_{j=0}^{k-1} 2^{k-1-j}\, JA_j \end{aligned}$$

This sum is in the generating functions. Factor $\frac{1}{1-2x}$ is such a sum of descending powers of 2. Factor x is an index shift since A_k in (152) has highest term JA_{k-1}.

$$gA(x) \;=\; x\,\frac{1}{1-2x}\,gJA(x) \qquad (153)$$

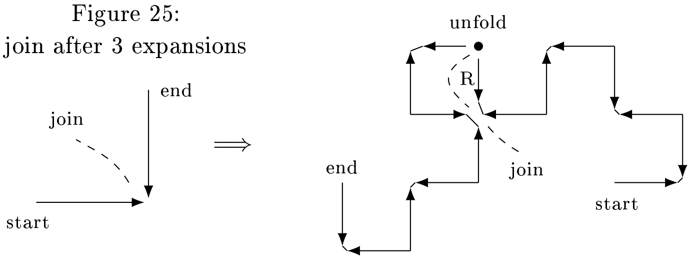
Theorem 26. *The join area is right boundary squares in successive 3-index steps,*

$$JA_k = RQ_{k-3} + RQ_{k-6} + \cdots + RQ_{0\,\mathrm{or}\,1\,\mathrm{or}\,2} \qquad (154)$$

These right boundary parts are directed at successive 180° *angles, starting from* −90° *relative to the start of the curve.*

Proof. After 3 expansions, the join meeting is

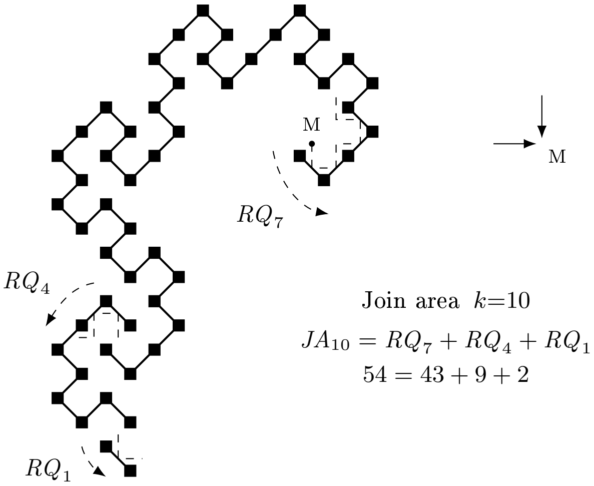
Figure 25:
join after 3 expansions



So the join is a right side turned −90° followed by a further join of the same kind as the original but rotated 180°,

$$\begin{aligned} JA_k &= RQ_{k-3} + JA_{k-3} \qquad \text{for } k \ge 3 \qquad (155) \\ &= RQ_{k-3} + RQ_{k-6} + \cdots + RQ_{0\,\mathrm{or}\,1\,\mathrm{or}\,2} + JA_{0\,\mathrm{or}\,1\,\mathrm{or}\,2} \end{aligned}$$

which is (154) since the initial joins are empty $JA_0 = JA_1 = JA_2 = 0$. $\square$



At point M the squares are the right side RQ_7 of a curve with initial segment downwards ($-90°$) and then curling around in its usual way. Those initial segments are shown dashed. They are at the end of the first joining sub-curve and are some of a right side since the end of that sub-curve is an unfolding of its previous level.

The next part RQ_4 starts in the opposite direction $+90°$ and the final part RQ_4 opposite again back to $-90°$.

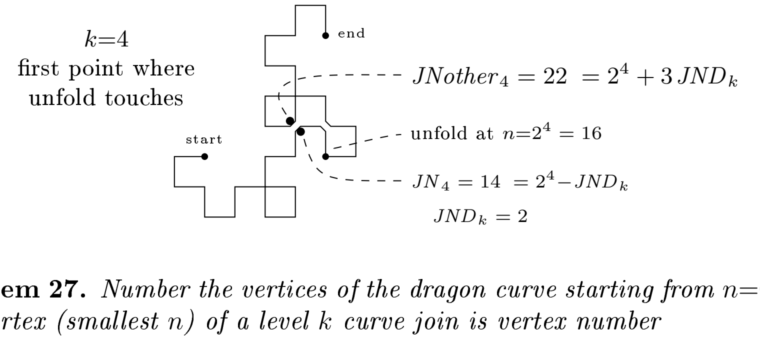
The index steps by 3 mean the RQ parts decrease in size quickly. The proportion of the first to total join area follows from (155) and JA growing as power r^k (section 2),

$$\frac{RQ_{k-3}}{JA_k} = \frac{JA_k - JA_{k-3}}{JA_k} \to 1 - \frac{1}{r^3} = 0.794877\ldots$$

Or simply the rest being JA_{k-3} so its proportion,

$$\frac{JA_k - RQ_{k-3}}{JA_k} = \frac{JA_{k-3}}{JA_k} \to \frac{1}{r^3} = 0.205122\ldots$$

The first point where a level k curve touches its unfolded copy is the smallest vertex number in the join and is the end-most part of that join area. The vertex numbers on each side can be calculated.



Theorem 27. *Number the vertices of the dragon curve starting from n=0. The first vertex (smallest n) of a level k curve join is vertex number*

$$\begin{aligned}
\mathit{JN}_k &= \tfrac{1}{7}(6.2^k + 2^{k \bmod 3}) \;\; = \lceil \tfrac{6}{7} 2^k \rceil && \textit{join } n && (156)\\
&= 1, 2, 4, 7, \;\; 14, 28, 55, 110, 220, 439, 878, \ldots && && \text{A057744}\\
&= \textit{binary } 011\,011 \ldots 011 \quad \textit{(zero or more repeats)}\\
&\qquad \textit{then } 1 \text{ or } 10 \text{ or } 100 \qquad \textit{total } k{+}1 \textit{ bits} && && (157)
\end{aligned}$$

and the opposing point it touches other(JN_k) *is*

$$\begin{aligned}
\mathit{JNother}_k &= \tfrac{1}{7}(10.2^k - 3.2^{k \bmod 3}) && \textit{join } n \textit{ other}\\
&= 1, 2, 4, 11, 22, 44, 91, 182, 364, 731, 1462, \ldots\\
&= \textit{binary } 101\,101 \ldots 101 \quad \textit{(zero or more repeats)}\\
&\qquad \textit{then } 1 \text{ or } 10 \text{ or } 100 \qquad \textit{total } k{+}1 \textit{ bits}
\end{aligned}$$

Or each measured back from the unfold point 2^k*,*

$$\begin{aligned}
\mathit{JN}_k &= 2^k - \mathit{JND}_k && \textit{join } n \textit{ from unfold}\\
\mathit{JNother}_k &= 2^k + 3\,\mathit{JND}_k && && (158)\\
\mathit{JND}_k &= \tfrac{1}{7}(2^k - 2^{k \bmod 3}) \;\; = \lfloor \tfrac{1}{7} 2^k \rfloor && && (159)\\
&= 0, 0, 0, 1, 2, 4, 9, 18, 36, 73, 146, 292, \ldots && && \text{A155803}\\
&= \textit{binary } 100\,100\,100 \ldots \quad \textit{total } k{-}2 \textit{ bits}
\end{aligned}$$

$2^{k \bmod 3}$ means $1, 2, 4$ according to $k \equiv 0, 1, 2 \bmod 3$. For $k \le 2$, there is nothing enclosed by the join and $\mathit{JN}_k = \mathit{JNother}_k = 2^k$. For $k \ge 3$, there is join area enclosed.

In the binary (157) for JN, the repeats of 011 have a high 0 bit. That bit is included in the $k{+}1$ total bit count.

Proof. Expand a join three times as per figure 25. The R side is a sub-curve of 2^{k-3} segments and then further join $k{-}3$ so

$$\begin{aligned}
\mathit{JND}_k &= 2^{k-3} + \mathit{JND}_{k-3}\\
&= 2^{k-3} + 2^{k-6} + \cdots + 2^{0 \text{ or } 1 \text{ or } 2} + \mathit{JND}_{0 \text{ or } 1 \text{ or } 2}
\end{aligned}$$

The power formula (159) follows from initial $\mathit{JND}_0 = \mathit{JND}_1 = \mathit{JND}_2 = 0$ which are no squares enclosed by join $k \le 2$.

For $\mathit{JNother}_k$, there are 3 lengths of 2^{k-3} from the unfold to the meeting. That factor 3 is in each such expansion so $\mathit{JNother}_k = 2^k + 3\mathit{JND}_k$ (158). □

The bit pattern of JND_k shows how it doubles each time plus 1 when $k \equiv 0 \bmod 3$. There is a new meeting only on every third expansion.

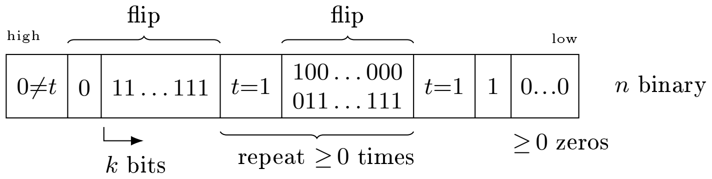
$$\mathit{JND}_k = 2\,\mathit{JND}_{k-1} + (1 \text{ if } k \equiv 0 \bmod 3)$$

For JN_k, this is double each time and subtract 1 when $k \equiv 0 \bmod 3$. For $\mathit{JNother}_k$, this is double each time and add 3 when $k \equiv 0 \bmod 3$.

Out of the total 2^{k+1} segments the join point limits are half the coefficients of their 2^k terms.

$$\frac{\mathit{JN}_k}{2^{k+1}} \to \frac{3}{7} \qquad\qquad \frac{\mathit{JNother}_k}{2^{k+1}} \to \frac{5}{7} \tag{160}$$

Join points n where curve k joins to its unfolded copy can be characterized from the $\mathit{other}(n)$ bit pattern of figure 13. A join point has n within its k bits but the corresponding $\mathit{other}(n)$ requiring k+1 bits. This occurs when the topmost flip is an 011…111 and that 0-bit is the next above the k bits of n, so flip to k+1 bits. This requires t=1 so that a 0 above the top of n is the $\neq t$ bit. This is a right turn, which is also since the join is on the left of the curve.



$$\mathit{JPpred}_k(n) = \begin{cases} 1 & \text{if } n < 2^k \text{ and } \mathit{other}(n) \ge 2^k \text{ same arm} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} &= 1 \text{ at } n = 7 && \text{for } k = 3 \\ &\qquad 14, 15 && \text{for } k = 4 \\ &\qquad 28, 30, 31 && \text{for } k = 5 \\ &\qquad 55, 56, 59, 60, 62, 63 && \text{for } k = 6 \end{aligned}$$

The smallest n with $\mathit{JPpred}_k(n) = 1$ is the join end point $n = \mathit{JN}_k$ from theorem 27.

All the even n in a given join are the previous level doubled. An n with $\mathit{JPpred}_k(n) = 1$ can have a low 0-bit added to its bit form to give $\mathit{JPpred}_{k+1}(2n)$ $= 1$. Conversely any $\mathit{JPpred}_{k+1}(2n) = 1$ can have a low 0-bit removed and satisfies $\mathit{JPpred}_k(n)$. For example k=4 points 14 and 15 become 28 and 30 in k=5.

Each join point completes a join area square so

$$\mathit{JA}_k = \sum_{n=0}^{2^k} \mathit{JPpred}_k(n)$$

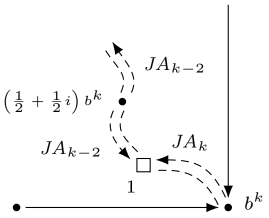
This sum can be calculated by counting *JPpred* bit patterns. Optional low 0-bits, lowest 1-bit and t=1 are a run of ≥ 2 bits. Each flipped run and t above are ≥ 3 bits and of 2 types each, except just 1 type for the highest. These run lengths are the compositions *TwoRuns* of section 3.4, but factor $\frac{1}{4}$ since the first and last runs are not of 2 types each.

$$\mathit{JA}_k = \tfrac{1}{4}\,\mathit{TwoRuns}_{k+2} \tag{161}$$

4.3 Join Area Increment

The relations in this section between *JA*, *dJA*, *LQ* and *RQ* follow numerically from their respective recurrences. The geometry shows their shapes are the respective parts too.

Theorem 28 (Daykin and Tucker). *The join area grows by a unit square at the join end and two copies of its second previous self at the middle* $\left(\frac{1}{2} + \frac{1}{2}i\right) b^k$.



$$JA_{k+1} = JA_k + 1 + 2\,JA_{k-2} \tag{162}$$

$$\begin{aligned} dJA_k &= JA_{k+1} - JA_k && \text{increment} \\ &= 2\,JA_{k-2} + 1 && k \geq 2 \\ &= dJA_{k-1} + 2\,dJA_{k-3} \end{aligned} \tag{163}$$

$$dJA_k = 0, 0, 1, 1, 1, 3, 5, 7, 13, 23, 37, 63, 109, 183, 309, 527, \dots$$

A077949

$$\textit{Generating function } gdJA(x) = x^2\,\frac{1}{1 - x - 2x^3}$$

Proof. Arrange four curves in a square. They expand as follows so that the two right sides touch in the middle. The two left sides never touch, otherwise they would overlap that middle.

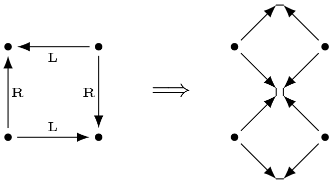


Figure 26:
square expansion
middle touch

The join area *JA* is where two curve ends touch. Expanding one level gives

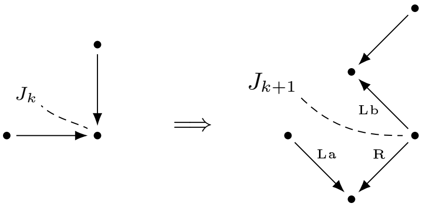


Figure 27:
join area
one expansion

The opposing left sides *La* and *Lb* do not touch as per figure 26 above. So the join JA_{k+1} is the area enclosed by the two adjacent sides *R* and *Lb*. This is the join between two curve starts.

Consider the unit squares on the boundaries of four curves arranged in a square,

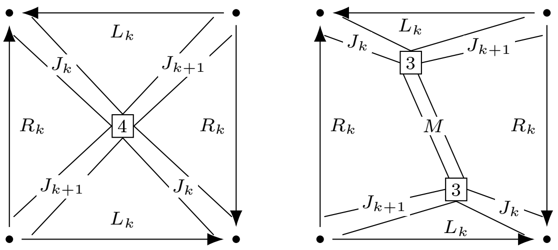


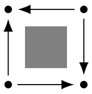
Figure 28: boundary square overlaps

Since dragon curves traverse all segments of the plane, this square arrangement must enclose all unit squares inside. The unit squares on the boundaries of the curves must overlap.

The top-left and bottom-right corners are curve end meetings and so join area JA_k. The top-right and bottom-left corners are curve start meetings and so JA_{k+1}.

There cannot be a single point in the middle where all 4 sides touch or it would be visited 4 times. There could be a single square in the middle common to all 4 sides, or there could be two squares which are each common to the two right sides and one left, with (possibly empty) middle squares M between the right sides.

For k=0, each curve is a single line segment which is the 4-overlap case. The joins are all empty $JA_0 = JA_1 = 0$.



4-overlap for k=0 unit sides

For $k \geq 1$, the 3-overlap occurs because the two right sides R touch per figure 26.

Expand the join from figure 27 a second time,

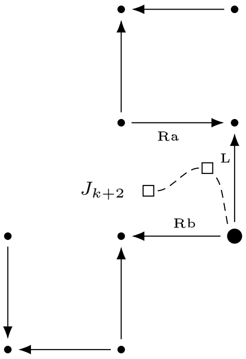


Figure 29: join second expansion

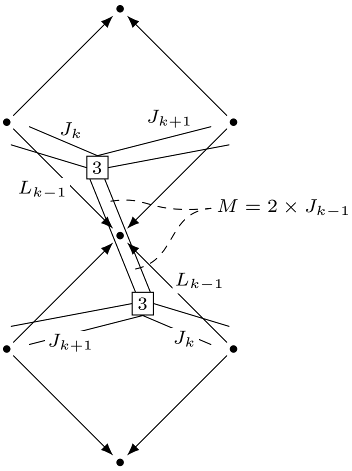
The join is the overlap of the three sides marked *Ra*, *L* and *Rb*. Each of those sides is a level k curve and the join is k+2.

This is the square arrangement of figure 28 turned 90°. If k=0 then it's the single-segment 4-overlap case. That single square is not enclosed and so $JA_2 = 0$.

If $k \geq 1$ then it's the 3-overlap case. The first 3-square is common to all of *Ra*, *L* and *Rb* and is part of the join. The second 3 square is common only to *Ra* and *Rb*. Its third side would be the absent vertical which would complete the square. That second 3 is therefore on the boundary and not part of the join. So

$$JA_{k+2} = JA_{k+1} + 1 + M$$

The nature of the middle part M is seen by expanding the 3-overlap square of figure 28 by one level as follows.



The right sides expand to touch in the middle as per figure 26. The middle squares M are therefore two endpoint meetings which are each join J_{k-1}. Hence the join area growing by copies of its previous self.

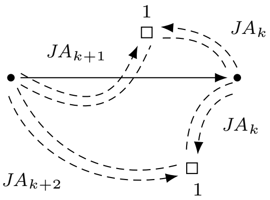
$$\begin{aligned} M &= 2\, J_{k-1} \\ JA_{k+2} &= JA_{k+1} + 1 + 2\, JA_{k-1} \end{aligned}$$

The generating function $gdJA(x)$ follows from the $gJA(x)$ formula (150).

$$gdJA(x) = (1-x)\; gJA(x) \qquad\qquad \text{term-wise differences}$$

The $\frac{1}{1-x}$ factor in $gJA(x)$ form (150) represents a cumulative sum, in this case a sum of dJA. ☐

Theorem 29. *The left and right boundary squares are two joins with a unit square in between*



$$LQ_k = JA_{k+1} + 1 + JA_k \tag{164}$$

$$RQ_k = JA_{k+2} + 1 + JA_k \tag{165}$$

Proof. In the square of figure 28, the left boundary is $JA_{k+1} + 1 + JA_k$.

In the second join expansion of figure 29, JA_{k+2} goes to the second "3" square. The missing vertical in that figure is a downward curve and so the rest of the Rb boundary is from the curve end to that points, which is JA_k. ☐

The proportion of join squares out of these left or right boundary squares follows from (164) and (165) and JA growing as a power r^k (section 2).

$$\frac{JA_k}{LQ_k} = \frac{JA_k}{JA_k + JA_{k+1} + 1} \;\to\; \frac{1}{1+r} \;\; = \tfrac{1}{4}r^3 - \tfrac{1}{2}r \;\; = 0.370972\ldots$$

$$\frac{JA_k}{RQ_k} = \frac{JA_k}{JA_k + JA_{k+2} + 1} \to \frac{1}{1+r^2} = r - \tfrac{1}{2}r^2 \ = 0.258055\ldots$$

The second r forms are division in $\mathbb{Q}[r]$ per section 2. Some equivalent recurrence or generating function manipulations gives identities expressing JA in terms of terms of LQ or RQ ready to divide.

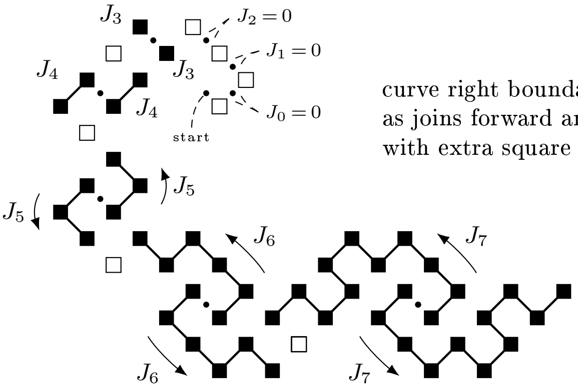
$$\begin{aligned} JA_k &= \tfrac{1}{4}LQ_{k+3} - \tfrac{1}{2}LQ_{k+1} - \tfrac{1}{2} \\ JA_k &= RQ_{k+1} - \tfrac{1}{2}RQ_{k+2} - \tfrac{1}{2} \end{aligned}$$

Daykin and Tucker give right and left boundary squares difference

$$RQ_k - LQ_k \ = \ dJA_{k+1} \tag{166}$$

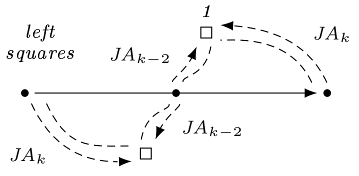
Geometrically this is the difference between JA_{k+1} in LQ and JA_{k+2} in RQ, which is join increment dJA_{k+1}.

The left and right boundary breakdowns of theorem 29 correspond to the four fractal boundary parts used by Chang and Zhang[9]. Their A_1A_∞ is JA_{k+2} on the right boundary taken as a sum of increments $JA_{k+2} = \sum_{j=0}^{k+1} dJA_j$. Each increment is $dJA = 2JA + 1$, with the $+1$ part $\to 0$ for the fractal. Their A_1B_1 and B_1A_2 sections are the $2JA$ of that increment, one forward and one rotated backward, so curling in or out. The last increment in JA_{k+2} is $dJA_{k+1} = 2JA_{k-1} + 1$ which is 3 levels down and hence shrink factor $(1/\sqrt{2})^3$ so $A_1B_1 = B_1A_2 = 2^{-3/2}A_1A_\infty$. Their algorithm 2 for expanding the parts corresponds to the increment again $JA_k = JA_{k-1} + 1 + 2JA_{k-3}$ (162).



curve right boundary squares
as joins forward and backward
with extra square in between.

Theorem 30. *For $k \geq 2$, the two symmetric halves of the left boundary squares are each two joins with a unit square in between.*

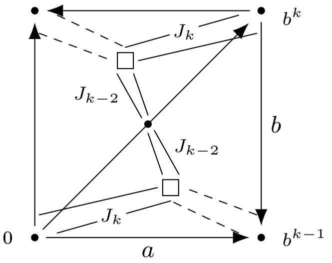


$$LQ_k = 2\left(JA_k + 1 + JA_{k-2}\right) \quad \textit{for } k \geq 2$$

The join parts may be empty. For $k{=}2$, both JA_k and JA_{k-2} are empty and the boundary is just the unit squares between, $LQ_2 = 2$. The rotational symmetry of LQ is per theorem 14.

Proof. This is as simple as the left boundary squares being two inward pointing right boundary squares as from figure 19, and theorem 29 above.

It can also be seen in a square arrangement as follows in the style of figure 28. The sides are level $k{-}1$. The diagonal from 0 to b^k is a level k curve.



Sides a and b are the expansion of the diagonal. The left boundary squares of a and right boundary squares of b are the joins JA_k and JA_{k-2} established above. ☐

4.4 Join End Square

Theorem 31. *The boundary square at the end of the join, which is the 3 or 4 overlap square of figure 28, is located at*

$$
\begin{aligned}
JEQ_k &= \tfrac{3+i}{5}\, b^k \;-\; \tfrac{1-3i}{10}\,(-1)^k && \textit{end square centre} \qquad (167)\\
&= \tfrac{1}{2}{+}\tfrac{1}{2}i,\ \tfrac{1}{2}{+}\tfrac{1}{2}i,\ -\tfrac{1}{2}{+}\tfrac{3}{2}i,\ -\tfrac{3}{2}{+}\tfrac{1}{2}i,\ -\tfrac{5}{2}{-}\tfrac{1}{2}i,\ -\tfrac{3}{2}{-}\tfrac{7}{2}i,\ \dots\\
JEC_k &= JEQ_k \;-\; \left(\tfrac{1}{2}{+}\tfrac{1}{2}i\right) && \textit{lower-left corner}\\
&= 0,\ 0,\ -1{+}i,\ -2,\ -3{-}i,\ -2{-}4i,\ 1{-}5i,\ 6{-}4i,\ \dots && -\,\mathrm{Re} = \mathrm{A077870}
\end{aligned}
$$

JEQ_k is the centre of the square so a Gaussian integer plus $\tfrac{1}{2}{+}\tfrac{1}{2}i$. The combination of b^k and $(-1)^k$ give real and imaginary parts a multiple of 5 leaving fraction $\tfrac{1}{2}$.

Proof. The following diagrams show how the end of the join expands. J is the end of the join (at $N = JN_k$). Q is the centre of the boundary square and is the JEQ_k to be calculated.

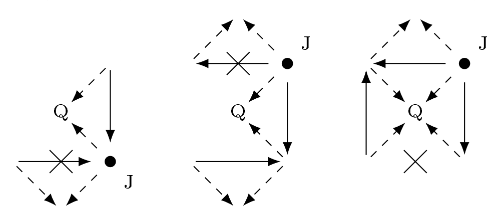
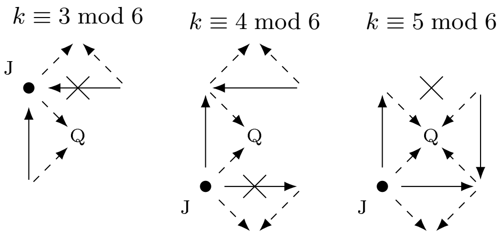


Figure 30: join end square expansions



The dashed lines are the expansion on the right of each line segment. X is the centre of the new boundary square. The expansions rotate by +45° for multiplying by b. For example the L shape of $k\equiv 0$ expands to a U shape pointing diagonally upward and then rotates +45° to be left in $k\equiv 1$.

For $k\equiv 2$, the join gains a new unit square and the Q point is the new join point J for $k\equiv 3$. Likewise $k\equiv 5$ rotated by 180°. Each of $k\equiv 3,4,5$ are 180° rotations of $k\equiv 0,1,2$ respectively. At $k\equiv 5$ the expansion returns to the pattern and orientation of $k\equiv 0$.

The distance Q to X is alternately up or down $\frac{1}{2}i$ so

$$\begin{aligned} JEQ_k &= b\left(JEQ_{k-1} + \tfrac{1}{2}i\,(-1)^k\right) \\ &= b^k\, JEQ_0 \;-\; b\tfrac{1}{2}i\sum_{j=0}^{k-1} b^j(-1)^{k-1-j} \\ &= b^k\,\left(\tfrac{1}{2}+\tfrac{1}{2}i\right) \;-\; b\tfrac{1}{2}i\,\frac{b^k-(-1)^k}{b-(-1)} \end{aligned} \qquad \square$$

Second Proof of Theorem 31. From figure 25 the join end square is given by the square 3 levels down, and with that level located at $\frac{1}{2}b^k$ and turned 180°

$$\begin{aligned} JEQ_k &= \tfrac{1}{2}b^k - JEQ_{k-3} \\ &= \tfrac{1}{2}b^k - \tfrac{1}{2}b^{k-3} + \tfrac{1}{2}b^{k-6} - \cdots \pm \tfrac{1}{2}b^{3\,\mathrm{or}\,4\,\mathrm{or}\,5} \mp JEQ_{0\,\mathrm{or}\,1\,\mathrm{or}\,2} \end{aligned} \qquad \square$$

The same argument can be made with the lower-left corner point JEC_k. The diagrams of figure 30 would have the corner point either unchanged or $+i$, instead of $\pm\frac{1}{2}i$ as for JEQ_k.

Figure 30 also show how the join endpoint goes anti-clockwise around the corners of the boundary square. The top-right corner is repeated in $k\equiv 1,2$. The bottom-right corner is repeated in $k\equiv 4,5$. So a period-6 pattern around the 4 corners.

The boundary square type goes in a pattern 2, 3e, 3o, repeating.

The above 3-repeat pattern is similar to *RQhalf* from theorem 20. The *RQ* half-way is in fact a join end,

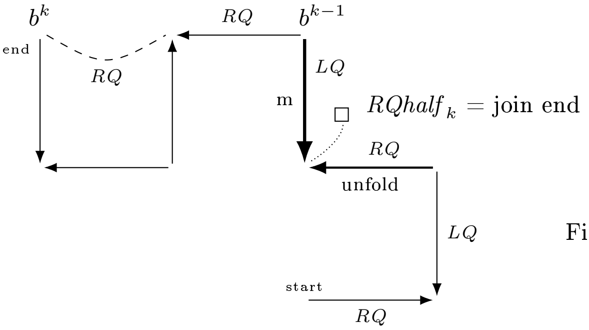


Figure 31

m is a $k{-}3$ sub-curve. The join end between it and the "unfold" sub-curve is $RQhalf_k$ because there is the same number of boundary squares before and after. Before and after are both two RQ and one LQ, less JA which is non-boundary. The U part is RQ per theorem 17.

$$RQ_k = 2\Big(2RQ_{k-3} + LQ_{k-3} - (JA_{k-3}{+}1)\Big) + 1 \qquad k \geq 3$$
$$RQhalf_k = b^{k-1} - i.JEQ_{k-3}$$

5 Points

Lemma 2. *Consider a path on a square grid which does not repeat any segment and which traverses all four sides of any enclosed unit square.*

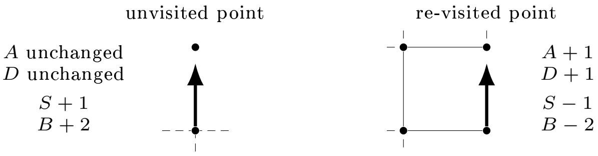
The number of single-visited points S, double-visited points D, enclosed area A and boundary length B are related

$$D = A \qquad \textit{double-visited = area} \tag{168}$$
$$S = B/2 + 1 \qquad \textit{single-visited and boundary} \tag{169}$$

The starting point of the path can be revisited. If it is then that point is double-visited.

Proof. A path of no line segments is taken to be a single point at its start. The relations hold with $D = A = B = 0$ and $S = 1$.

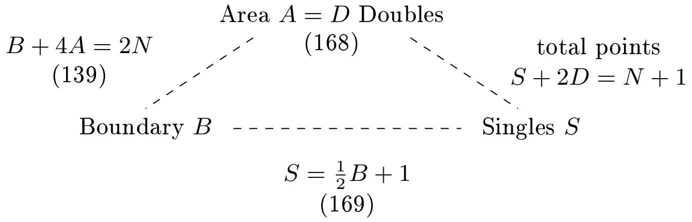
When a further line segment is added to the end of the path it either goes to an unvisited point or it re-visits a point,



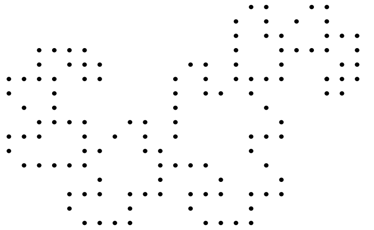
On going to an unvisited point a new single is added so $S{+}1$, and D unchanged. No new area is enclosed so A unchanged. The boundary increases by 2 (one on each side of the new line) so $B{+}2$. These new values satisfy the relations.

On re-visiting, a single-visited point becomes a double, so $S{-}1$ and $D{+}1$. A new unit square is enclosed so $A{+}1$. The boundary changes by -3 enclosed and $+1$ new outside so $B{-}2$. These new values satisfy the relations.

It also suffices here to show just one of (168) or (169) then the other follows from $4A + B = 2N$ of lemma 1 with segments N being $N{+}1$ points as singles and doubles $N{+}1 = S + 2D$. $\square$



5.1 Single and Double Points



dragon $k = 8$
single-visited points
$S_8 = 123$

Theorem 32. *The number of single-visited points in dragon curve k is*

$$S_k = BQ_k \quad = \tfrac{1}{2}B_k + 1 \qquad \text{A003476}$$

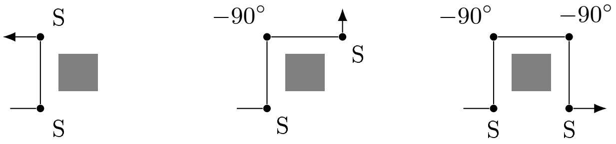
and double-visited points

$$D_k = \tfrac{1}{2}(2^k + 1 - S_k) \quad = A_k \qquad \textit{doubles = area} \qquad \text{A003230}$$

Proof. The dragon curve does not repeat any segment and always traverses all four sides of any unit square so lemma 2 applies, giving $D_k = A_k$ and $S_k = \frac{1}{2}B_k + 1$.

For $S_k = BQ_k$, all single-visited points are on the boundary since they have only two segments going to them, or one segment at the start and end of the curve. So each single-visited point is at the corner of some boundary square.

At a single-visited point the boundary may turn either left +90° or right −90°. The first and last corner of a boundary square are where the curve turns away from that square. This is a +90° turn since the boundary never goes straight ahead (as from figure 15).



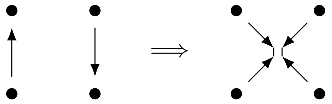
So there is a +90° single-visited between each boundary square.

The 2 and 3-side boundary squares also have corners with −90° turns. If those points are single-visited then they are on the opposite side boundary. Following the curve around anti-clockwise they are +90° on that side and so fall between boundary squares and are counted there. So all single-visited points are between boundary squares on one or the other side of the curve, giving $S_k = BQ_k$. ☐

Second Proof of Theorem 32. Single and double points can also be counted by how line segments expand into adjacent unit squares.

Every line segment expands to a new vertex. Visits to the original vertices are unchanged so the new ones add to the counts of singles and doubles.

A new double is formed when there are opposing right sides,



opposing sides expand to
new double-visited point

This occurs in 4-sided enclosed unit squares A_k and in *BQ3e* type squares as from figure 20. There are no 2-sided squares with opposing sides since it would be impossible for a subsequent unfold or adjacent curve to traverse both the absent sides without overlap when filling the plane (theorem 1 and theorem 2).

The new vertices are odd points, being odd n (and odd $x{+}y$). So new *Dodd* points in k, and all existing points becoming even, are

$$\begin{aligned}
\mathit{Dodd}_k &= \begin{cases} 0 & \text{if } k = 0 \\ A_{k-1} + \mathit{BQ3e}_{k-1} & \text{if } k \ge 1 \end{cases} \quad = AL_k \\
\mathit{Deven}_k &= \begin{cases} 0 & \text{if } k = 0 \\ D_{k-1} & \text{if } k \ge 1 \end{cases} \\
D_k &= \mathit{Deven}_k + \mathit{Dodd}_k \quad = \sum_{j=0}^{k} \mathit{Dodd}_j \quad = A_k
\end{aligned} \tag{170}$$

$\mathit{Dodd} = AL$ follows from $\mathit{BQ3e} = JA$ as at (151) and relation JA to AL at (149) of theorem 25. Then cumulative sum of AL is per (147) for A_k. Single-visited points follow by difference. ☐

Singles can be taken odd and even too. New singles *Sodd* are where boundary squares 1e, 2, 3o expand to a new single within the square. Expansion to a new point outside the square might be single-visited too, but if so then it is in a boundary square on the opposite side of the curve, so counted there.

$$\begin{aligned}
\mathit{Sodd}_k &= \begin{cases} 1 & \text{if } k{=}0 \\ \mathit{BQ1e}_{k-1} + \mathit{BQ2}_{k-1} + \mathit{BQ3o}_{k-1} & \text{if } k \ge 1 \end{cases} \quad \text{odd singles} \\
&= LQ_k \\
\mathit{Seven}_k &= \begin{cases} 1 & \text{if } k = 0 \\ S_{k-1} & \text{if } k \ge 1 \end{cases} \quad \text{even singles}
\end{aligned}$$

$$S_k = Seven_k + Sodd_k \quad = Seven_0 + \sum_{j=0}^{k} Sodd_j \tag{171}$$

S_0 through S_3 are $S_k = 2^k + 1$ since for $k \le 3$ all points are singles. For $k \ge 4$, some points double so there are fewer singles.

S_k count is odd for $k \ge 1$ since $S + 2D =$ total points means it has the same parity as $2^k + 1$. Any double point is made by taking away from the total points what would have been 2 singles.

When existing singles and doubles are preserved in expansion to $2n$, they gain an extra low 0-bit on their point numbers. The sums (170),(171) are therefore over how many singles or doubles there in level k with a given number of low 0-bits. Terms $Dodd_k$ and $Sodd_k$ are odd n, terms $Dodd_{k-1}$ and $Sodd_{k-1}$ for how many with a single low 0-bit, etc.

Geometrically those m low zero bits make the point locations a multiple of b^m, as from *point* formula (66). So the sums are also over locations which are odd $x{+}y$, etc.

Theorem 33. *For the first few curve levels there are more single-visited points than double-visited, but doubles come to dominate*

$$D_k > S_k \qquad \text{iff } k \ge 11$$

If each double-visited point is counted as 2, for total $2D_k + S_k = 2^k + 1$ *vertices then those which are part of a double exceed those which are singles,*

$$2\,D_k > S_k \qquad \text{iff } k \ge 8$$

Proof. Seeking $D_k > S_k$ means $\frac{1}{2}(2^k + 1 - S_k) > S_k$ so want

$$S_k \;\le\; \tfrac{1}{3}2^k \tag{172}$$

Suppose this is true at $k{-}1$ and $k{-}3$. Applying the $S = BQ$ recurrence of theorem 32 to those bounds shows it true at k too

$$S_k = S_{k-1} + 2S_{k-3} \;\le\; \tfrac{1}{3}2^{k-1} + 2\tfrac{1}{3}2^{k-3} \;= \tfrac{1}{4}2^k$$

(172) is first true at $k{=}11, 12, 13$ and so is then true of all $k \ge 11$.

Similarly for $2D_k > S_k$ which becomes $S_k \le \frac{1}{2}2^k$ and first true at $k = 8, 9, 10$. □

In general this sort of question when and whether $D_k - S_k > 0$ is a linear recurrence positivity problem. In this case an easy one since a single largest root 2.

The *other*(n) procedure of section 1.5 identifies single and double points by whether *other*(n) is on the same arm for double-visited or different arm for single-visited.

$$Spred_k(n) = \begin{cases} 1 & \text{if } other(n) \text{ different arm or } > 2^k \\ 0 & \text{otherwise} \end{cases}$$

$$Dpred_k(n) = \begin{cases} 1 & \text{if } other(n) \text{ same arm and} \le 2^k \\ 0 & \text{otherwise} \end{cases}$$

$$Dpred = 1 - Spred \qquad \text{opposites}$$

Dpred is 1 at both n and $other(n)$, so half is D,

$$S_k = \sum_{n=0}^{2^k} Spred_k(n) \qquad D_k = \tfrac{1}{2} \sum_{n=0}^{2^k} Dpred_k(n) \tag{173}$$

These sums can be calculated from the bit fields of $other(n)$ from figure 13. This is more complicated than the proofs of theorem 32 above, but gives a combinatorial interpretation to S and D.

Dpred is bits in the form of figure 13. Low 0-bits, lowest 1-bit and t are a run of ≥ 2 bits of 2 possible types ($t = 0 \,\text{or}\, 1$). Each flip run and t above it is a run of ≥ 3 bits of 2 types (run of 0s or 1s). These runs and types are the same as *TwoRuns* from section 3.4.

$$D_k = \tfrac{1}{2} \sum_{l=5}^{k+1} 2^{k-l}\, TwoRuns(l)$$

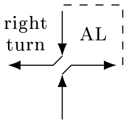
l is the number of bits comprising the runs up to and including the $\neq t$ bit. The further $k-l$ bits above are arbitrary so factor 2^{k-l}.

There are only k bits of n but the sum extends to $l = k+1$ since when $t=1$ the $\neq t$ bit is 0 and is satisfied by high 0s understood above the top of n. This does not apply for $t=0$ where the $\neq t$ bit would be 1. So $\frac{1}{2} TwoRuns(l)$ when $l = k+1$. The 2^{k-l} factor gives that $\frac{1}{2}$.

At a double-visited point, the curve turns either left or right. The turn is *BitAboveLowestOne* which is t in figure 13.

$$DpredLeft_k(n) = Dpred_k(n) \text{ and } turn(n) = 1$$
$$DpredRight_k(n) = Dpred_k(n) \text{ and } turn(n) = -1$$

A double with a right turn encloses area on the left of the curve since the curve must eventually curl around to revisit the point and it cannot encircle the curve origin (that would overlap the 4-arm plane filling of theorem 2). Similarly a double with a left turn encloses area on the right of the curve.



double-visited point
right turn encloses
square on left of curve

Each such double corresponds to an enclosed unit square as per lemma 2, so similar to (173)

$$AR_k = \tfrac{1}{2} \sum_{n=0}^{2^k} DpredLeft_k(n) \qquad AL_k = \tfrac{1}{2} \sum_{n=0}^{2^k} DpredRight_k(n)$$

The predicates can be applied to the curve continued infinitely by asking just for $other(n)$ in the same arm for double-visited, or different arm for single-visited.

$$\begin{aligned}
\mathit{Spred}_\infty(n) &= \begin{cases} 1 & \text{if } \mathit{other}(n) \text{ different arm} \\ 0 & \text{otherwise} \end{cases} \\
&= 1,1,1,1,1,1,1,0,1,1,1,0,1,1,0,0,1,1,1,0,\dots \\
\text{=1 at } \ n &= 0\dots6,\ 8,9,10,\ 12,13,\ 16,17,18,\ 20,\ 23,24,\ \dots
\end{aligned}$$

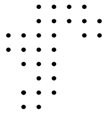
$$\begin{aligned}
\mathit{Dpred}_\infty(n) &= \begin{cases} 1 & \text{if } \mathit{other}(n) \text{ same arm} \\ 0 & \text{otherwise} \end{cases} \\
&= 0,0,0,0,0,0,0,1,0,0,0,1,0,0,1,1,0,0,0,1,\dots \\
\text{=1 at } \ n &= 7,\ 11,\ 14,15,\ 19,\ 21,22,\ 25,\ 28,\ 30,31,\ 35,\ \dots
\end{aligned}$$

Spred_∞ has runs of at most 3 consecutive single-visited points after the initial 7 of n=0 to 6. The start and end of each blob are double-visited and have just 2 singles between. Within a blob, a single-visited is on the boundary and there are at most 3 consecutive singles (a hanging square) since all segments of the enclosed area are traversed.

The number of visits to the location of an n is 1 for *Spred* or 2 for *Dpred*,

$$\begin{aligned}
\mathit{Visits}_k(n) &= \mathit{Dpred}_k(n) + 1 \qquad (174) \\
\mathit{Visits}_\infty(n) &= \mathit{Dpred}_\infty(n) + 1 \\
&= 1,1,1,1,1,1,1,2,1,1,1,2,1,1,2,2,1,1,1,2,1,\dots
\end{aligned}$$

5.2 Distinct Points



$k=5$
distinct visited points
$P_5 = 29$

The number of distinct points visited by the dragon curve k is singles plus doubles.

$$\begin{aligned}
P_k &= S_k + D_k \qquad (175) \\
&= 2^k + 1 - A_k \qquad \text{with doubles = area} \qquad (176) \\
&= \begin{cases} 2,3,5,9,16 & \text{for } k=0 \text{ to } 4 \\ 4P_{k-1} - 5P_{k-2} + 4P_{k-3} - 6P_{k-4} + 4P_{k-5} & \text{for } k \ge 5 \end{cases} \\
&\qquad \text{same recurrence as } A_k \text{ area, different initial values} \\
&= 2,3,5,9,16,29,54,101,190,361,690,1325,2558,\dots
\end{aligned}$$

$$\text{Generating function} \quad gP(x) = \tfrac{1}{2}\left(\frac{1}{1-2x} + \frac{1}{1-x} + \frac{2+x+2x^2}{1-x-2x^3} \right)$$

The generating function has the same form as $gA(x)$ partial fractions (141) except for sign of the cubic term. That term is S_k, as in the relations

$$\begin{aligned}
A_k &= \tfrac{1}{2}(2^k + 1 - S_k) \qquad \text{from } S + 2D = 2^k + 1 \qquad (177) \\
P_k &= \tfrac{1}{2}(2^k + 1 + S_k) \qquad \text{and (176)} \qquad (178)
\end{aligned}$$

If there were no singles then it would be D doubles = P distinct = $\frac{1}{2}(2^k+1)$

half total points. Every 2 singles reduces the doubles by 1 and increases the distinct points by 1 (as +2 singles, −1 double).

As noted above, S_k is the same parity as the total points $2^k + 1$ so (177), (178) are integers.

P and A relation (176) is Euler's formula for regions of a connected planar graph. Vertices are the points P, edges are the 2^k segments, and regions are A enclosed unit squares.

$$\textit{vertices} + \textit{inside regions} = \textit{edges} + 1$$

Points are at locations $x+y$ even or odd. Even locations are a multiple of b so are the expansions of the preceding level points. The odd locations are the new points.

$$\begin{aligned}
\textit{Peven}_k &= \textit{Seven}_k + \textit{Deven}_k = \begin{cases} 1 & \text{if } k{=}0 \\ P_{k-1} & \text{if } k \geq 1 \end{cases} \\
\textit{Podd}_k &= \textit{Sodd}_k + \textit{Dodd}_k = \begin{cases} 1 & \text{if } k{=}0 \\ dP_{k-1} & \text{if } k \geq 1 \end{cases}
\end{aligned}$$

$$\begin{aligned}
dP_k &= P_{k+1} - P_k && (179) \\
&= 2^k - \big(A_k + \textit{BQ3e}_k\big) && (180) \\
&= 2^k - dA_k \qquad \text{where } dA = A_{k+1} - A_k && (181) \\
&= 1, 2, 4, 7, 13, 25, 47, 89, 171, 329, 635, 1233, \ldots
\end{aligned}$$

At (180), there is a new point in $k{+}1$ between of the each 2^k segments of k, except in each enclosed unit square and each *BQ3e* type boundary square the opposing segments touch on expansion, so reducing new points.

Each such touch is a new enclosed unit square, so (181). That follows also simply from difference of P form (176) for $k{+}1$ and k.

5.3 Point Differences

Theorem 34. *The distinct differences* $\big|n - \textit{other}(n)\big|$ *in dragon curve* k *occur at the join points. The sum of all these differences is*

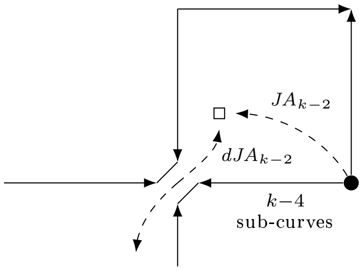
$$\begin{aligned}
\textit{Odistinct}_k &= \sum\nolimits_{d=0}^{4\,JND_{k-1}} d.\textit{Opred}(d) \\
&= \begin{cases} 0 & \textit{if } k{=}0 \\ \frac{1}{68} 2^k \Big(2JA_k - 13JA_{k+1} + 9JA_{k+2} - 1\Big) - \frac{4}{17} & \textit{if } k{\geq}1 \end{cases} && (182) \\
&= 0, 0, 0, 0, 4, 12, 28, 124, 444, 1340, 4668, \ldots
\end{aligned}$$

$$\textit{gOdistinct}(x) = \frac{4x^4}{(1-x)\,(1-2x-2(2x)^3)} \qquad (183)$$

Proof. The join is the only place new double-visited points occur on unfolding, so differences elsewhere are replications of differences occurring in joins of previous levels.

In curve k, the join of previous levels is JA_{k-1}. In the manner of figure 25, and with join increment from theorem 28, JA_{k-1} comprises its preceding JA_{k-2}

and increment dJA_{k-2}



dJA_{k-2} is symmetric each side of the new touching point. The $\big|n - other(n)\big|$ differences are offsets from the difference 4.2^{k-4} around the big square shown. The differences forward and backward cancel out, leaving just 4.2^{k-4} for each point there, which is dJA_{k-2} many. So

$$\begin{aligned} Odistinct_k &= Odistinct_{k-1} + 4.2^{k-4}.dJA_{k-2} \qquad \text{for } k \geq 4 \\ &= \sum_{j=0}^{k-2} 2^j \, dJA_j \end{aligned}$$

This sum is generating function (183). The cubic part there is $gdJA(2x)$ which is the power 2^j, and factor $\frac{1}{1-x}$ for cumulative sum.

Recurrence or generating function manipulations give (182). □

$Odistinct_k$ at (182) uses JA with a view to taking a mean, since there are JA_{k-1} many distinct differences. The mean distinct difference n to $other(n)$, as a fraction of the total length 2^k of the curve, is

$$\frac{Odistinct_k}{2^k.JA_{k-1}} \to \tfrac{1}{68}\big(2r - 13r^2 + 9r^3\big) = \frac{1}{r^2+4} = 0.145451\ldots$$

The inverted $1/(r^2{+}4)$ is by dividing, or by an identity,

$$2^k.JA_{k-1} = \tfrac{1}{4}\,Odistinct_{k+2} + 4\,Odistinct_k - 2^{k-1} + 1$$

The distinct differences range from 4 up to maximum $4\,JND_{k-1}$ at the end of the join per theorem 27. As a fraction of the length 2^k this maximum is

$$\frac{4\,JND_{k-1}}{2^k} \to \frac{2}{7} = 0.285714\ldots \qquad \text{A021039}$$

The mean difference as a fraction of this maximum difference is then

$$\frac{Odistinct_k}{JA_{k-1}\,.\,4\,JND_{k-1}} \to \frac{7}{2r^2+8} = 0.509081\ldots$$

The total of all differences in curve k follows from the distinct differences, since all double-visited points arise from joins. This total takes a pair of visits n_1, n_2 to a point just one way around, not also n_2, n_1.

$$Oall_k = \sum_{n=0}^{2^k} n - other(n) \quad \text{when } n \geq other(n) \text{ same arm}$$

$$\begin{aligned}
&= Odistinct_k + 2\,Oall_{k-1} \quad = \sum_{j=0}^{k} 2^j\ Odistinct_{k-j} && (184)\\
&= \tfrac{1}{68}\,2^k\Big(8JA_{k+2} - 4JA_{k+1} - 2JA_k - 16\Big) - \tfrac{4}{17}\\
&= 0,0,0,0,4,20,68,260,964,3268,11204,\ldots
\end{aligned}$$

$$gOall(x) = \frac{4x^4}{(1-x)\,(1-2x)\,(1-2x-2(2x)^3)}$$

This generating function is a further factor $\frac{1}{1-2x}$ on *gOdistinct* to sum descending powers of 2 at (184), similar to *gA* at (153).

$$gOall(x) = \frac{1}{1-2x}\,gOdistinct(x)$$

The number of differences in *Oall* is the number of double-visited points D_k. *Oall* grows only as *JA* so a mean difference $Oall_k/D_k \to 0$.

Another measure of the differences can be made by considering how many bits are flipped. On the same set of join points as *Odistinct* above, the number of bits flipped is

$$\begin{aligned}
OdistinctX_k &= \sum_{n=0}^{2^{k-1}} CountOneBits\big(\mathrm{BITXOR}(n, other(n))\big)\\
&\qquad\qquad \text{where } other(n) > 2^{k-1} \text{ and same arm}\\
&= \tfrac{1}{2}\sum_{m=1}^{k-1}\ \sum_{t=0}^{m-3-2t\,\geq\,0} \big(m{-}t{-}1\big)\,2^{t+1}\binom{m{-}3{-}2t}{t} && (185)\\
&= \tfrac{1}{58}\Big(-(k{+}3)JA_k - (16k{+}6)JA_{k+1} + (19k{-}20)JA_{k+2} + k\Big) && (186)\\
&= 0,0,0,0,2,5,9,22,48,91,179,352,\ldots
\end{aligned}$$

$$gOdistinctX(x) = \frac{2x^4 - x^5}{(1-x)\,(1-x-2x^3)^2}$$

For example in $k{=}4$ there is one double-visited point $n = 7, 11$. These are binary 111 and 1011 which differ at 2 bit positions, so $OdistinctX_4 = 2$.

The binomial sum (185) is the possible bit patterns for $other(n)$ ending with the high bit flipped so n in the first half and $other(n)$ in the second. The bit runs are m bits comprising $t{+}1$ runs, each of which has one unflipped bit so $m{-}t{-}1$ bits flipped, and binomial ways to arrange the runs.

Form (186) is by recurrence or generating function manipulations, and using *JA* since there are $k.JA_{k-1}$ total bits in the points, giving a limit for the proportion flipped

$$\begin{aligned}
\frac{OdistinctX_k}{k.JA_{k-1}} &\to \tfrac{1}{58}\big(-r - 16r^2 + 19r^3\big) \quad = \tfrac{1}{58}\big(38 - r + 3r^2\big)\\
&= 0.774651\ldots \quad \text{bit flip proportion}
\end{aligned}$$

Total bits flipped by all n to $other(n)$ pairs in level k are, similar to (184),

$$
\begin{aligned}
\mathit{OallX}_k &= \sum_{n=0}^{2^k} \mathit{CountOneBits}\big(\mathrm{BITXOR}(n, \mathit{other}(n))\big) \\
&\qquad \text{where } n > \mathit{other}(n) \text{ and same arm} \\
&= \mathit{OdistinctX}_k + 2\,\mathit{OallX}_{k-1} \quad = \sum_{j=0}^{k} 2^j\ \mathit{OdistinctX}_{k-j} \\
&= 3.2^k - \tfrac{1}{4} - \tfrac{1}{116}\Big((38k{+}46)\mathit{JA}_k{+}({-}k{+}39)\mathit{JA}_{k+1}{+}(3k{+}22)\mathit{JA}_{k+2}\Big) \\
&= 0, 0, 0, 0, 2, 9, 27, 76, 200, 491, 1161, 2674, \ldots
\end{aligned}
$$

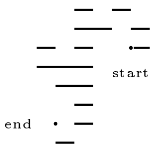
$$
g\mathit{OallX}(x) = \frac{1}{1-2x}\, g\mathit{OdistinctX}(x) \ = \frac{2x^4 - x^5}{(1-2x)\,(1-x)\,(1-x-2x^3)^2}
$$

OallX grows as its 2^k term, so among double-visited points, the mean number of bits flipped to go to $\mathit{other}(n)$ has limit

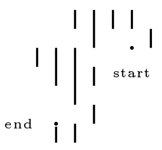
$$
\frac{\mathit{OallX}_k}{D_k} \to 6 \qquad \text{bits flipped}
$$

5.4 Lines

Some line segments in the dragon curve are consecutive and they can be considered in runs making horizontal and vertical lines.



$\mathit{Horizontals}_5 = 12$ lines



$\mathit{Verticals}_5 = 12$ lines

$k{=}5$
lines

The end of each line is a single-visited point. Since the curve always turns left or right 90°, every single-visited point is an end of a vertical and an end of a horizontal, except the curve start and end. The curve start is only an end of a horizontal since the first segment is horizontal. The curve end for $k \geq 1$ is only the end of a vertical since the last segment is vertical (an unfold of the first segment).

$$
\begin{aligned}
\mathit{Horizontals}_k &= \begin{cases} 1 & \text{if } k = 0 \\ \tfrac{1}{2}(S_k - 1) \ = \tfrac{1}{4}B_k & \text{if } k \geq 1 \end{cases} \\
&= 1, 1, 2, 4, 7, 12, 21, 36, 61, 104, \ldots \qquad\qquad k{\geq}1 \quad \tfrac{1}{2}\,\text{A164395} \\
\mathit{Verticals}_k &= \begin{cases} 0 & \text{if } k = 0 \\ \mathit{Horizontals}_k & \text{if } k \geq 1 \end{cases} \\
\mathit{Lines}_k &= \mathit{Horizontals}_k + \mathit{Verticals}_k \\
&= S_k - 1 \ = \tfrac{1}{2}B_k \\
&= 1, 2, 4, 8, 14, 24, 42, 72, 122, 208, \ldots \qquad\qquad \text{A164395}
\end{aligned}
$$

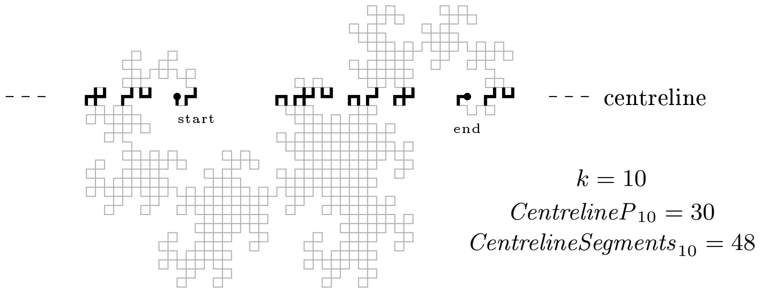
The number of horizontals and verticals are equal for $k \geq 1$ since they use each single-visited point and the respective curve ends the same ways.

Equality can also be seen from the unfolding. Points where the curve touches on unfolding have both a horizontal and a vertical on each side so reduce the same. If there's the same horizontals and verticals in the preceding level then the unfolding doubles them both and the touching reduces them both. There is one join point for each join square JA_k.

$$\begin{aligned} \mathit{Horizontals}_k &= \mathit{Horizontals}_{k-1} + \mathit{Verticals}_{k-1} - JA_{k-1} \\ \mathit{Verticals}_k &= \mathit{Horizontals}_{k-1} + \mathit{Verticals}_{k-1} - JA_{k-1} \end{aligned}$$

5.5 Centreline

The line passing through curve start and end is the "centreline". The curve has various segments with one or other end on that line, and some segments entirely on it when k even.



Point n is on the centreline of curve k when its imaginary part is 0 after rotating back for curve start to end horizontal.

$$\mathit{CentrelinePointPred}_k(n) = \begin{cases} 1 & \text{if } \operatorname{Im} \mathit{point}(n).\overline{\omega_8}^k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{187}$$

$$\begin{aligned} k{=}0 &\quad = 1,1 \\ k{=}1 &\quad = 1,0,1 \\ k{=}2 &\quad = 1,0,0,1,1 \end{aligned}$$

Segments touch the centreline when either end is on the line,

$$\begin{aligned} \mathit{CentrelineSegmentPred}_k(n) = {} & \mathit{CentrelinePointPred}_k(n) \\ & \text{or } n < 2^k \text{ and } \mathit{CentrelinePointPred}_k(n{+}1) \end{aligned}$$

$$\begin{aligned} k{=}0 &\quad = 1 \\ k{=}1 &\quad = 1,1 \\ k{=}2 &\quad = 1,0,1,1 \end{aligned}$$

Theorem 35. *$\mathit{CentrelineSegmentPred}_k(n)$ segments of dragon curve k are characterized by the following state machine on n written in k many bits taken high to low. An n ever reaching "non" is not on the centreline.*

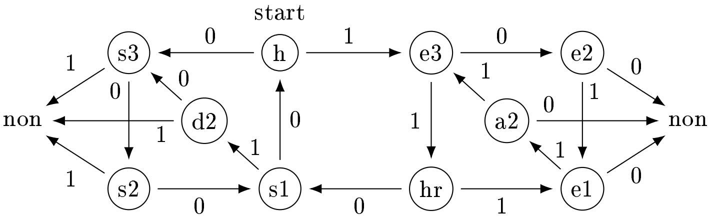
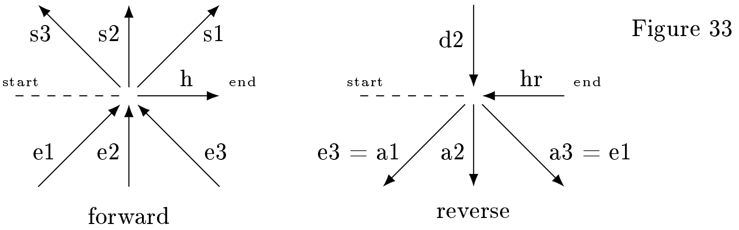
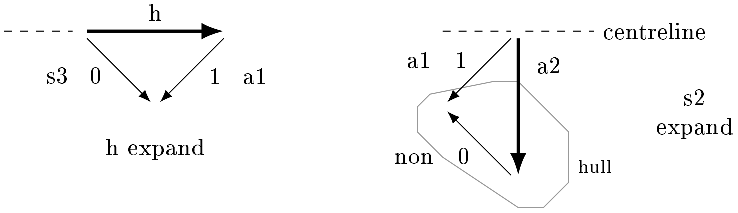


Figure 32: n segment on centreline, bits high to low

Proof. The states are segments in the following orientations relative to the centreline.



Arrows are in direction of expansion, so all expand on the right. Forward segments are forward along the curve (even n) so their expansion is bits 0,1. Reverse (odd n) are reverse along curve so 1,0. For example,



For h, bit 0 is orientation s3, or rather s3 rotated 180°. All states include their 180° rotation since the centreline is unchanged by 180° rotate. Bit 1 is a1.

For a2, bit 0 is end-most since a2 is a reverse state. The expanded segment at 0 is "non". A segment which doesn't have an end on the centreline will never touch the centreline on further expansion. This is since all the centreline segments and orientations of figure 33 are from expansion of a $k-1$ segment which was touching the centreline. A non-touching never produces a touching.

Or alternatively, the "non" is too far away from the centreline. Any further expansions of it are within the convex hull (ahead in section 7), shown drawn around the segment, scaled to that sub-curve length. Working through the hull vertex formulas shows the hull never extends to the centreline. Similarly other "non" expansions.

State a1 expands to the same new states as e3 does, so a1 and e3 are combined as e3 in the state machine. Similarly a3 expands to the same as e1 and they are combined as e1. ☐

The number of n in each state follows by mutual recurrences on the transitions. The total not in "non" is

$$\begin{aligned} \mathit{CentrelineSegments}_k &= \begin{cases} 1 & \text{if } k{=}0 \\ [\tfrac{3}{2}, 2] \,.\, 2^{\lfloor k/2 \rfloor} & \text{if } k \geq 1 \end{cases} \\ &= 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 64, \ldots \qquad \text{A029744} \end{aligned}$$

Segments h,s1,s2,s3 and reverses hr,d2 (arrow end) have their n start on the centreline. Reckoning only those as accepting is *CentrelinePointPred* (187).

$$\mathit{CentrelinePointPred}_k(n) = \text{centreline states h,s1,s2,s3, hr,d2}$$

Counting these states, and +1 for the curve endpoint, is number of n visits to the centreline. This is visits in the sense that double-visited points are counted for both n.

$$\begin{aligned} \mathit{CentrelineVisits}_k &= 2^{\lfloor n/2 \rfloor} + 1 \\ &= 2, 2, 3, 3, 5, 5, 9, 9, 17, 17, \ldots \qquad \text{A051032} \end{aligned}$$

For k even, this count can also be derived from h and hr states. Together they are

$$\begin{aligned} \mathit{CentrelineH}_k &= \begin{cases} 1 & \text{if } k{=}0 \\ \tfrac{1}{2} \,.\, 2^{\lfloor k/2 \rfloor} & \text{if } k \text{ even} \geq 2 \\ 0 & \text{if } k \text{ odd} \end{cases} \qquad \text{h or hr} \\ &= 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, \ldots \qquad k{\geq}2 \ \text{A077957} \end{aligned}$$

The curve always turns left or right, so each point which is a turn on the centreline is start or end of h or hr. Curve start and end are not turns. Curve start is h (so counted) when $k \equiv 0 \bmod 4$ but not when $k \equiv 2 \bmod 4$. Curve end is opposite, hr when $k \equiv 2 \bmod 4$ and not when $k \equiv 0 \bmod 4$, $k \geq 4$. So +1 for all even $k \geq 2$.

$$\mathit{CentrelineVisits}_k = 2 \, \mathit{CentrelineH}_k + 1 \qquad k \text{ even} \geq 2 \tag{188}$$

Single and double visited points on the centreline can be distinguished by some state machine manipulations for intersection of *CentrelinePointPred* and *Spred*. The result as high to low is 34 states, or is simplified a little by reversing low to high. Mutual recurrences on the transitions give counts of singles. Counts of doubles follow by subtracting from *CentrelineVisits* at (189). *CentrelineD* is number of double-visited locations, like the whole curve D.

$$\begin{aligned} \mathit{CentrelineS}_k &= [\tfrac{5}{3}, \tfrac{5}{3}, 3, 3] \,.\, 3^{\lfloor k/4 \rfloor} && \text{single-visited} \\ &= 2, 2, 3, 3, 5, 5, 9, 9, 15, 15, 27, 27, \ldots && k{\geq}2 \text{ dup A083658} \\ \mathit{CentrelineD}_k &= \tfrac{1}{2} \big(2^{\lfloor k/2 \rfloor} + 1 - \mathit{CentrelineS}_k \big) && \text{double-visited} \qquad (189) \\ &= 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 3, 3, \ldots \\ \mathit{CentrelineP}_k &= \mathit{CentrelineS}_k + \mathit{CentrelineD}_k && \text{total points} \\ &= \tfrac{1}{2} \big(2^{\lfloor k/2 \rfloor} + 1 + \mathit{CentrelineS}_k \big) \\ &= 2, 2, 3, 3, 5, 5, 9, 9, 16, 16, 30, 30, \ldots \end{aligned}$$

The centreline just from curve start to end has $2^{\lfloor k/2 \rfloor}+1$ points, so for curve scaled to unit length start to end, a limit measure for the amount of centreline in the curve, both within and beyond curve start to end, is

$$\frac{\mathit{CentrelineP}_k}{2^{\lfloor k/2 \rfloor}+1} \to \frac{1}{2}$$

In the manner of section 5.4, for even k, some curve segments are consecutive and so form contiguous lines. Each end of such a line on the centreline is a *CentrelineS* single, except curve start and end when perpendicular so as at (188) but -1.

$$\begin{aligned} \mathit{CentrelineLines}_k &= \begin{cases} 1 & \text{if } k{=}0 \\ \tfrac{1}{2}\big(\mathit{CentrelineS}_k - 1\big) & \text{if } k \text{ even} \geq 2 \\ 0 & \text{if } k \text{ odd} \end{cases} \qquad \text{contiguous} \\ &= 1, 0, 1, 0, 2, 0, 4, 0, 7, 0, 13, 0, \ldots \end{aligned}$$

CentrelineH is power $4^{k/4}$ but *CentrelineS* is only $3^{k/4}$ so a mean line length grows

$$\begin{aligned} \frac{\mathit{CentrelineH}_k}{\mathit{CentrelineLines}_k} &= [\tfrac{3}{5}, 0, \tfrac{2}{3}, 0] \,.\, (\tfrac{4}{3})^{\lfloor k/4 \rfloor} \qquad\qquad k \text{ even} \geq 2 \\ &\quad + \frac{1}{[\frac{25}{9}, 0, 3, 0] \,.\, (\frac{3}{2})^{\lfloor k/2 \rfloor} - [\frac{5}{3}, 0, \frac{3}{2}, 0] \,.\, (\frac{3}{4})^{\lfloor k/4 \rfloor}} \\ &= 1, 1, 1, 1, \tfrac{8}{7}, \tfrac{16}{13}, \tfrac{16}{11}, \tfrac{8}{5}, \tfrac{128}{67}, \tfrac{256}{121}, \ldots \end{aligned}$$

The centreline state machine figure 32 also identifies points or segments on lines at ±45° or 90° from curve start by starting in states s1,s2,s3. s3 has its target centreline at +45°, so starting in that state answers for a line at −45° relative to the segment. Similarly s1 having the centreline at +135°. And similarly curve end and the e states.

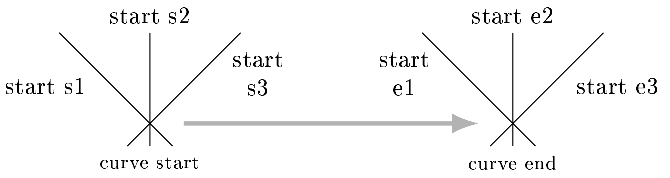
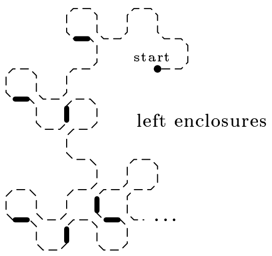
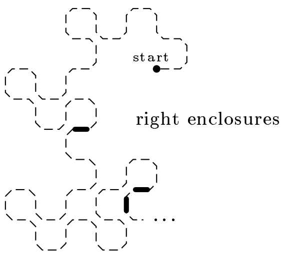


Figure 34: starts for other lines

6 Enclosure Sequence

As each segment is successively appended to the dragon curve it may enclose a new unit square on the right or left of the curve, or not.





In the manner of lemma 2, a new enclosed unit square is formed when a point is re-visited. So a segment enclosing a unit square has a second-visit point at its end. A second visit in the bit patterns of figure 13 has highest run 1000 which is flipped to 0111 for the first visit. Other run flips are arbitrary.

$$
\begin{aligned}
&\mathit{DpredFirst}_k(n) \ = \mathit{Dpred}_k(n) \text{ and } n < \mathit{other}(n) \\
&\mathit{DpredFirst}_\infty(n) \ = \mathit{Dpred}_\infty(n) \text{ and } n < \mathit{other}(n) \\
&\qquad = 0,0,0,0,\ 0,0,0,1,\ 0,0,0,0,\ 0,0,1,1,\ 0,0,0,0,\ldots \\
&\qquad =1 \text{ at } \ n = \ \ 7,14,15,21,28,30,31,39,42,45,53,\ldots \\
&\mathit{DpredSecond}(n) \ = \mathit{Dpred}_\infty(n) \text{ and } n > \mathit{other}(n) \\
&\qquad = 0,0,0,0,\ 0,0,0,0,\ 0,0,0,1,\ 0,0,0,0,\ 0,0,0,1,\ldots \\
&\qquad =1 \text{ at } \ n = 11,19,22,25,35,38,43,44,49,50,57,\ldots
\end{aligned}
$$

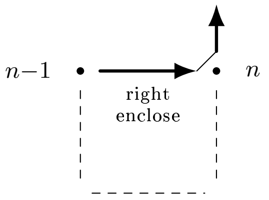
$\mathit{DpredFirst}_k$ is first visit to a point which will be re-visited within level k. $\mathit{DpredFirst}_\infty$ is first visit to a point which will eventually be re-visited by the curve continued infinitely, which for n of bit length k means a revisit either within the same k, or in $k{+}1$ across the join.

For *DpredSecond*, no distinction is needed between a level k and continuing infinitely since the other visit precedes n.

Total through to 2^k is the number of double-visited points D_k,

$$D_k = \sum_{n=0}^{2^k} \mathit{DpredFirst}_k(n) \quad = \sum_{n=0}^{2^k} \mathit{DpredSecond}(n)$$

At each second-visit, the curve turns either left or right. When it turns left it is away from a unit square just enclosed on the right. When it turns right it is away from a unit square just enclosed on the left. The turn is never to the same side as the square as that would overlap a side of that square.



turn left at right side enclosure
or would overlap segment
of square just enclosed

Second-visit with turn left or right is

$$\mathit{DpredSecondL}(n) = \mathit{DpredSecond}(n) \text{ and } \mathit{turn}(n) = +1$$

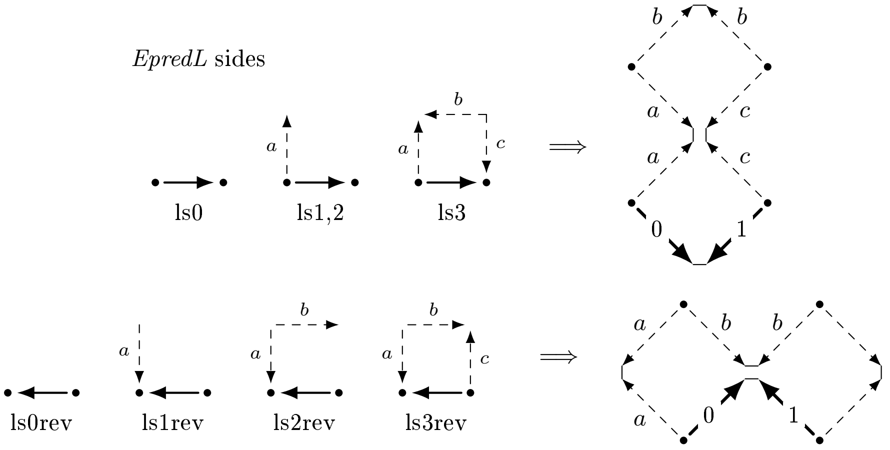
turn to left, encloses on right

$$
\begin{aligned}
&=\!1 \text{ at } \; n = 25, 49, 50, 57, 89, 97, 98, 100, 113, 114, 121, \ldots \\
&\mathit{DpredSecondR}(n) = \mathit{DpredSecond}(n) \text{ and } \mathit{turn}(n) = -1 \\
&\qquad \text{turn to right, encloses on left} \\
&=\!1 \text{ at } \; n = 11, 19, 22, 35, 38, 43, 44, 67, 70, 75, 76, \ldots
\end{aligned}
$$

The totals of these are left and right side areas

$$
AL_k = \sum_{n=0}^{2^k} \mathit{DpredSecondR}(n) \qquad AR_k = \sum_{n=0}^{2^k} \mathit{DpredSecondL}(n)
$$

An explicit calculation of enclosure bit patterns can also be made by considering how segments expand. On the left side a line segment may have $0, 1, 2$ or 3 segments. Configuration *ls0* has no side segments. This is initial segment n=0.



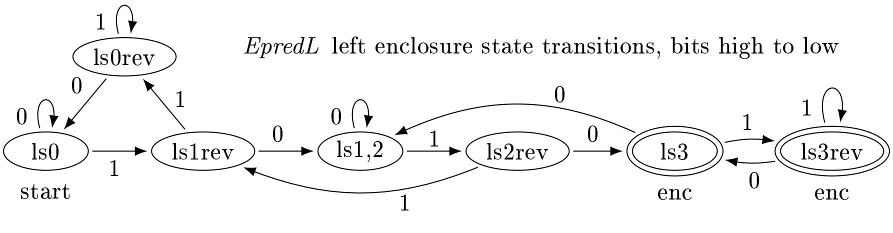
Side c cannot occur without b since that would require the curve to curl around on the right to reach c, and that would overlap 4-arm plane filling (theorem 2). Likewise b cannot occur without a.

The arrows show the direction of segment expansion. An odd n segment is the reverse direction. Each expansion is a new low bit on n. The sides are on the left of the curve in each case.

Configurations *ls1* and *ls2* expand the same. They differ only in side b and it is not a side of either 0 or 1 on expansion.

Each side a, b, c, when present, precedes the segment n. The new configuration for the 0-bit segment does not include the 1-bit segment since that segment is after.

The state transitions on expansion are as follows. Both *ls3* and *ls3rev* are where segment n encloses a unit square on the left.


EpredL left enclosure state transitions, bits high to low

Usual DFA state machine manipulations can reverse to match bits from low to high.

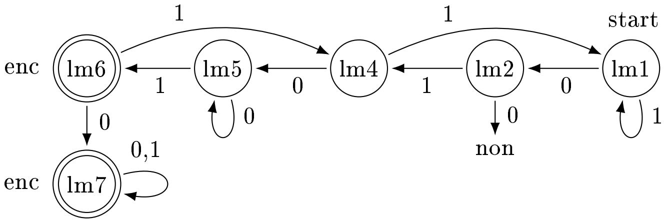


Figure 35: *EpredL* bits low to high

States *lm6* and *lm7* are both a segment enclosing a square on the left. A high 0-bit above the top of n could be understood to send *lm6* to *lm7* if a single enclosing state is desired.

These state transitions are based on segment numbers with the first segment numbered 0. Point $n{+}1$ is the segment end so

$$\begin{aligned} \mathit{EpredL}(n) &= \mathit{DpredSecondR}(n{+}1) \\ &=1 \text{ at } \ n = 10, 18, 21, 34, 37, 42, 43, 66, 69, 74, 75, \ldots \end{aligned}$$

The bit pattern of $\mathit{other}(n{+}1)$ from figure 13 can be seen in the low-to-high states of figure 35. $\mathit{other}(n{+}1)$ seeks the bit above lowest 1 in $n{+}1$, which means bit above lowest 0 in n. This is at *lm2*. A 1-bit is required there for right turn. A 0-bit is a left turn so not a left-side enclosure and it goes to ‘‘non’’ for not-enclosed.

The loops match 1000 or 0111 bit runs with $t{=}1$ above. 0111 is the same as the initial bit above lowest 0. The loop ending *lm6* is the 1000 case and only it gives an enclosing segment *lm6*, being the bigger of the two visits to the point.

Up to 3 unit squares can be enclosed consecutively on the same side. There cannot be more than 3 or the turns away would make segments overlap.

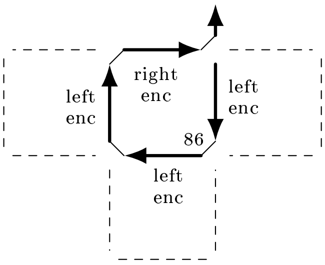
Some state machine manipulations can test whether $n{+}1$ is also the respective left or right enclosure, and intersection $n, n{+}1, n{+}2$ for a triple. Taking that low to high shows they are the original digit forms with certain fixed extra low bits. *EpredR* even means the bits there must be *EpredR* and lowest bit 0.

$$\mathit{EpredR3} = \begin{array}{|l|c|} \multicolumn{1}{l}{\scriptstyle\text{high}} & \multicolumn{1}{r}{\scriptstyle\text{low}} \\ \hline \quad\mathit{EpredR}\qquad\qquad & 1111 \\ \hline \end{array} \qquad \mathit{EpredL3} = \begin{array}{|l|c|} \multicolumn{1}{l}{\scriptstyle\text{high}} & \multicolumn{1}{r}{\scriptstyle\text{low}} \\ \hline \quad\mathit{EpredL}\quad & 101 \\ \hline \end{array}$$

$$\text{or} \ \begin{array}{|l|c|} \hline \quad\mathit{EpredR}\text{ even} & 0111 \\ \hline \end{array}$$

$$\begin{aligned} \mathit{EpredR3}(n) &= \mathit{EpredR}(n) \text{ and } \mathit{EpredR}(n{+}1) \text{ and } \mathit{EpredR}(n{+}2) \\ &=1 \text{ at } \ n = 391, 399, 775, 783, 799, 903, 911, 1415, \ldots \\ \mathit{EpredL3}(n) &= \mathit{EpredL}(n) \text{ and } \mathit{EpredL}(n{+}1) \text{ and } \mathit{EpredL}(n{+}2) \\ &=1 \text{ at } \ n = 85, 149, 173, 277, 301, 341, 349, 533, \ldots \end{aligned}$$

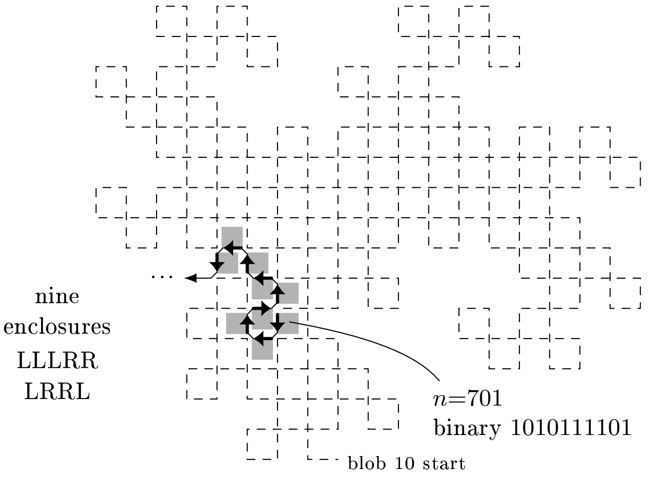
The next segment encloses the square between those 3 on the other side.



3 left enclosures
are 3 right turns
so next segment
is a right enclosure

Runs of left and right enclosures can occur. A run of 9 enclosures is the longest which occurs. A run of 10 or more do not occur since the intersection of 10 predicates n through $n{+}9$ is empty.

The first run of 9 enclosures occurs starting at segment $n{=}701$. The following diagram is where it occurs within the part of blob 10 up to that point (blobs ahead in section 12).



$$\begin{aligned} \mathit{EpredNine}(n) &= \text{all } \mathit{Epred}(n) \text{ through } \mathit{Epred}(n{+}8) \\ &=1 \text{ at } \ n = 701, 1213, 1405, 2237, 2429, 2749, 2813, 4285, \ldots \end{aligned}$$

State machine manipulations forming *EpredNine* as an intersection show also that the enclosure sides (and corresponding turns) are always the same sequence LLLRR LRRL.

The number of runs of 9 within a given level k follows from the state machine intersection too by the transitions as mutual recurrences on how many strings in each state,

$$\mathit{EncNine}_k = \begin{cases} 0 & \text{if } k \le 7 \\ \left.\begin{array}{l} \frac{7}{3}2^{k-9} + \frac{1}{8}n - \frac{1}{96}[61, 83, 61, 35] \\ \qquad + \frac{1}{32}\big(dJA_{k+1} + dJA_k\big) \end{array}\right\} & \text{if } k \ge 8 \end{cases} \tag{190}$$
$$= \text{(10 zeros)}, 1, 3, 7, 17, 40, 90, 197, 424, 900, 1889, \ldots$$

The last point of a level k is single-visited so the last segment is non-enclosing and a run of enclosures does not extend across levels. The cubic part at (190) is expressed using *dJA* only for convenience. The same cubic part arises ahead in *BlobBnohang* (369), but without obvious geometric relationship to that.

The coefficient of the 2^k term is the limit for how many 9s occur. Each of them is 9 unit squares so as a fraction of total area this is

$$\frac{9\,\mathit{EncNine}_k}{A_k} \to \frac{21}{2^8} = \frac{21}{256} = 0.08203125$$

Let $A(n)$ be the area enclosed by the first n many segments, and $AL(n)$, $AR(n)$ similarly area enclosed on the left and right sides. These are total *Epred*,

$$A(n) = AL(n) + AR(n) = \sum_{j=0}^{n-1} \mathit{Epred}(j) \;= \sum_{j=0}^{n} \mathit{DpredSecond}(j)$$
$$AL(n) = \sum_{j=0}^{n-1} \mathit{EpredL}(j) \qquad AR(n) = \sum_{j=0}^{n-1} \mathit{EpredR}(j)$$

The 2^k segments of level k are

$$A(2^k) = A_k \qquad AL(2^k) = AL_k \qquad AR(2^k) = AR_k$$

The area of the first n segments can be calculated efficiently by following the enclosure states. For $AL(n)$, let $AL_k(\mathit{state})$ be the area enclosed on the left side of a level k curve in configuration *state* of *EpredL*. The adjacent sub-curves of each state add join area or boundary squares so

$$\begin{aligned} AL_k(\mathit{ls0}) &= AL_k & AL_k(\mathit{ls0rev}) &= AR_k \\ AL_k(\mathit{ls12}) &= AL_k + JA_{k+1} & AL_k(\mathit{ls1rev}) &= AR_k + JA_k \\ AL_k(\mathit{ls3}) &= AL_k + LQ_k & AL_k(\mathit{ls2rev}) &= AR_k + JA_{k+1} \\ & & AL_k(\mathit{ls3rev}) &= AR_k + RQ_k \end{aligned} \tag{191}$$

For example, $AL_k(\mathit{ls3})$ is the left area AL_k plus all of the left boundary squares LQ because side *ls3* is entirely enclosed. *ls1rev* is the *JA* join area configuration. *ls12* is the join after one expansion.

At a 1-bit of n, $AL(n)$ counts the whole of the sub-curve with a 0-bit there, with forward/reverse and surrounds according as the *EpredL* state there. Then remaining n is the bits of n below there, likewise in their respective *EpredL* states. So for $n = a_{k-1}a_{k-2}\cdots a_1a_0$ binary bits,

$$\begin{aligned} AL(n) = {} & a_{k-1}\,.\,AL_{k-1}(\mathit{ls0}) && \text{high bit} \\ & + a_{k-2}\,.\,AL_{k-2}(\mathit{EpredLstate}(a_{k-1}0)) \\ & + a_{k-3}\,.\,AL_{k-3}(\mathit{EpredLstate}(a_{k-1}a_{k-2}0)) \\ & + \cdots \\ & + a_0 \quad .\,AL_0(\mathit{EpredLstate}(a_{k-1}a_{k-2}\cdots a_2a_1 0)) && \text{low bit} \end{aligned} \tag{192}$$

$AR(n)$ can be written in a corresponding way from similar *EpredR* states.

At (143), the limit area of the whole curve scaled to a unit length is $\frac{1}{2}$. The same limit holds within a level too.

Theorem 36. *The proportions of enclosing segments have limits*

$$\lim \frac{AL(n)}{n} \to \lim \frac{AR(n)}{n} \to \frac{1}{4} \qquad \lim \frac{A(n)}{n} \to \frac{1}{2} \tag{193}$$

Proof. Among the AL area by *EpredL* states (191), the smallest is either $AL_k = \frac{1}{4}.2^k - \frac{1}{4}L_k$ or $AR_k = \frac{1}{4}.2^k - \frac{1}{4}R_k$. Working through the L, R cubics shows R is the bigger subtraction so

$$AR_k \le AL_k \tag{194}$$

A lower bound for $AL(n)$ can be formed from (192) by reducing each of its area terms to this minimum $AL(\mathit{ls0rev}) = AR_j$. The sum of each $\frac{1}{4}2^j$ part of AR_j gives $\frac{1}{4}n$, then less the $\frac{1}{4}R_j$ parts. A simple lower bound is to subtract all those possible R, irrespective which bits of n actually set, so

$$AL(n) \ge \tfrac{1}{4}n - \sum_{j=0}^{k-1} \tfrac{1}{4}R_j \ = \tfrac{1}{4}n + \tfrac{1}{2}k + \tfrac{1}{4} - \tfrac{1}{8}R_{k+2} \tag{195}$$

$$\frac{AL(n)}{n} \ge \tfrac{1}{4} - \frac{R_{k+2}}{n} \ \ge \tfrac{1}{4} - \frac{R_{k+2}}{2^{k-1}}$$

The sum to R_{k+2} at (195) follows by R is $\frac{1}{2}L$ (114) and sum L is R (107). Then R grows only as r^k so that $R_{k+2}/2^{k-1} \to 0$.

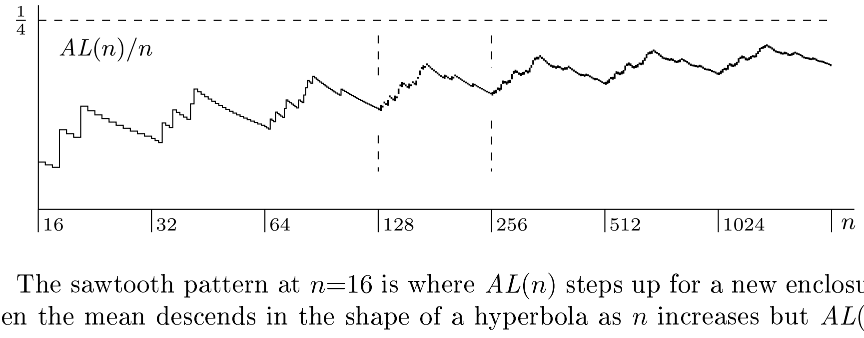
An upper bound on $AL(n)$ follows in a similar way, but taking the biggest of the state areas. Working through the cubics shows this is *ls3rev*, but they all differ from $\frac{1}{4}2^k$ by at most a cubic which $/2^k \to 0$.

The same argument holds for $AR(n)$ which has similar quantities and bounds. $A(n)$ is their sum. $\square$

Knowing AR smaller at (194) is not needed if took $B = L + R$ for the subtraction. It sums to B which again grows only as a cubic.

Roughly speaking, convergence at (193) depends on the enclosure sequence being uniform enough that $A(n)$ etc don't go too far away within blocks. It's possible for a sequence to have mean $\frac{1}{2}$ in 2^k blocks, but not converge. For example if all A enclosures were in the first half of each 2^k block then the middle would be always exactly $\frac{2}{3}$. In the case of A, every 2^j sub-block encloses at least AR_j so total enclosures only ever depart from the overall mean by a cubic.

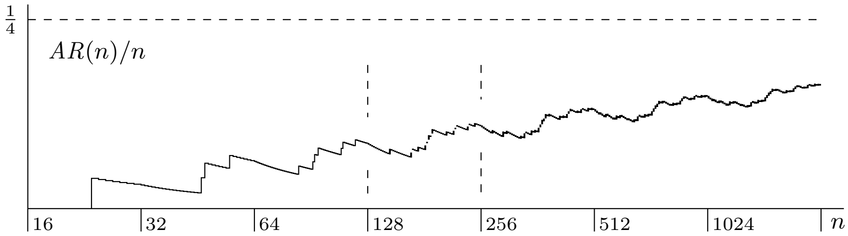
The way $AL(n)/n \to \frac{1}{4}$ can be illustrated in a plot. Blocks of $n = 2^k$ to $2.2^k{-}1$ are scaled to the same width in order to show successive refinements.



The sawtooth pattern at $n{=}16$ is where $AL(n)$ steps up for a new enclosure then the mean descends in the shape of a hyperbola as n increases but $AL(n)$

unchanged. Each run of non-enclosures is a descent this, but in log scale they are soon too small to see.

AR can be illustrated similarly. It differs in dropping to a low within each 2^k block whereas AL rises to a peak.



7 Convex Hull

A convex hull is the smallest convex polygon which can be drawn around a given set of points. Benedek and Panzone[7] show that the convex hull around the dragon curve fractal is a 10-sided polygon. The convex hulls around finite iterations k=6 onwards are 10-sided polygons too.

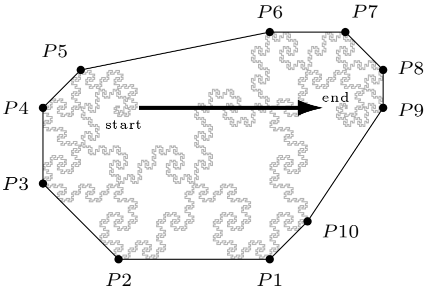


Figure 36: convex hull for $k \ge 6$

Vertices P1 to P10 are numbered per Benedek and Panzone. Curve expansion here is on the right here whereas they expand on the left, so figure 36 is a vertical mirror image of their drawing.

Theorem 37 (Finite form of Benedek and Panzone)**.** *The convex hull around dragon curve level $k \ge 6$ is a set of 10 vertices*

$$\begin{aligned}
P1(k) &= -\tfrac{1}{3}\left(b^{k+3} + p(k{+}3)\right) && \textit{for } k \ge 6 && (196)\\
P2(k) &= -\tfrac{1}{3}\left(b^{k+2} + p(k{+}2)\right)\\
P3(k) &= -\tfrac{1}{3}\left(b^{k+1} + p(k{+}1)\right)\\
P4(k) &= -\tfrac{1}{3}\left(b^{k} \quad + p(k) \quad\right)\\
P5(k) &= -\tfrac{1}{3}\left(b^{k-1} + p(k{-}1)\right)\\
P6(k) &= \tfrac{1}{3}\left(\ (2{+}i)b^{k} \quad + i\,p(k{+}1)\right) && = b^k - i\,P3(k) && (197)\\
P7(k) &= \tfrac{1}{3}\left(\ (3{+}i)b^{k} \quad + i\,p(k) \quad\right) && = b^k - i\,P4(k) && (198)\\
P8(k) &= \tfrac{1}{3}\left((3{+}4i)b^{k-1} + i\,p(k{-}1)\right) && = b^k - i\,P5(k) && (199)
\end{aligned}$$

$$
\begin{aligned}
P9(k) &= \tfrac{1}{3}\big(\quad 7i\; b^{k-2} + i\,p(k{-}2)\big) \\
P10(k) &= \tfrac{1}{3}\big(\ (4{+}i)b^{k-1} +\ p(k) \qquad\big) \\
b &= 1{+}i \\
p(m) &= -1,\ -b,\ -b^2,\ -b^3{-}3,\ 1,\ b,\ b^2,\ b^3{+}3 \\
&\quad \textit{for } m \equiv 0 \text{ to } 7 \bmod 8
\end{aligned}
$$

Re = A118831

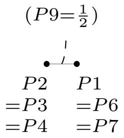
For $k < 6$, the above points are the hull vertices but with some duplications and some points excluded.

k	*vertices*	*duplication*	*exclude*
0	*2*	*$P2{=}P3{=}P4$ and $P1{=}P6{=}P7$ ($P9$ on boundary)*	$P5, P8, P10$
1	*3*	*$P1{=}P2{=}P10$ and $P3{=}P4{=}P5$ and $P6{=}P7{=}P8$*	$P9$
2	*4*	*$P1{=}P10$ and $P2{=}P3$ and $P4{=}P5$ and $P7{=}P8{=}P9$ ($P6$ on boundary)*	
3	*6*	*$P3{=}P4$ and $P8{=}P9$ ($P10$ on boundary)*	
4	*7*	*$P1{=}P10$ and $P4{=}P5$ and $P7{=}P8$*	
5	*9*	*$P8{=}P9$*	

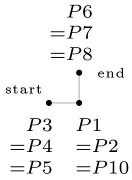
The vertices are all Gaussian integers, each being a point visited by the curve. The various b^k and $p(k)$ terms result in multiples of 3 so factor $\frac{1}{3}$ gives integers.

The fractal hull vertices of Benedek and Panzone are limits $\overline{P1(k)/b^k} \to (2+2i)/3$ etc as $k{\to}\infty$. (Conjugates due to left rather than right expansion noted above.)

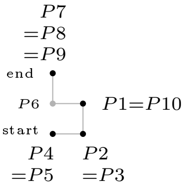
Proof. For k=0 to k=5, the convex hulls and point formulas are as follows. In k=0 the hull is empty. $P9(0){=}\frac{1}{2}$ is on its boundary but not a vertex.



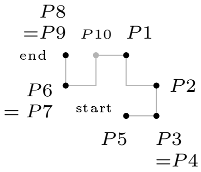
k=0



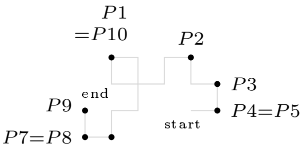
k=1



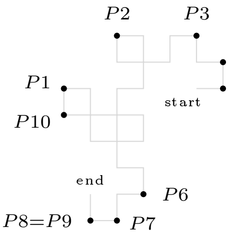
k=2



k=3



k=4



k=5

For k=6, the hull is calculated explicitly and the vertices are per the formulas and are all distinct.

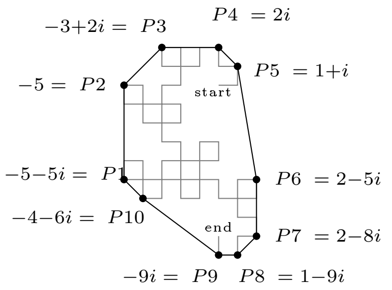


Figure 37:
k=6
convex hull

Proceed then by induction. Suppose the formulas are true of level $k-1$. Level k comprises level $k-1$ and an unfolded copy of $k-1$. The convex hull around level k is the hull around the hulls of those two copies.

The unfolding is shown in figure 38 below. O is the origin. $E = b^k$ is the endpoint of level k. $M = b^{k-1}$ is the midpoint where the two level $k-1$ copies meet. O to M is the first copy with vertices P1a etc. E to M is the unfolded copy with vertices P1b etc.

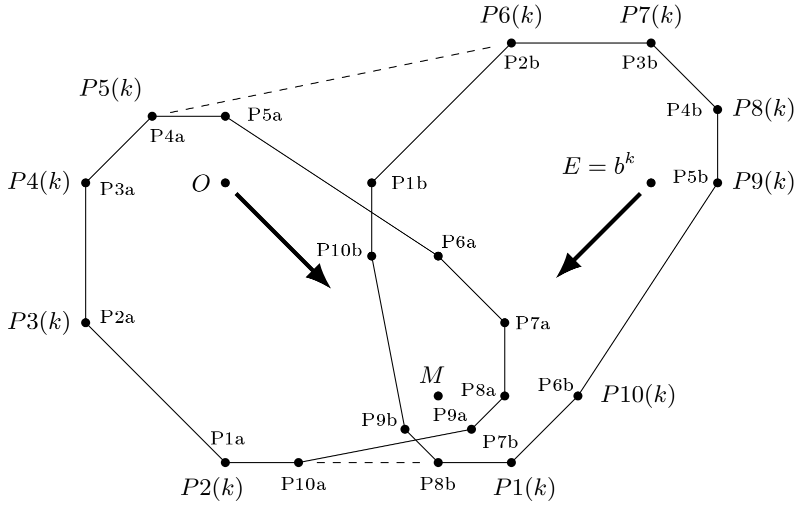


Figure 38: convex hull k from two unfolded copies of $k-1$

For the top $P5(k)$--$P6(k)$ shown dashed, the sub-hull side P4a--P5a relative to the b^k direction is

$$\frac{P5(k-1) - P4(k-1)}{b^k} \;=\; \frac{1}{6} + \frac{p(k-1) - p(k-2)}{3\,b^k} \qquad k \geq 7 \qquad (200)$$

and this $p(k)$ difference has imaginary part

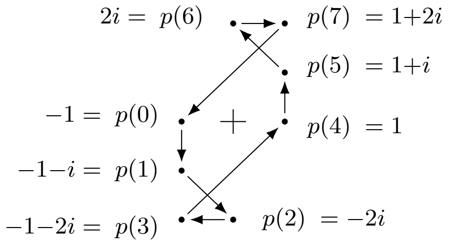
$$\operatorname{Im} \frac{p(k-1) - p(k-2)}{b^k} \;=\; 0 \qquad \text{all } k \qquad (201)$$

So side P4a--P5a is horizontal. Point $P2b$ is above that line since

$$\operatorname{Im} \frac{P6(k) - P5(k)}{b^k} = \tfrac{1}{6} + \tfrac{1}{3}\,[1,-1,-1,1]\,(\tfrac{1}{2})^{\lceil k/2 \rceil} \quad > 0 \qquad k \geq 7$$

so the hull for level k is $P4a$ to $P2b$.

As a remark, (201) can be illustrated by



The direction of each arrow is a difference $p(k)-p(k-1)$. At each point the direction turns $+45°$. For example $p(4)$ to $p(5)$ is vertical and turns $+45°$ to go to $p(6)$. At $p(6)$ the turn is again $+45°$ to become horizontal but with a 180° negation so horizontal to the right rather than the left. Such 180° negations are at $p(2)$, $p(3)$, $p(6)$ and $p(7)$.

These $+45°$ turns correspond to $+45°$ from each b in the denominator. The numerator starts $0°$ at $p(-1)-p(-2)$ (which is $p(7)-p(6)$) then $+k{\times}45°$. The denominator is $k{\times}45°$, for net $0°$.

For the bottom side of figure 38 shown dashed, the points P1a, P10a, P8b and P7b relative to the b^k endpoint are

$$\begin{aligned} P10(k{-}1)/b^k &= \tfrac{1}{6} - \tfrac{2}{3}i \ + \ \tfrac{1}{3}\,p(k{-}1)/b^k \\ P1(k{-}1)/b^k &= \quad\ - \tfrac{2}{3}i \ + \ \tfrac{1}{3}\,p(k{-}2)/b^k \\ P8'(k{-}1)/b^k &= \tfrac{1}{2} - \tfrac{2}{3}i \ + \ \tfrac{1}{3}\,p(k{-}2)/b^k \\ P7'(k{-}1)/b^k &= \tfrac{2}{3} - \tfrac{2}{3}i \ + \ \tfrac{1}{3}\,p(k{-}1)/b^k \end{aligned} \tag{202}$$

By (201) the imaginary parts are all equal. So these four points are on a horizontal line and the hull for level k is the ends $P1a$ and $P7b$.

So the level k hull points are given from the $k{-}1$ points by mutual recurrences.

$$\begin{aligned} P1(k) &= P7'(k{-}1) &&= b^k - i\,P7(k{-}1) \\ P2(k) &= \ P1(k{-}1) \\ P3(k) &= \ P2(k{-}1) \\ P4(k) &= \ P3(k{-}1) \\ P5(k) &= \ P4(k{-}1) \\ P6(k) &= P2'(k{-}1) &&= b^k - i\,P2(k{-}1) \\ P7(k) &= P3'(k{-}1) &&= b^k - i\,P3(k{-}1) \\ P8(k) &= P4'(k{-}1) &&= b^k - i\,P4(k{-}1) \\ P9(k) &= P5'(k{-}1) &&= b^k - i\,P5(k{-}1) \\ P10(k) &= P6'(k{-}1) &&= b^k - i\,P6(k{-}1) \end{aligned} \tag{203}$$

The power forms (196) of the theorem satisfy these recurrences and so completes the induction. The power forms can be found by writing these recurrences in generating functions and solving simultaneously with some linear algebra, or directly from expanding. The chain of dependencies is

$$\begin{tikzcd}
\mathrm{P9} \arrow[r] & \mathrm{P5} \arrow[r] & \mathrm{P4} \arrow[r] & \mathrm{P3} \arrow[r] & \mathrm{P2} \arrow[d] & \mathrm{P6} \arrow[l] & \mathrm{P10} \arrow[l] \\
& \mathrm{P8} \arrow[ur] & & \mathrm{P7} \arrow[u] & \mathrm{P1} \arrow[l] & &
\end{tikzcd}$$

It's convenient to begin at $P1(k)$ and expand to reach $P1(k{-}4)$ again,

$$\begin{aligned}
P1(k) &= b^k - iP7(k{-}1) \\
&= b^k - i\big(b^{k-1} - iP3(k{-}2)\big) \\
&= b^k - i\big(b^{k-1} - iP2(k{-}3)\big) \\
&= b^k - i\big(b^{k-1} - iP1(k{-}4)\big) \\
P1(k) &= b^{k-1} - P1(k{-}4) \qquad\qquad k \ge 10 \qquad (204)
\end{aligned}$$

Apply (204) repeatedly until reaching $k = 6, 7, 8 \,\mathrm{or}\, 9$. Let this be $q \ge 0$ many times so that $k{-}6 = 4q + r$ with $0 \le r \le 3$ and ending at $P1(6{+}r)$.

$$\begin{aligned}
P1(k) &= b^{k-1} - b^{k-5} + b^{k-9} - \cdots b^{r+9} \;\; + (-1)^q P1(6{+}r) \\
&= (-1)^q P1(6{+}r) \;\; + b^{r+9} \sum_{j=0}^{q-1} (b^4)^{q-1-j}\,(-1)^j \qquad (205) \\
&= (-1)^q P1(6{+}r) \;+\; b^{r+9}\, \frac{(b^4)^q - (-1)^q}{b^4 - (-1)} \\
&= -\tfrac{1}{3}\Big(b^{k+3} \quad - b^{r+9}(-1)^q - 3\,(-1)^q P1(6{+}r)\Big) \qquad (206) \\
&\qquad \text{since } b^4 - (-1) = -3 \quad \text{and } r{+}9 + 4q = k{+}3
\end{aligned}$$

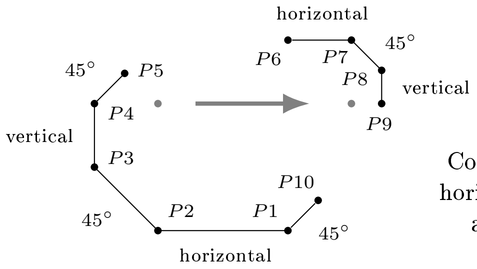
In (205), when $q{=}0$ the sum is taken as empty so 0. b^{r+9} is the lowest power of b in the sum since ending at $P1(6{+}r)$ means the corresponding b power per (204) is $6{+}r + 3$.

In (206), the right hand part is periodic in $r = 0, 1, 2, 3$ and q odd or even. It uses the initial $P1(6)$ through $P1(9)$. $P1(6)$ is the base case in figure 37. The others can be calculated from the recurrences (203) or by explicitly forming those level hulls. The result is the 8 terms of $p(k)$.

These terms could be numbered starting anywhere mod 8. The choice here is to number matching the b power in each $P1$ etc. So the expression in (206) is reckoned as $p(k{+}3)$ to match its b^{k+3}.

$$p(k{+}3) \;=\; -b^{r+9}(-1)^q \;-\; 3\,(-1)^q P1(6{+}r) \qquad \square$$

Eight of the ten hull sides are always vertical, horizontal or 45° diagonal.



Convex hull sides horizontal, vertical and diagonal

This can be seen in k=6 and thereafter from the way each level hull is 45° rotations of the preceding level. Algebraically P6--P7 and P1--P2 are horizontal since

$$\begin{aligned} \operatorname{Im} \frac{P6(k) - P7(k)}{b^k} &= \operatorname{Im} \frac{-1}{3} + i\frac{p(k+1) - p(k)}{3b^k} &&= 0 \qquad (207) \\ \operatorname{Im} \frac{P1(k) - P2(k)}{b^k} &= \operatorname{Im} \frac{2}{3} - \frac{p(k+3) - p(k+2)}{3b^k} &&= 0 \end{aligned}$$

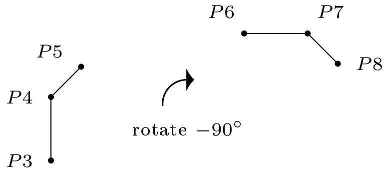
P3--P4 is vertical since it's a factor of i on P6--P7 per the cross forms of (197),(198). P8--P9 is vertical since

$$\operatorname{Re} \frac{P8(k) - P9(k)}{b^k} = \operatorname{Re} \frac{i}{6} + i\frac{p(k{-}1) - p(k{-}2)}{3b^k} = 0$$

P4--P5 is diagonal per (200). P7--P8 and P1--P10 are diagonal per (202). P2--P3 similarly

$$\operatorname{Re} \frac{P2(k) - P3(k)}{b^{k+1}} = \operatorname{Re} \frac{-1}{3}i - \frac{p(k+2) - p(k+1)}{3b^{k+1}} = 0 \qquad (208)$$

The cross formulas $P6(k) = b^k - i\,P3(k)$ etc in (197)(198)(199) show P6--P7--P8 is a shift and 90° rotate of P3--P4--P5. Geometrically this is simply the previous level unfolding of points P2--P3--P4 (figure 38).

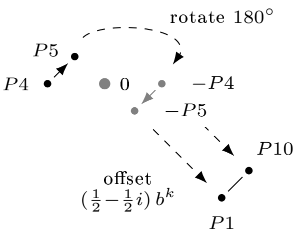


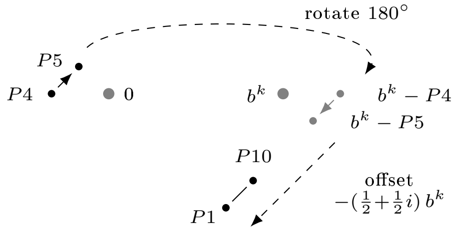
Benedek and Panzone have further correspondences $P6 = R(P2)$ through $P10 = R(P6)$ which are a shrink in addition to shift and rotate 45°. The same correspondences exist in finite iterations but are not exact, only approached as $k{\to}\infty$. For example P7 and P3 are related by shift b^k, rotate and shrink $-b/2$, but an additional period-8 offset.

$$\begin{aligned} P7(k) &= b^k - \tfrac{1}{2}b\,P3(k) \; + \mathit{offset}(k) \\ \mathit{offset}(k) &= \tfrac{1}{6}b\Big(b\,p(k) - p(k{+}1)\Big) \\ &= 0, 0, \tfrac{b}{2}, \tfrac{\overline{b}}{2}, 0, 0, \tfrac{-b}{2}, \tfrac{-\overline{b}}{2} \qquad \text{for } k \equiv 0 \,\text{to}\, 7 \bmod 8 \end{aligned}$$

The sides P4--P5 and P1--P10 are the same length and are at a rotation by 180° and offset.

$$\begin{aligned} P10(k) &= -P4(k) \;+\; (\tfrac{1}{2}-\tfrac{1}{2}i)\, b^k \qquad (209)\\ P1(k) &= -P5(k) \;+\; (\tfrac{1}{2}-\tfrac{1}{2}i)\, b^k \\ P5(k) - P4(k) &= P10(k) - P1(k) \qquad \text{lengths} \end{aligned}$$





Theorem 38. *Number the points of the dragon curve starting from n=0 at the origin. The point numbers of the convex hull vertices are*

$$\begin{aligned} P1N(k) &= \tfrac{8}{15}\,2^k + \tfrac{1}{15}\,[7,-1,-2,-4] \qquad (210)\\ &= 1,1,2,4,9,17,34,68,137,273,546,1092,\ldots \qquad \text{A115451}\\ &= \textit{binary } 1000\ldots1000 \textit{ ending } 1 \textit{ or } 10 \textit{ or } 100 \textit{ or } 1001\\ &\qquad \textit{for } k \textit{ bits, } \; k \ge 1\\ P2N(k) &= \tfrac{4}{15}\,2^k + \tfrac{1}{15}\,[-4,7,-1,-2] \quad = P1N(k{-}1)\\ P3N(k) &= \tfrac{2}{15}\,2^k + \tfrac{1}{15}\,[-2,-4,7,-1] \quad = P1N(k{-}2)\\ P4N(k) &= \tfrac{1}{15}\,2^k + \tfrac{1}{15}\,[-1,-2,-4,7] \quad = P1N(k{-}3)\\ P5N(k) &= \tfrac{1}{30}\,2^k + \tfrac{1}{15}\,[7,-1,-2,-4] \quad = P1N(k{-}4)\\ P6N(k) &= \tfrac{13}{15}\,2^k + \tfrac{1}{15}\,[2,4,-7,1] \qquad = 2^k - P1N(k{-}2)\\ &= 1,2,3,7,14,28,55,111,222,444,887,1775,\ldots\\ P7N(k) &= \tfrac{14}{15}\,2^k + \tfrac{1}{15}\,[1,2,4,-7] \qquad = 2^k - P1N(k{-}3)\\ &= 1,2,4,7,15,30,60,119,239,478,956,1911,\ldots\\ P8N(k) &= \tfrac{29}{30}\,2^k + \tfrac{1}{15}\,[-7,1,2,4] \qquad = 2^k - P1N(k{-}4)\\ P9N(k) &= \tfrac{59}{60}\,2^k + \tfrac{1}{15}\,[4,-7,1,2] \qquad = 2^k - P1N(k{-}5)\\ P10N(k) &= \tfrac{17}{30}\,2^k + \tfrac{1}{15}\,[-1,-2,-4,7] \quad = 2^{k-1} + P1N(k{-}3)\\ gP1N(x) &= \frac{1}{(1+x)\,(1+x^2)\,(1-2x)} \end{aligned}$$

Proof. The same sub-part breakdown of figure 38 is a set of mutual recurrences for $k \ge 7$ similar to (203). In the second sub-part, the point numbers count back from the end 2^k.

$$\begin{aligned} P1N(k) &= 2^k{-}P7N(k{-}1) & \qquad P6N(k) &= 2^k{-}P2N(k{-}1)\\ P2N(k) &= \;P1N(k{-}1) & P7N(k) &= 2^k{-}P3N(k{-}1)\\ P3N(k) &= \;P2N(k{-}1) & P8N(k) &= 2^k{-}P4N(k{-}1) \end{aligned}$$

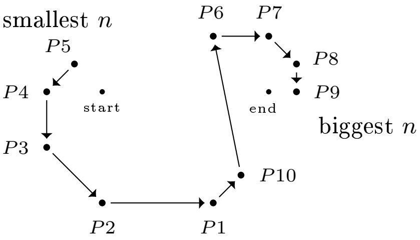
$$\begin{aligned} P4N(k) &= P3N(k{-}1) & \qquad P9N(k) &= 2^k{-}P5N(k{-}1) \\ P5N(k) &= P4N(k{-}1) & \qquad P10N(k) &= 2^k{-}P6N(k{-}1) \end{aligned}$$

The power forms (210) etc satisfy these recurrences. They can be found by expanding or some linear algebra on generating functions. It can be verified explicitly that the power forms also give the point numbers for $2 \le k \le 6$, and the integer points of $k{=}0$ and $k{=}1$. □

The point numbers for a given k go in sequence smallest to biggest as

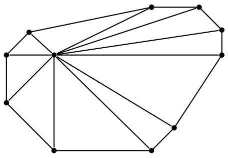
$$P5N < P4N < P3N < P2N < P1N < P10N < P6N < P7N < P8N < P9N$$

This is the curve first reaching the hull boundary at P5 then curling around anti-clockwise from P5 to P10 before switching across to the end part curling clockwise P6 to P9.



Points P5 to P10 are on the right boundary (so left turns). P6 to P9 are on the left boundary (so right turns). For a few initial levels some points are on both boundaries.

The area of the hull for $k{=}0$ is 0 for and $k{=}1$ is $\frac{1}{2}$ (half a unit square). The area for $k \ge 2$ can be calculated by taking triangles from the origin to successive vertices $P1$ through $P10$. Each triangle area is $\frac{1}{2}\mathrm{Im}\big(z_1.\overline{z_2}\big)$ in the usual way for z_2 clockwise around from z_1.



For $k < 6$, there are some duplications in the P point values giving empty triangles, and some points on the boundary which split a triangle into two parts, but this doesn't change the result.

$$HA_k = \tfrac{1}{2}\,\mathrm{Im}\Big(P1(k).\overline{P2(k)} + P2(k).\overline{P3(k)} + \cdots + P10(k).\overline{P1(k)}\Big) \tag{211}$$

$$= \begin{cases} 0,\ \frac{1}{2} & \text{for } k = 0,1 \\ \frac{7}{6}2^k - \frac{1}{12}[22,29,22,31].2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{6}[1,2,3,2] & \text{for } k \ge 2 \end{cases} \tag{212}$$

$$= 0,\ \tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{9}{2},\ \tfrac{23}{2},\ 28,\ \tfrac{121}{2},\ 129,\ \tfrac{539}{2},\ 559,\ \tfrac{2273}{2},\ 2307, \ldots$$ $\frac{1}{2}$A341029

Each point P is of the form $b^k + p(k{+}n)$ so the products are $b^k.\bar{b}^k = 2^k$ and 8-period terms in b^k, $\bar{b}^k$, and constant. $\operatorname{Im} b^k$ and $\operatorname{Im} \bar{b}^k$ are 8-periodic in the half power $2^{\lfloor k/2 \rfloor}$. Adding the factors on those terms shows the 8-periodic factors repeat themselves so just 4-periodic.

Scaling by $1/\sqrt{2}^k$ so endpoints a unit length gives the area of the convex hull around the dragon curve fractal. This is the area of the polygon given by Benedek and Panzone.

$$\frac{HA_k}{|b^k|^2} = \frac{HA_k}{2^k} \quad \to \frac{7}{6} = 0.1666\ldots \qquad \text{A177057}$$

This limit is approached from below since the half-power term in (212) is negative and bigger than the constant. On expanding, some of the new points in the curve push out beyond the existing hull, so the new hull is bigger than just doubling due to factor b.

$$\begin{aligned} HA_{k+1} - 2HA_k \;&= \begin{cases} \frac{1}{2} & \text{if } k \le 1 \\ \frac{1}{12}\,[15, 14, 13, 18].2^{\left\lfloor \frac{k}{2} \right\rfloor} - \frac{1}{6}\,[0, 1, 4, 3] & \text{if } k \ge 2 \end{cases} \\ &= \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{5}{2},\ 5,\ \tfrac{9}{2},\ 8,\ \tfrac{23}{2},\ 20,\ \tfrac{37}{2}, \ldots \end{aligned}$$

Theorem 39. *The two points of dragon curve k furthest apart are $P3$ and $P8$ of the convex hull, except for $k{=}0$ where $P3$ and $P7$. The distance apart in each case is*

$$\begin{aligned} Hdiam_k &= \sqrt[2]{\begin{cases} 1 & \text{if } k = 0 \\ \frac{5}{2} 2^k - [3, 4].2^{\left\lfloor \frac{k}{2} \right\rfloor} + 1 & \text{if } k \ge 1 \end{cases}} \qquad (213) \\ &= \sqrt[2]{\ }\ 1, 2, 5, 13, 29, 65, 137, 289, 593, 1217, 2465, \ldots \end{aligned}$$

Distances $Hdiam_0 = 1$ and $Hdiam_7 = \sqrt{289} = 17$ are integers and otherwise $Hdiam_k$ is irrational.

Proof. The points furthest apart must be vertices of the convex hull. For $k < 6$, the maximum distances can be verified explicitly and are per the formula.

For $k \ge 6$, points $P1$ through $P10$ of the convex hull are at various factors of b^k, and offsets $p(m)$ from those powers. The offsets are at most

$$pmax \;=\; \max(\tfrac{1}{3}\,|p(m)|) \;= \tfrac{1}{3}\sqrt{5}$$

Comparing factors of b^k on the points, P3--P8 are the furthest apart. Their distance is at least

$$|P3(k) - P8(k)| \;> \left|(-\tfrac{3}{2} - \tfrac{1}{2} i) b^k\right| - 2pmax \;= \sqrt{\tfrac{5}{2}}\sqrt{2}^k - 2pmax$$

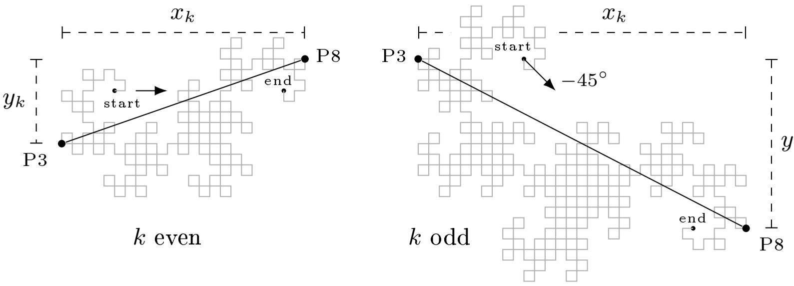
The second furthest by b^k factors is P3--P9 and their distance, and the distance of any with smaller b^k factor, is at most

$$|P3(k) - P9(k)| \;< \left|(-\tfrac{3}{2} - \tfrac{1}{3} i) b^k\right| + 2pmax \;= \sqrt{\tfrac{85}{36}}\sqrt{2}^k + 2pmax$$

For $k < 13$, it can be verified explicitly that P3--P8 is the maximum of all vertices. For $k \geq 13$, the difference between the two distance bounds is positive.

$$\left(\sqrt{\tfrac{5}{2}} - \sqrt{\tfrac{85}{36}}\right)\sqrt{2}^{k} \; > \; 4pmax \qquad\qquad \text{for } k \geq 13$$

To show $Hdiam$ irrational or not, it can be verified explicitly for $k \leq 7$ that only $Hdiam_0$ and $Hdiam_7$ are rational. For $k \geq 8$, align the curve to its segments and endpoint horizontal for k even, or endpoint $-45°$ for k odd.



k even $\qquad$ k odd

These alignments are rotation by factor $i^{-\lceil k/2 \rceil}$. x and y distances between P3 and P8 are

$$\begin{aligned} x_k &= \text{Re}\,(P8(k){-}P3(k)).i^{-\lceil k/2\rceil} &&= [\tfrac{3}{2},1].2^{\lceil k/2\rceil} - 1 && \scriptstyle k\geq 1 \text{ A052955} \\ y_k &= \left|\text{Im}\,(P8(k){-}P3(k)).i^{-\lceil k/2\rceil}\right| &&= \tfrac{1}{2}2^{\lceil k/2\rceil} \\ Hdiam_k^2 &= x_k^2 + y_k^2 \qquad \text{for } k \geq 1 && && (214) \end{aligned}$$

Write (214) as sum and difference

$$(Hdiam_k + x_k)(Hdiam_k - x_k) \; = \; y_k^2 \tag{215}$$

x_k and y_k are integers so if $Hdiam_k$ is an integer then sum and difference must give the factors of y_k^2. y_k is a power of 2 so those factors must be powers of 2.

$$\begin{aligned} Hdiam_k + x_k &= 2^{\alpha} \\ Hdiam_k - x_k &= 2^{\beta} \end{aligned} \qquad\qquad 2\,Hdiam_k = 2^{\alpha} + 2^{\beta}$$

So $Hdiam_k$ would contain at most 2 1-bits, and $Hdiam_k^2$ at most 3 1-bits. But the expression in (213), with its subtraction of middle 3 or 4 term, is bit pattern

$$Hdiam_k^2 = \begin{array}{llll} & \overbrace{}^{\text{middle run}} & & \\ 100 & 111\ldots1101 & 00\ldots001 & k \text{ even} \\ 100 & 111\ldots1100 & 00\ldots001 & k \text{ odd} \\ \uparrow & \qquad\qquad\;\uparrow & & \\ 2^{k} & \qquad\quad 2^{\lfloor k/2 \rfloor} & & \end{array}$$

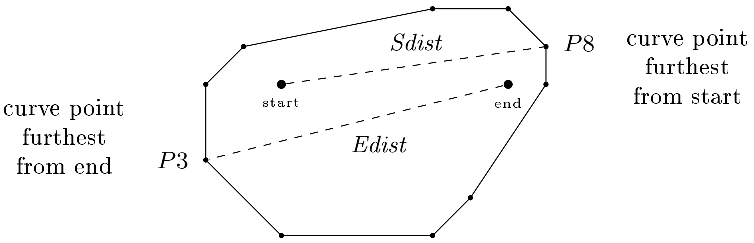
So the bits are high and low 1-bits and when $k \geq 8$ have $k - \lfloor k/2 \rfloor \geq 4$ so the middle run of 1s is at least 2 more for total at least 4. ☐

For $k{=}7$, $Hdiam_7{=}17{=}10001$ binary and $Hdiam_7^2{=}289{=}100100001$ binary have 2 and 3 1-bits respectively as is necessary. $x_7{=}15$ and $y_7{=}8$ are primitive Pythagorean triple $15, 8, 17$.

For the curve scaled to endpoints a unit length, the limit for *Hdiam* is

$$\frac{Hdiam_k}{\sqrt{2}^k} \to Hdiamf = \sqrt{\tfrac{5}{2}} = 1.581138\ldots \qquad \text{A020797}$$

The points of the curve furthest from the start and end are those hull vertices where the angles each side are $< 90°$, or by a pairwise calculation similar to the proof above. $P8$ is furthest from the start for $k \geq 1$. $P3$ is furthest from the curve end b^k for all k.



$$\begin{aligned} Sdist_k &= \begin{cases} 1 & \text{if } k=0 \\ \big|P8(k)\big| & \text{if } k \geq 1 \end{cases} \\ &= \sqrt{\tfrac{25}{18}2^k - \tfrac{1}{9}[13,8,8,16]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{9}[5,1,2,4]} \qquad k \geq 1 \\ &= \sqrt{\ } \ 1,2,4,\ \ 8,17,41,\ \ 82,164,333,697,\ldots \\ Edist_k &= \big|b^k - P3(k)\big| \\ &= \sqrt{\tfrac{17}{9}2^k - \tfrac{2}{9}[5,10,7,5]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{9}[2,4,5,1]} \\ &= \sqrt{\ } \ 1,2,5,13,26,52,109,233,466,932,\ldots \end{aligned} \tag{216}$$

$Sdist_0 = Edist_0 = 1$ are integers and $Sdist_2 = 2$ is an integer. Otherwise *Sdist* and *Edist* are both irrational. That can be seen explicitly for $k < 9$, and for $k \geq 9$ apply the same rotation as *Hdiam* above for x, y distances.

For k odd, y is a power of 2, the same value as from *Hdiam* since k odd has $P3$ aligned to curve start and $P8$ to curve end. But then too many 1-bits in $Sdist^2$ and $Edist^2$.

For k even, if *Sdist* or *Edist* is rational then it and its x, y are a Pythagorean triple. Any Pythagorean triple has at least one x, y leg a multiple of 4, as from $2pq$ in the usual parameterization of a primitive triple from Euclid X.29.1,

$$\text{legs } p^2{-}q^2,\ 2pq \quad \text{hypotenuse } p^2{+}q^2 \qquad \text{with } p, q \text{ not both odd} \tag{217}$$

But neither x, y for *Sdist* and *Edist* are multiples of 4.

Scaling the curve to endpoints a unit length gives limits for the distances, being the coefficients of their $\sqrt{2}^k$ terms

$$\begin{aligned} \frac{Sdist_k}{\sqrt{2}^k} &\to \tfrac{5}{6}\sqrt{2} &&= 1.178511\ldots \qquad \text{A020829} \\ \frac{Edist_k}{\sqrt{2}^k} &\to \tfrac{1}{3}\sqrt{17} &&= 1.374368\ldots \end{aligned}$$

7.1 Convex Hull Boundary

Theorem 40. *The boundary length of the convex hull around dragon curve k is*

$$HB_k = \begin{cases} 2,\ 2{+}\sqrt{2} & \textit{if } k = 0 \,\text{or}\, 1 \\ \Big(\tfrac{1}{6}\,[9,10].2^{\lfloor k/2 \rfloor} + \tfrac{2}{3}\,[0,-1,0,1]\Big) & \textit{if } k \ge 2 \\ +\Big(\tfrac{1}{6}\,[5,9].2^{\lfloor k/2 \rfloor}\ \ - \tfrac{1}{3}\,[7,6,5,6]\Big)\sqrt{2} & \\ +\sqrt{\mathit{P910norm}(k)}\ \ + \sqrt{\mathit{P910norm}(k{+}1)} & \end{cases}$$

$$= 2,\ 2{+}\sqrt{2},\ \ 4{+}\sqrt{2},\ \ 5{+}\sqrt{2}{+}\sqrt{5},\ \ 6{+}\sqrt{2}{+}\sqrt{5}{+}\sqrt{17},\ldots$$

$$\begin{aligned} \mathit{P910norm}(k) &= \big|P9(k){-}P10(k)\big|^2 \\ &= \tfrac{13}{36}\,2^k + \tfrac{1}{9}\,[-2,11,2,-11]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{9}\,[1,5] \\ &= 2,1,5,17,25,37,89,205,377,701,\ldots \qquad k \ge 2 \end{aligned}$$

P910norm is a square at $\mathit{P910norm}(3) = 1$ and $\mathit{P910norm}(6) = 25$, but otherwise not a square and for $k \ge 3$ it is odd so is not combined with the $\sqrt{2}$ term.

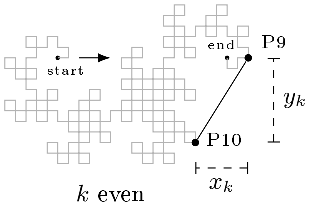
Proof. For $k \le 1$, the hull boundary is calculated explicitly. For $k \ge 2$, the integer and $\sqrt{2}$ parts of HB are the straight and 45° diagonal sides of the hull.

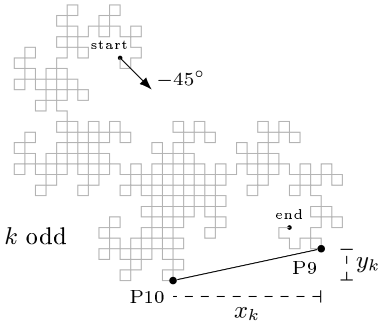
Side P9--P10 is $\sqrt{\mathit{P910norm}(k)}$. Side P5--P6 is $\sqrt{\mathit{P910norm}(k{+}1)}$ since as from figure 38 side P9--P10 is the previous level P5--P6. For $2 \le k < 6$, some sides are zero length but the formulas still hold.

$\mathit{P910norm}(k)$ is odd since multiplying by 36 and taking terms mod 72 is

$$\Big(36\,\mathit{P910norm}(k)\Big) \equiv 36 \bmod 72 \qquad k \ge 3$$

To show $\sqrt{\mathit{P910norm}(k)}$ rational or not, it can be verified explicitly for $2 \le k < 10$ that only $k{=}3$ and $k{=}6$ are rational. For $k \ge 10$, align the curve to its segments with endpoint horizontal for k even or $-45°$ for k odd.





These alignments are rotation by factor $i^{-\lceil k/2 \rceil}$ so x and y distances between P9 and P10 are

$$\begin{aligned} x_k &= \mathrm{Re}\ (P9(k){-}P10(k)).i^{-\lceil k/2 \rceil}\ \ = \tfrac{1}{6}[2,5].2^{\lfloor k/2 \rfloor} + \tfrac{1}{6}[-1,2,1,-2] \\ &= 1,1,1,4,3,6,5,14,11,26,21,54,\ldots \qquad k \ge 2 \end{aligned}$$

$$\begin{aligned} y_k &= \text{Im }(P9(k) - P10(k)).i^{-\lceil k/2 \rceil} \;=\; \tfrac{1}{2}[1, \tfrac{1}{3}].2^{\lfloor k/2 \rfloor} + \tfrac{1}{3}[0, 1, 0, -1] \\ &= 1, 0, 2, 1, 4, 1, 8,\;\; 3, 16,\;\; 5, 32, 11, \ldots \qquad k \ge 2 \end{aligned}$$

$$P910norm(k) = x_k^2 + y_k^2 \tag{218}$$

For k even, write (218) as sum and difference similar to (215)

$$(\sqrt{P910norm_k} + x_k)(\sqrt{P910norm_k} - x_k) \;=\; y_k^2$$

x_k and y_k are integers so if $\sqrt{P910norm_k}$ is rational then sum and difference are the factors of y_k^2. For k even, $y_k = 2^{k/2-1}$ so they must be powers of 2.

$$\begin{aligned} \sqrt{P910norm_k} + x_k &= 2^\alpha \\ \sqrt{P910norm_k} - x_k &= 2^\beta \end{aligned} \qquad 2\sqrt{P910norm_k} = 2^\alpha + 2^\beta$$

So $\sqrt{P910norm_k}$ would be most 2 bits and $P910norm_k$ at most 3 bits. But its factor $\frac{13}{36}$ is bit pattern 111000... and the low end has 4 bits since

$$P910norm_k \bmod 64 \;\equiv 57 \;= \text{binary } 111001 \qquad k \text{ even } \ge 10$$

For k odd, if $\sqrt{P910norm_k}$ is an integer then x_k, y_k are legs of a Pythagorean triple. Any Pythagorean triple has at least one leg a multiple of 4 as from (217). But y_k is odd and x_k is not a multiple of 4,

$$x_k \equiv 2 \bmod 4 \qquad y_k \equiv 1 \bmod 2 \qquad k \text{ odd} \ge 7 \qquad \square$$

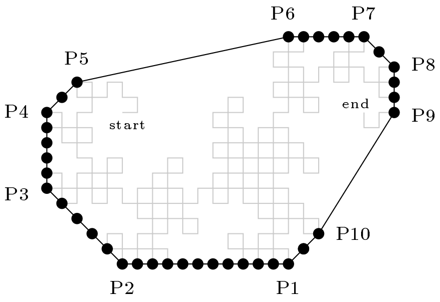
The two rational $\sqrt{P910norm_k}$ are $P910norm_3 = 1$ and $P910norm_6 = 25$. The latter has x_6, y_6 as primitive Pythagorean triple $3, 4, 5$.

For the curve scaled to endpoints a unit length, the hull boundary length limit is the high terms. This is the length around the fractal hull vertices from Benedek and Panzone.

$$\begin{aligned} \frac{HB_k}{\sqrt{2^k}} \to HBf &= \tfrac{3}{2} + \tfrac{5}{6}\sqrt{2} + \tfrac{1}{6}\sqrt{13} + \tfrac{1}{6}\sqrt{26} \qquad (219) \\ &= 4.129273\ldots \end{aligned}$$

A341030

Another boundary length measure can be made by counting all Gaussian integer points on the hull boundary.



k=8 hull boundary integer points $HBP_8 = 37$

Theorem 41. *The number of integer points on the boundary of the convex hull around dragon curve k is*

$$\begin{aligned} HBP_k &= \tfrac{1}{6}\Big([14,19].2^{\left\lfloor \frac{k}{2} \right\rfloor} + [-2,-4,\,2,4]\Big) && \textit{hull boundary integers} \\ &= 2,3,5,7,9,12,19,26,37,50,75,102,149,202,\ldots && k \text{ even} \;\; \text{A062092} \end{aligned}$$

Proof. For $k < 6$ the points can be counted explicitly. For $k \geq 6$, the number of points on the straight and 45° sides are their respective x, y extents.

Side P5--P6 has no integer points other than its ends since the real and imaginary parts of difference $s = P5(k) - P6(k)$ have no common factor. This is seen by writing multiples which give ±1.

$$\begin{aligned} \operatorname{Re} s - 5 \operatorname{Im} s &= (-1)^{k/4} && k \equiv 0 \bmod 4 \\ 3 \operatorname{Re} s - 2 \operatorname{Im} s &= -(-1)^{(k-1)/4} && k \equiv 1 \bmod 4 \\ 5 \operatorname{Re} s + \operatorname{Im} s &= -(-1)^{(k-2)/4} && k \equiv 2 \bmod 4 \\ 2 \operatorname{Re} s + 3 \operatorname{Im} s &= (-1)^{(k-3)/4} && k \equiv 3 \bmod 4 \\ \text{so} \quad \gcd(\operatorname{Re} s, \operatorname{Im} s) \; &= \; 1 \end{aligned}$$

Side P9--P10 side is the previous level P5--P6 (per figure 38 side) so it too has no integer points. ☐

Pick's theorem for a polygon on a square lattice is

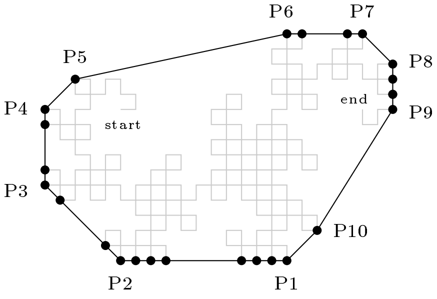
$$\text{Area} = \text{InsidePoints} + \tfrac{1}{2}\text{BoundaryPoints} - 1 \tag{220}$$

So with area HA and boundary points HBP the interior and total hull integer points are

$$\begin{aligned} HIP_k &= HA_k - \tfrac{1}{2} HBP_k + 1 && \text{hull interior points} \\ &= \begin{cases} 0 & \text{if } k = 0 \\ \frac{7}{6} 2^k - \frac{1}{6}[18,24,18,25]\, 2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{3}[4,5,4,3] & \text{if } k \geq 1 \end{cases} \\ &= 0,0,0,2,8,23,52,117,252,535,1100,2257,\ldots \end{aligned}$$

$$\begin{aligned} HP_k &= HIP_k + HBP_k && \text{hull total integer points} \\ &= \begin{cases} 2,3 & \text{if } k = 0 \,\text{or}\, 1 \\ \frac{7}{6} 2^k - \frac{1}{6}[4,5,4,6]\, 2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{3}[3,3,5,5] & \text{if } k \geq 2 \end{cases} \\ &= 2,3,5,9,17,35,71,143,289,585,1175,2359,\ldots \end{aligned}$$

The dragon curve touches its convex hull only at some points.



k=8 curve
touching hull boundary
vertices $HBV_8 = 24$
gaps $HBVgaps_8 = 9$

Theorem 42. *The number of dragon curve vertices on the convex hull is*

$$\begin{aligned} HBV_k &= \begin{cases} 2,3 & \textit{if } k = 0 \,\text{or}\, 1 \\ \frac{1}{2}\,[15,17,22,26].2^{\left\lfloor \frac{k}{4} \right\rfloor} - 6 & \textit{if } k \geq 1 \end{cases} \\ &= 2,3,5,7,9,11,16,20,24,28,38,46,54,62,82,98,\ldots \end{aligned}$$

Proof. For $k \leq 6$, the vertices on the hull can be counted explicitly.

For $k \geq 6$, let $P1v(k)$ be the number of curve vertices on side P1--P2. P1 is included and P2 is excluded. Similarly the other sides through to wrapping around P10--P1.

From figure 38 the sides of hull k comprise sides of $k{-}1$ sub-hulls. Side P5--P6 has no vertices except its endpoints since both sub-hull sides turn away from it. Side P1--P2 is the two sub-hull sides P1--P10 and P8′--P7′, plus the vertex at P8′ which is not included in $P7v(k{-}1)$. So mutual recurrences

$$\begin{aligned} P1v(k) &= P7v(k{-}1) + 1 + P10v(k{-}1) & P6v(k) &= P2v(k{-}1) \\ P2v(k) &= P1v(k{-}1) & P7v(k) &= P3v(k{-}1) \\ P3v(k) &= P2v(k{-}1) & P8v(k) &= P4v(k{-}1) \\ P4v(k) &= P3v(k{-}1) & P9v(k) &= P5v(k{-}1) \\ P5v(k) &= 1 & P10v(k) &= P6v(k{-}1) \end{aligned} \qquad (221)$$

The chain of dependencies is similar to the vertex positions (203) but $P1v$ contributes to both $P7v$ and $P10v$.

$$\begin{tikzcd} & & P8v \arrow[r] & P4v \arrow[r] & P3v \arrow[r] & P2v \arrow[d] & P6v \arrow[l] \\ P9v \arrow[r] & P5v = 1 & & & P7v \arrow[u] & P1v \arrow[l] \arrow[r] & P10v \arrow[u] \end{tikzcd}$$

With initial values calculated explicitly and expanding 4 times, or a little linear algebra on generating functions,

$$\begin{aligned} P1v(k) &= [2,1,2,2].2^{\lfloor k/4 \rfloor} - 1 \\ &= 3,3,\ 7,3,7,7,\ 15,7,15,15,\ldots \qquad k \geq 6 \end{aligned}$$

The other sides are the same with index delays and the total is HBV.

$$HBV_k = P1v(k) + P2v(k) + \cdots + P10v(k) \qquad \square$$

Hull boundary points HBP grow as a half power $2^{k/2}$ whereas curve vertices HBV on that boundary grow only as a quarter power $2^{k/4}$, so the proportion $\to 0$ as $k{\to}\infty$.

Curve segments are on the straight sides of the hull boundary (aligned to b^k) when k even, or the 45° diagonal sides when k odd. Every 2 vertices on a side are a segment, and when an odd number of vertices the last vertex goes to the side endpoint for a further segment, so $\frac{1}{2}$ rounded up.

$$HBS_k = \begin{cases} 1,2,3,4,5,5 & k = 0 \,\text{to}\, 5 \\ \left\lceil \frac{1}{2}P1v(k) \right\rceil + \left\lceil \frac{1}{2}P3v(k) \right\rceil + \left\lceil \frac{1}{2}P6v(k) \right\rceil + \left\lceil \frac{1}{2}P8v(k) \right\rceil & k \text{ even } \geq 6 \\ \left\lceil \frac{1}{2}P2v(k) \right\rceil + \left\lceil \frac{1}{2}P4v(k) \right\rceil + \left\lceil \frac{1}{2}P7v(k) \right\rceil + \left\lceil \frac{1}{2}P10v(k) \right\rceil & k \text{ odd } \geq 6 \end{cases}$$

$$= \begin{cases} 1,2,3 & k = 0 \text{ to } 2 \\ \frac{1}{2}\,[5,5,7,8]\,2^{\lfloor k/4 \rfloor} & k \geq 3 \end{cases} \qquad \text{curve segments on hull}$$

$$= 1,2,3,4,\ \ 5,5,7,8,\ \ 10,10,14,16,\ \ 20,20,28,32,\dots$$

The vertices on the hull boundary are in runs and gaps. For example in the k=6 figure 37 left side P1--P2 has segments at each end and a gap in the middle, whereas side P3--P4 is a continuous run.

Theorem 43. *The number of gaps in the dragon curve vertices on its convex hull boundary is*

$$HBVgaps_k = \begin{cases} 0,0,0,1,2 & \textit{if } k = 0 \text{ to } 4 \\ \frac{1}{4}\,[15,18,20,24].2^{\left\lfloor \frac{k}{4} \right\rfloor} - 6 & \textit{if } k \geq 5 \end{cases}$$

$$= 0,0,0,1,2,3,4,6,9,12,14,18,24,30,\dots \qquad \text{odd } k{\geq}3 \ \ \text{A061776}$$

Proof. Take hull k as sub-hulls of $k{-}1$ from figure 38 again. The sub-hull points $P10(k{-}1)$ and $P8'(k{-}1)$ are separated by a distance bigger than a single vertex point since

$$\begin{aligned} \frac{b^k - iP8(k{-}1) \ - \ P10(k{-}1)}{\omega_8^k} &= \tfrac{1}{3}\sqrt{2}^k + \tfrac{1}{3}[-1,\ 2\sqrt{2},\ 1,\ \sqrt{2}] \\ &\geq 3\sqrt{2} \quad \text{for } k \geq 7 \end{aligned}$$

Let *P1gaps*(k) be the number of gaps in side P1--P2, and the other sides similarly. The endpoints are always visited by the curve so gaps are within each side. The mutual recurrences (221) for the curve vertices are the same for the gaps, including side P1--P2 with +1 extra gap. The initial values at k=6 are not the same, being instead *P1gaps* = *P2gaps* = *P5gaps* = *P9gaps* = 1. The result is

$$P1gaps(k) = 2^{\left\lfloor \frac{k-1}{4} \right\rfloor} - 1$$

The other sides are the same with index delays. The total is *HBVgaps*.

$$HBVgaps_k = P1gaps(k) + P2gaps(k) + \dots + P10gaps(k) \qquad \square$$

7.2 Minimum Area Rectangle

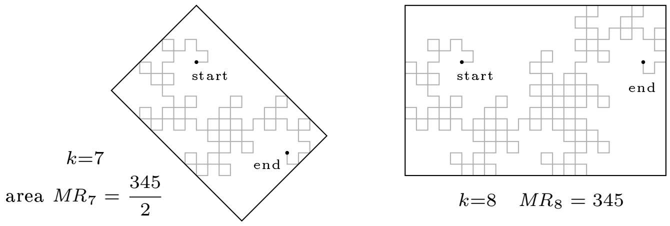


Figure 39: minimum area rectangle k=7 and k=8

Theorem 44. *The minimum-area rectangle around the dragon curve level k has area*

$$MR_k = \begin{cases} 0, 1, 2, 6, 15, \frac{192}{5} & \textit{for } k{=}0 \textit{ to } k{=}5 \\ \dfrac{\left(\frac{3}{2}.2^{\lceil k/2 \rceil} - 1\right)\left(2^{\lceil k/2 \rceil} - 1\right)}{2^{k \bmod 2}} & \textit{for } k \ge 6 \\ \quad = \frac{3}{2}2^k - \frac{5}{2}2^{\lfloor k/2 \rfloor} + \frac{1}{2^{k \bmod 2}} & \end{cases} \tag{222}$$

$$= 0, 1, 2, 6, 15, \tfrac{192}{5}, 77, \tfrac{345}{2}, 345, \tfrac{1457}{2}, 1457, \tfrac{5985}{2}, 5985, \ldots$$

For $k{=}0$ to $k{=}4$, the rectangle is aligned vertically/horizontally. For $k{=}5$, the rectangle is aligned to a 3,4 slope. For $k \ge 6$, the rectangle is aligned to the b^k endpoint.

$2^{k \bmod 2}$ means $2^0 = 1$ or $2^1 = 2$ according to k even or odd. The first $k \ge 6$ form (222) shows how the numerator is the same for pairs of k, as for example 345 of $k{=}7$ and $k{=}8$ in figure 39 above. The second $k \ge 6$ form shows how the area grows as $\frac{3}{2}2^k$ less a half power.

Proof. A minimum area rectangle has at least one side aligned to a side of the convex hull, so it suffices to consider rectangles on the hull sides.

For $k{=}0$, the curve is a unit line segment with area $MR_0 = 0$.

For $k{=}1$ the curve is an L shape. The two possible rectangle alignments both have area $MR_1 = 1$.

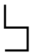
$k = 1$
$MR_1 = 1$

area $= 1$

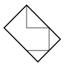
area $= \sqrt{2} \times \frac{1}{2}\sqrt{2} = 1$

For $k{=}2$ to $k{=}4$, the possible alignments and areas are as follows. In each case the first is the minimum.

$k = 2$
$MR_2 = 2$

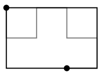


area $= 2$

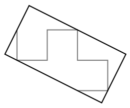


area $= \frac{3}{2}\sqrt{2} \times \sqrt{2} = 3$

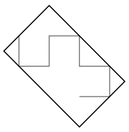
$k = 3$
$MR_3 = 6$



area $= 6$

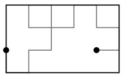


$$\operatorname{Re}\frac{1-(-2{+}2i)}{2{-}i} \times \operatorname{Im}\frac{2i-0}{2{-}i} \times 5 = 6{+}\tfrac{2}{5}$$

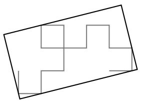


area
$\frac{3}{2}\sqrt{2} \times \frac{5}{2}\sqrt{2}$
$= 7{+}\frac{1}{2}$

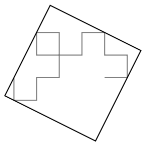
$k = 4$
$MR_4 = 15$



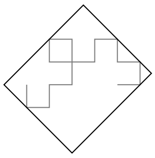
3×5
$= 15$



$$\operatorname{Re} \frac{1{+}i - (-4{-}i)}{4{+}i} \times \operatorname{Im} \frac{-3{+}2i - (-3{-}i)}{4{+}i} \times 10 \;= 15{+}\tfrac{9}{17}$$

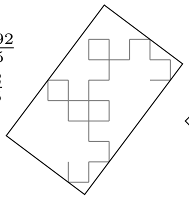


$$\operatorname{Re} \frac{2i - (-4{-}i)}{1{+}2i} \times \operatorname{Im} \frac{-3{+}2i - 1}{1{+}2i} \times 5 \;= 20$$

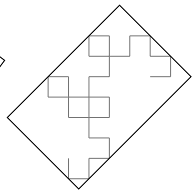


$3\sqrt{2} \times \frac{7}{2}\sqrt{2}$
$= 21$

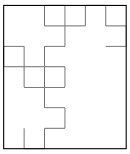
$k = 5$
$MR_5 = \frac{192}{5}$
$= 38 + \frac{2}{5}$



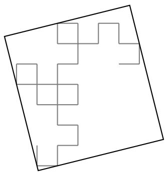
$$\operatorname{Re} \frac{2i - (-4{-}5i)}{3{+}4i} \times \operatorname{Im} \frac{-5 - 1}{3{+}4i} \times 25 = 38 + \tfrac{2}{5}$$



$\frac{11}{2}\sqrt{2} \times \frac{7}{2}\sqrt{2}$
$= 38 + \frac{1}{2}$



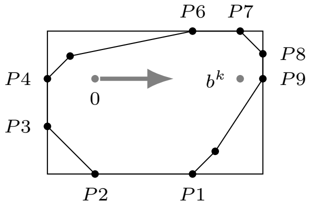
$6 \times 7 = 42$



$$\operatorname{Re} \frac{1{+}i - (-5{-}i)}{4{+}i} \times \operatorname{Im} \frac{-3{+}2i - (-3{-}5i)}{4{+}i} \times 17 = 42 + \tfrac{14}{17}$$

As a remark, for k=4 the minimum area $MR_4 = 15$ is per the general formula (222). For k=5, the diagonal case $38{+}\frac{1}{2}$ is also per the formula but it is not the minimum. The 3,4 slope rectangle there is $\frac{1}{10}$ smaller.

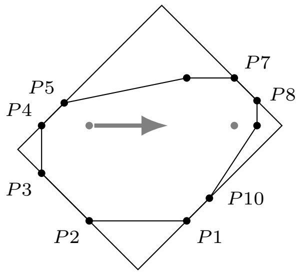
For $k \geq 6$, the hull vertices $P1(k)$ through $P10(k)$ from the convex hull (196) are used. A rectangle aligned to the endpoint b^k is the claimed minimum,



The four sides P3--P4 etc are vertical as at (207) so area

$$\begin{aligned} MR_k &= 2^k . \operatorname{Re} \frac{P9(k) - P4(k)}{b^k} . \operatorname{Im} \frac{P6(k) - P1(k)}{b^k} \\ &= 2^k \left(\frac{3}{2} - \frac{1}{2^{\lceil k/2 \rceil}} \right) \left(1 - \frac{1}{2^{\lceil k/2 \rceil}} \right) \end{aligned}$$

A rectangle aligned 45° diagonally to the b^k endpoint is



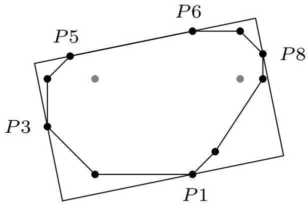
The four sides P4--P5 etc are diagonal as per (208). The area is

$$\begin{aligned}
MRdiag(k) &= 2^{k+1} . \operatorname{Re} \frac{P8(k) - P2(k)}{b^{k+1}} . \operatorname{Im} \frac{P4(k) - P1(k)}{b^{k+1}} \\
&= \frac{\left(\frac{5}{3} 2^{\lceil k/2 \rceil} - \frac{2}{3} [1,2,2,4]\right) \left(2^{\lceil k/2 \rceil} - \frac{1}{2} 2^{k \bmod 2}\right)}{2^{k \bmod 2}} \qquad (223) \\
&> MR_k \qquad \text{for } k \ge 6
\end{aligned}$$

Compared to the MR form (222) the right term of (223) has $-\frac{1}{2} 2^{k \bmod 2} \ge -1$ always. The left term has $\frac{5}{3}$ exceeding $\frac{3}{2}$ by $\frac{1}{6}$ and putting that to the half-power has

$$\tfrac{1}{6} 2^{\lceil k/2 \rceil} - \tfrac{2}{3} [1,2,2,4] \ > \ -1 \qquad \text{for } k \ge 6$$

A rectangle aligned to the P5--P6 side is

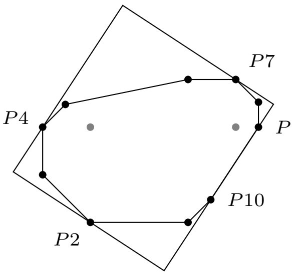


The area is the extents in the direction of the P5--P6 line. Dividing by that to rotate,

$$\begin{aligned}
MRtwo(k) &= \left| P6(k) - P5(k) \right|^2 . \operatorname{Re} \frac{P8(k) - P3(k)}{P6(k) - P5(k)} . \operatorname{Im} \frac{P5(k) - P1(k)}{P6(k) - P5(k)} \\
&= \frac{\operatorname{Re} (P8 - P3) (\overline{P6 - P5}) \, . \operatorname{Im} (P5 - P1) (\overline{P6 - P5})}{|P6 - P5|^2} \\
&= \frac{20}{13} 2^k \ + \ a(k) \, 2^{\lfloor k/2 \rfloor} \ + \ \frac{b(k) \, 2^k + c(k) \, 2^{\lfloor k/2 \rfloor} + d(k)}{\frac{13}{18} 2^k + e(k) \, (-2)^{\lfloor k/2 \rfloor} + f(k)} \\
&\qquad a(k) = [-\tfrac{427}{169}, -\tfrac{498}{169}, -\tfrac{418}{169}, -\tfrac{516}{169}] \\
&\qquad b(k) = [\tfrac{2231}{3042}, \tfrac{358}{507}, \tfrac{862}{1521}, \tfrac{20}{507}] \\
&\qquad c(k) = [\tfrac{1966}{1521}, \tfrac{166}{507}, -\tfrac{1459}{1521}, \tfrac{172}{507}] \\
&\qquad d(k) = [\tfrac{4}{9}, 0] \qquad e(k) = [\tfrac{11}{9}, \tfrac{4}{9}] \qquad f(k) = [\tfrac{5}{9}, \tfrac{1}{9}] \\
&> MR_k
\end{aligned}$$

Factor $\frac{20}{13} > \frac{3}{2}$. The difference $\frac{1}{26}2^k$ is enough to exceed $a(k) > -\frac{7}{2}$ and the fraction part > -1.

A rectangle aligned to the P9--P10 side is



The area is the extents in the direction of the P9--P10 line.

$$\begin{aligned}
\mathit{MRthree}(k) &= \big|P9(k) - P10(k)\big|^2 \,.\, \mathrm{Re}\,\frac{P7(k) - P2(k)}{P9(k) - P10(k)} \,.\, \mathrm{Im}\,\frac{P4(k) - P10(k)}{P9(k) - P10(k)} \\
&= \frac{\mathrm{Re}\,(P7{-}P2)(\overline{P9{-}P10}) \,.\, \mathrm{Im}\,(P4{-}P10)(\overline{P9{-}P10})}{|P9 - P10|^2} \\
&= \frac{45}{26}\,2^k + a(k)\,2^{\lfloor k/2 \rfloor} + \frac{b(k)\,2^k + c(k)\,2^{\lfloor k/2 \rfloor} + d(k)}{\frac{13}{36}2^k + e(k)\,(-2)^{\lfloor k/2 \rfloor} + f(k)} \\
&\qquad a(k) = [-\tfrac{615}{338}, -\tfrac{1109}{338}, -\tfrac{867}{338}, -\tfrac{1361}{338}] \\
&\qquad b(k) = [-\tfrac{365}{2028}, \tfrac{941}{6084}, \tfrac{935}{2028}, \tfrac{245}{1521}] \\
&\qquad c(k) = [\tfrac{205}{1014}, \tfrac{3179}{3042}, \tfrac{289}{1014}, -\tfrac{2321}{3042}] \\
&\qquad d(k) = [0,\, \tfrac{4}{9}] \qquad e(k) = [-\tfrac{2}{9}, \tfrac{11}{9}] \qquad f(k) = [\tfrac{1}{9}, \tfrac{5}{9}] \\
&> \mathit{MR}_k
\end{aligned}$$

Factor $\frac{45}{26} > \frac{3}{2}$. The difference $\frac{3}{13}2^k$ is enough to exceed $a(k) > -\frac{9}{2}$ and the fraction part > -1. ☐

The number of curve vertices on the minimum area rectangle can be calculated from the sides in theorem 42. For k=2, there are two rectangles of equal minimum area touching 4 or 5 vertices respectively. Take that as 5 so that for $k \le 2$ all curve vertices are touched.

$$\begin{aligned}
\mathit{MRV}_k &= P1v(k) + P3v(k) + P6v(k) + P8v(k) + 4 \qquad\qquad k \ge 6 \\
&= \begin{cases} 2, 3, 5, 7, 9, 5 & \text{if } k = 0 \,\text{to}\, 5 \\ \frac{1}{2}[10, 7, 14, 10] . 2^{\left\lfloor \frac{k}{4} \right\rfloor} & \text{if } k \ge 6 \end{cases} \\
&= 2, 3, 5, 7, 9, 5, 14, 10, 20, 14, 28, 20, 40, 28, 56, 40, 80, \ldots
\end{aligned}$$

Odd and even values of MRV repeat with offset 3,

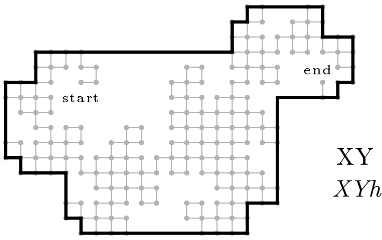
$$\begin{aligned}
\mathit{MRV}_{2k} &= \mathit{MRV}_{2k+3} \qquad \text{for } 2k \ge 6 \\
&= 14, 20, 28, 40, 56, 80, 112, 160, 224, 320, \ldots \qquad \text{A063920, A070875}
\end{aligned}$$

Or similarly the number of gaps. The corners of the rectangle are 4 gaps in addition to those within the hull sides.

$$\begin{aligned} \mathit{MRVgaps}_k &= \mathit{P1gaps}(k) + \mathit{P3gaps}(k) \qquad\qquad k \ge 6 \\ &\quad + \mathit{P6gaps}(k) + \mathit{P8gaps}(k) + 4 \\ &= \begin{cases} 0,1,2,3,5 & \text{if } k = 0 \,\text{to}\, 4 \\ [5,5,7,7]_{k-5} \,.\, 2^{\left\lfloor \frac{k-5}{4} \right\rfloor} & \text{if } k \ge 5 \end{cases} \\ &= 0{,}1{,}2{,}3{,}5,\ \ 5,5,7,7,\ \ 10,10,14,14,\ \ 20,20,28,28,\ldots \quad \text{undup A070875} \end{aligned}$$

7.3 XY Convex Hull

An XY convex hull is the smallest polygon surrounding a set of points which is "convex" in the X and Y directions. Any line parallel to the X or Y axes intersects the polygon in a single contiguous segment. (Whereas the full convex hull requires this for a line at any angle.)



k=8
XY convex hull
$\mathit{XYhullA}_8 = 223$

The boundary length is simply the width and height extents of the convex hull in the direction of curve endpoint, since the XY hull goes up and down each extent, in stair-steps. This is the alignment and hence boundary length of the minimum area rectangle general case (but not some of the initial cases).

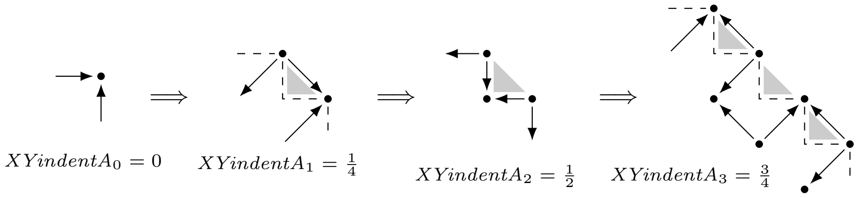
$$\begin{aligned} \mathit{XYhullB}_k &= \begin{cases} 2 & \text{if } k{=}0 \\ 2\left(\operatorname{Re} \frac{\mathit{P8}_k - \mathit{P3}_k}{\omega_8^k} + \operatorname{Im} \frac{\mathit{P6}_k - \mathit{P1}_k}{\omega_8^k} \right) & \text{if } k \ge 1 \end{cases} \\ &= \sqrt{2} \,.\, \Big(\mathit{XYhullB}_{k-1} + [0,2] \Big) \qquad k \ge 2 \\ &= \begin{cases} 2 & \text{if } k{=}0 \\ 5\sqrt{2}^k - [4, 2\sqrt{2}] & \text{if } k \ge 1 \end{cases} \\ &= 2,\ 3\sqrt{2},\ 6,\ 8\sqrt{2},\ 16,\ 18\sqrt{2},\ 36,\ 38\sqrt{2},\ldots \qquad k{\ge}1 \text{ sans } \sqrt{2} \text{ A123208} \end{aligned}$$

The limit for the curve scaled to unit length is $\mathit{XYhullB}/\sqrt{2}^k \to 5$. This can be compared to the limit for the full hull (219). The full hull boundary is shorter since it cuts across corners for $k \ge 1$.

Theorem 45. *The XY convex hull around the points of dragon curve k aligned to curve endpoint has area*

$$\begin{aligned} \mathit{XYhullA}_k &= \begin{cases} 0,0,1 & \textit{for } k = 0,1,2 \\ \frac{1027}{1008} 2^k - \frac{1}{9} [23,22,22,23]\, 2^{\lfloor k/2 \rfloor} & \\ \quad + \frac{1}{7} [16,-17,22,-26]\, 2^{\lfloor k/4 \rfloor} - \frac{1}{18} [14,7] & \textit{for } k \ge 3 \end{cases} \\ &= 0,0,1,\tfrac{5}{2},8,22,48,\tfrac{217}{2},223,\tfrac{961}{2},969,\ldots \end{aligned}$$

Proof. The XY hull is contained within the full hull. The vertical and horizontal sides of the full hull are unchanged. The diagonal sides are reduced by stair-step indentations. Initial segment expansions of a diagonal are



The grey triangles are indentations where the XY hull is smaller than the diagonal of the full hull. The XY hull is around just the points, not the curve segments between them, hence the dashed lines in $XYindentA_1$ and $XYindentA_3$. Each segment is unit length so the triangles are $\frac{1}{4}$ or $\frac{1}{2}$.

Curve $k \geq 10$ comprises sub-curves $k{-}4$ and for $k{-}4 \geq 6$ each of those sub-curves has a 10-vertex convex hull,

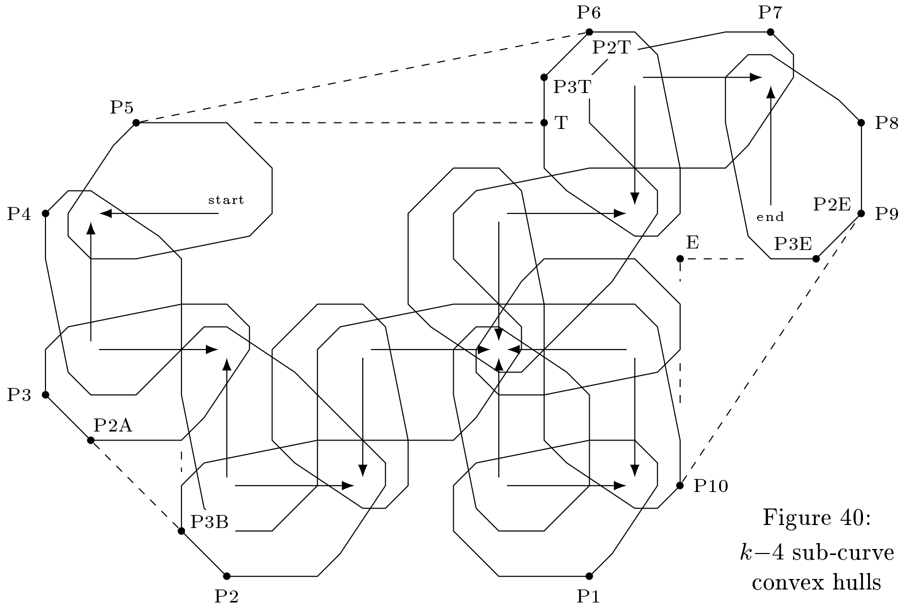
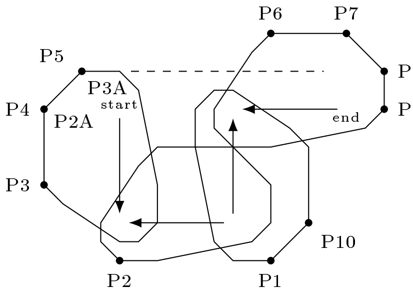


Figure 40: $k{-}4$ sub-curve convex hulls

On diagonal side P2--P3 at lower left, the triangular region shown dashed is not in the XY hull. The rest of that side is two further $k{-}4$ sides P2--P3. So recurrence for $XYindentA$, starting k=6 side P2--P3 is $XYindentA_4$,

$$\begin{aligned} XYindentA_k &= 2\,XYindentA_{k-4} \qquad\qquad \text{for } k \geq 4 \\ &\qquad + \tfrac{1}{4}\Big(\big|P2(k{+}2){-}P3(k{+}2)\big| - 2\big|P2(k{-}2){-}P3(k{-}2)\big|\Big)^2 \\ &= \tfrac{4}{63}\,2^k + \tfrac{1}{9}\,[4,2,2,-2]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{14}\,[-4,-1,-2,10]\,2^{\lfloor k/4 \rfloor} - \tfrac{1}{36}\,[8,1,2,1] \\ &= 0, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 2, \tfrac{11}{4}, \tfrac{11}{2}, \tfrac{31}{4}, 22, \tfrac{143}{4}, \tfrac{143}{2}, \ldots \end{aligned}$$

Sides P1-P10, P4--P5 and P7--P8 are all the same. They are 2 levels further down from P2--P3, as can be seen from expanding twice so that P1-P10 etc are sides P2--P3 of each sub-curve.



In figure 40, the top polygon region P5, P6, P3T, T is all absent from the XY hull, and in addition indents in the side P2T--P3T which is a $k-4$ sub-curve. Similarly the bottom polygon region P10, P9, P3E, E and its P2E--P3E side.

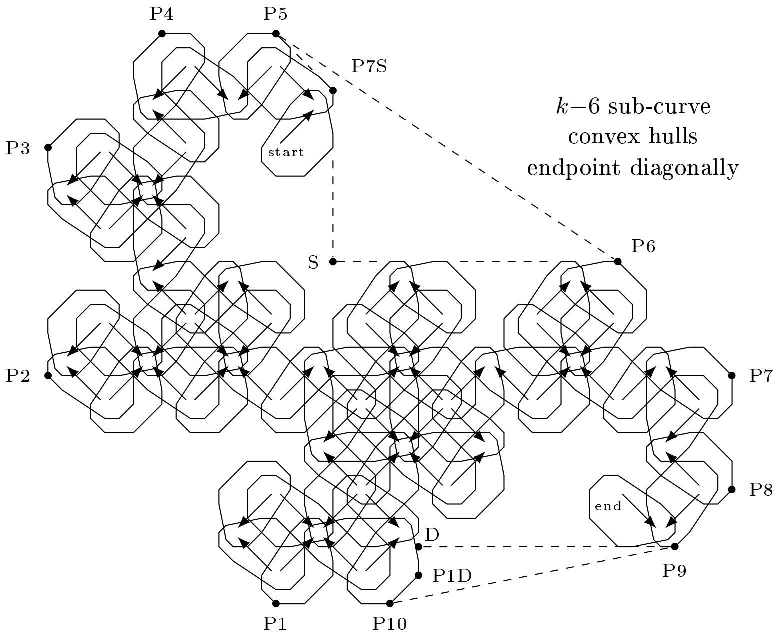
Total reduction over the full hull is then as follows, for $k \geq 10$. Smaller k hulls can be formed explicitly.

$$\begin{aligned} XYhullA_k = HA_k &- XYindentA_{k-2} - 3XYindentA_{k-4} - 2XYindentA_{k-6} \\ &- PolygonArea\big(P5(k),\ T,\ P3T,\ P6(k)\big) \\ &- PolygonArea\big(P10(k),\ P9(k),\ P3E,\ E\big) \qquad \square \end{aligned}$$

A similar calculation can be made for an XY convex hull around the curve with endpoints aligned diagonally. The boundary from diagonal hull extents is

$$\begin{aligned} DXhullB_k &= 2\left(\operatorname{Re}\frac{P7_k - P2_k}{\overline{\omega_8}^{k+1}} + \operatorname{Im}\frac{P4_k - P1_k}{\overline{\omega_8}^{k+1}}\right) \\ &= \sqrt{2}\,.\big(DXhullB_{k-1} + [3,0,1,0]\big) \\ &= \tfrac{11}{3}\sqrt{2}\,.\,\sqrt{2}^k\ - [\tfrac{5}{3}\sqrt{2},\ \tfrac{10}{3},\ \tfrac{7}{3}\sqrt{2},\ \tfrac{14}{3}] \\ &= 2\sqrt{2},\ 4,\ 5\sqrt{2},\ 10,\ 13\sqrt{2},\ 26,\ 27\sqrt{2},\ 54, \ldots \end{aligned}$$

For area, the vertical and horizontal hull sides become the diagonals with indents. The P5--P6 and P9--P10 sides are polygon shapes, with different points. It's convenient to take $k-6$ sub-curves to see which sub-curve sides those parts follow.



In k=6, sides P3--P4 and P6--P7 are $XYindentA_3$, so k−3. Side P1--P2 is 2 levels bigger. Sides P8--P9 and P5--P7S in region S is 2 levels smaller. Side P10--P1D in region D is 2 levels yet smaller, being a P1--P2 at 6 levels down. So

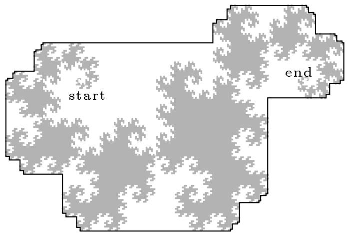
$$\begin{aligned} DXhullA_k = HA_k &- \ \ XYindentA_{k-1} - 2XYindentA_{k-3} \qquad \text{for } k{\ge}12 \\ &- 2XYindentA_{k-5} - \ \ XYindentA_{k-7} \\ &- PolygonArea\big(\ P5(k),\ P7S,\ S,\ P6(k)\big) \\ &- PolygonArea\big(P10(k),\ P9(k),\ D,\ P1D\big) \end{aligned}$$

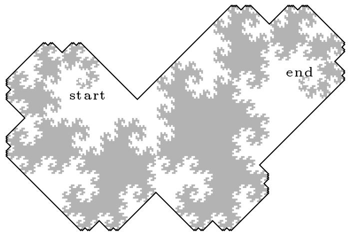
Hulls $k < 12$ can be calculated explicitly, giving

$$DXhullA_k = \begin{cases} 0, 0, \frac{1}{2}, 3 & \text{for } k \le 3 \\ \frac{557}{576}\,2^k - \frac{1}{72}\,[143, 286, 121, 242]\,2^{\lfloor k/2 \rfloor} & \\ \quad + \frac{1}{8}\,[-5, 6, -6, 8]\,2^{\lfloor k/4 \rfloor} - \frac{1}{18}\,[2, 4, 5, 10] & \text{for } k \ge 4 \end{cases}$$

$$= 0, 0, \tfrac{1}{2}, 3, \tfrac{13}{2}, 17, \tfrac{95}{2}, 100, \tfrac{427}{2}, 435, 934, \ldots$$

Area limits for the curve scaled to endpoints a unit length are the coefficients of the 2^k terms, and boundary length limits are coefficients of their $\sqrt{2}^k$ terms (as already noted for $XYhullB$).





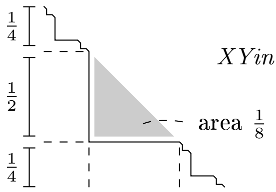
$$\frac{XYhullA_k}{2^k} \to \frac{1027}{1008} = 1.018849\ldots \qquad \frac{DXhullA_k}{2^k} \to \frac{557}{576} = 0.967013\ldots$$

$$\frac{XYhullB_k}{2^k} \to 5 \qquad \frac{DXhullB_k}{2^k} \to \frac{11}{3}\sqrt{2} = 5.185449\ldots$$

The limit for the indent area *XYindentA* on the diagonal of a square of unit side is (with P2--P3 for scaling),

$$\frac{XYindentA_{k-2}}{\left(\operatorname{Re} \overline{\omega_8}^k (P2(k) - P3(k))\right)^2} \to XYindentAf = \frac{1}{7}$$

The self-similar ends of P2--P3 in figure 40 are 4 levels smaller so limits $\frac{1}{4}$ the length, leaving middle triangle $\frac{1}{2}$. For a unit square, this is middle triangle area $\frac{1}{8}$ and 2 ends of $\frac{1}{4}$ size so $2.\frac{1}{16}$ area, etc.



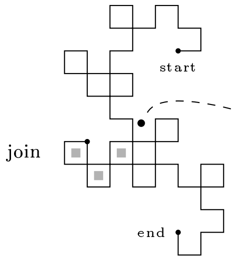
$$XYindentAf = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \cdots = \frac{1}{7}$$

In *XYhullA*, the regions T and E are different shapes but the limits for their area are the same 17/288 (not including the further *XYindentA* on the diagonals within them).

The above calculations are XY hulls around just the points, so XY hull sides are all vertical and horizontal. If curve segments are included then *XYhullA* odd k has segments on the diagonals which increase the area. Indent $XYindentA_1$ reduces 0 and $XYindentA_3$ reduces to $\frac{1}{4}$. Putting those smaller indents through the calculation increases the general case hull area by

$$[0, \tfrac{7}{8}, 0, \tfrac{5}{4}]\, 2^{\lfloor k/4 \rfloor} \quad = 0,\ \tfrac{7}{8},\ 0,\ \tfrac{5}{4},\ 0,\ \tfrac{7}{4},\ 0,\ \tfrac{5}{2},\ 0,\ \tfrac{7}{2},\ \ldots$$

8 Centroid



k=6 segments centroid

$GS_6 = \dfrac{-26-51i}{16}$

within boundary square JEQ_{k-1} at end of join

Theorem 46. *Consider the dragon curve to have mass uniformly distributed along its length. The centroid (centre of gravity) of level k is*

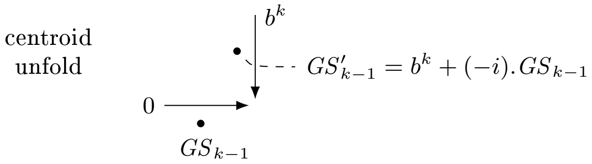
$$GS_k = \left(\tfrac{2}{5} - \tfrac{1}{5}i\right) b^k \ + \ \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) \left(\tfrac{\overline{b}}{2}\right)^k \qquad \textit{segments centroid}$$

$$\textit{where } b = 1{+}i$$

$$= \tfrac{1}{2},\ \tfrac{3+i}{4},\ \tfrac{2+3i}{4},\ \tfrac{-3+9i}{8},\ \tfrac{-13+6i}{8},\ \tfrac{-39-13i}{16},\ \tfrac{-26-51i}{16},\ \ldots$$

Proof. For k=0, the curve is a single line segment and $GS_0 = \frac{1}{2}$ per the formula is its midpoint.

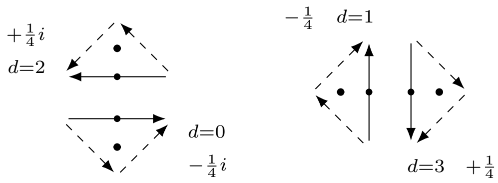
For $k \geq 1$, the centroid is the midpoint of the centroids of level $k-1$ and the unfolded copy of $k-1$.



$$\begin{aligned}
GS_k &= \tfrac{1}{2}\left(GS_{k-1} + GS'_{k-1}\right) && \text{midpoint} \\
&= \tfrac{1}{2}b^k + \tfrac{\bar{b}}{2}\,GS_{k-1} \\
&= \sum_{j=0}^{k} \tfrac{1}{2}b^j\left(\tfrac{\bar{b}}{2}\right)^{k-j} && \text{repeated substitution and } GS_0{=}\tfrac{1}{2} \qquad (224) \\
&= \tfrac{1}{2}\left(b^{k+1} - \left(\tfrac{\bar{b}}{2}\right)^{k+1}\right) / \left(b - \tfrac{\bar{b}}{2}\right) && \square
\end{aligned}$$

Second Proof of Theorem 46. The centroid can also be calculated from the number of segments in each direction $S(k,d)$ of theorem 10 and how they expand.

The curve always turns ±90° so even numbered segments are horizontal and odd numbered segments are vertical. Even numbered segments expand on the right and odd numbered segments expand on the left of their direction along the curve which is d in $S(k,d)$.



Relative to the midpoint of the existing segment the expanded segments have centroid $-\frac{1}{4}i$, $-\frac{1}{4}$, $+\frac{1}{4}i$, $+\frac{1}{4}$ for d = 0,1,2,3 respectively. Write that as $-\frac{1}{4}i(-i)^d$. There are 2^{k-1} segments in total in the previous level so divide by that for the mean, and multiply b to expand the curve.

$$GS_k = b\left(GS_{k-1} + \tfrac{1}{2^{k-1}}\sum_{d=0}^{3} -\tfrac{1}{4}i(-i)^d\,S(k{-}1,d)\right) \qquad k \geq 1 \qquad (225)$$

$$= b\,GS_{k-1} + b\left(\tfrac{1}{2}\right)^{k-1}.\tfrac{-i}{4}\,\bar{b}^{k-1} \qquad (226)$$

$$= b\,GS_{k-1} + \tfrac{1}{2}\left(\tfrac{\bar{b}}{2}\right)^k \qquad (227)$$

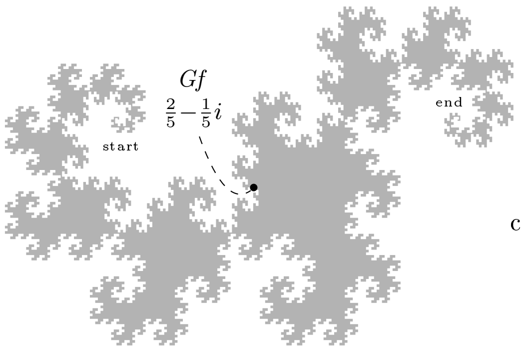
For k=1, the sum part of (225) is $-\frac{1}{4}i$ since the only segment is $S(0,d) = 1$ at d=0. This is per the expression in (226).

For $k \geq 2$, the sum part of (225) is conveniently handled by the $S(k,d)$ form (84) with b powers. Its 2^k powers cancel out since the $(-i)^d$ factor has them equally in all four directions. Its b^k term cancels out too since $(-i)^d.(-i)^d = (-1)^d$ and $\sum_{d=0}^{3}(-1)^d = 0$. Its $\overline{b^k}$ term remains since $(-i)^d.i^d = 1$ always.

Expanding (227) repeatedly gives a sum of the same terms as the previous (224) for GS, but in reverse order. (224) has $\overline{b}/2$ rising and b falling. (227) has $\overline{b}/2$ falling and b rising. $\square$

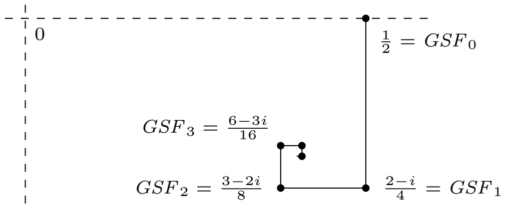
The centroid scaled relative to the endpoint b^k is

$$\begin{aligned}
GSF_k &= GS_k/b^k \\
&= \left(\tfrac{2}{5} - \tfrac{1}{5}i\right) \;+\; \left(\tfrac{1}{10} + \tfrac{1}{5}i\right)\left(\tfrac{-i}{2}\right)^k && (228) \\
&= \tfrac{1}{2},\ \tfrac{2-i}{4},\ \tfrac{3-2i}{8},\ \tfrac{6-3i}{16},\ \tfrac{13-6i}{32},\ \tfrac{26-13i}{64},\ \tfrac{51-26i}{128},\ \dots \\
& \hspace{6em} \text{numerator}\quad \mathrm{Re}_k = -\,\mathrm{Im}_{k+1} = \text{A007910} \\
&\to Gf = \tfrac{2}{5} - \tfrac{1}{5}i \qquad \text{as } k{\to}\infty && (229)
\end{aligned}$$



centroid limit

The power $(\frac{-i}{2})^k$ in GSF_k spirals clockwise towards the limit.



The numerators of GSF_k are integers (Gaussian integers) over 2^{k+1}. The real and imaginary parts are the same sequence, with the real part one ahead,

$$\begin{aligned}
GSF_k &= \frac{GSFnum_{k+1} - i\,GSFnum_k}{2^{k+1}} \\
GSFnum_k &= \tfrac{1}{5}\left(2^{k+1} + [-2, 1, 2, -1]\right) \\
&= 0, 1, 2, 3, 6, 13, 26, 51, 102, 205, 410, 819, 1638, 3277, \dots \qquad \text{A007910} \\
&= \text{binary } k{-}1 \text{ bits } 1100, 1100, \dots \text{ and } {+1} \text{ when } k \equiv 1 \,\text{or}\, 2 \bmod 4 \\
&= 0, 1, 10, 11, 110, 1101, 11010, 110011, 1100110, 11001101, \dots
\end{aligned}$$

$GSFnum_k$ is an integer since $[-2, 1, 2, -1] \equiv -2^{k+1} \bmod 5$.

Theorem 47. *The centroid of the dragon curve segments is within the boundary unit square at the end of the join of unfolded component sub-curves.*

Proof. The centre of the join end square is JEQ_k from theorem 31. The centroid of the two copies of the curve joining is the next level GS_{k+1}. Let δ_k be the offset from JEQ_k to GS_{k+1}.

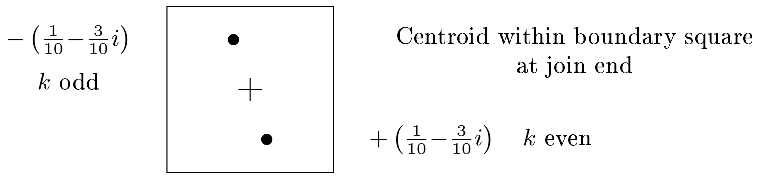
$$\begin{aligned} JEQ_k + \delta_k &= GS_{k+1} \\ \delta_k &= \left(\tfrac{2}{5} - \tfrac{1}{5}i\right) b^{k+1} \;+\; \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) \left(\bar{b}/2\right)^{k+1} \\ &\quad - \Big(\left(\tfrac{3}{5} + \tfrac{1}{5}i\right) b^k \;-\; \left(\tfrac{1}{10} - \tfrac{3}{10}i\right) (-1)^k \Big) \\ &= \left(\tfrac{1}{10} - \tfrac{3}{10}i\right)(-1)^k \;+\; \left(\tfrac{3}{20} + \tfrac{1}{20}i\right) \left(\bar{b}/2\right)^k \end{aligned}$$

Since $|\bar{b}/2| = 1/\sqrt{2} < 1$ the real and imaginary parts of δ_k are at most the bigger two fractions in the coefficients

$$|\operatorname{Re} \delta_k|\,, |\operatorname{Im} \delta_k| \;\le\; \tfrac{3}{10} + \tfrac{3}{20} \;=\; \tfrac{9}{20} \;<\; \tfrac{1}{2}$$

So the centroid is always less than $\tfrac{1}{2}$ away from the middle of the square. ☐

δ_k term $(\bar{b}/2)^k \to 0$ as $k \to \infty$ so δ_k tends to alternate $\pm \left(\tfrac{1}{10} - \tfrac{3}{10}i\right)$ from the centre of the square.



A variation on the centroid can be made by considering each of the $2^k + 1$ curve vertices as having a unit weight. Double-visited points have weight 2. This can be calculated by considering the line segments as weight $\tfrac{1}{2}$ at each of their ends then adding an extra $\tfrac{1}{2}$ at the origin and $\tfrac{1}{2}$ at the endpoint b^k. Those extras give a vertex centroid differing from GS_k by a small amount.

$$\begin{aligned} GV_k &= \left(GS_k 2^k + 0 + b^k/2\right) / (2^k + 1) && \text{vertices centroid} \\ &= GS(k) \;+\; (b^k/2 - GS(k))/(2^k + 1) \\ &= GS(k) \;+\; \left(\tfrac{1}{10} + \tfrac{1}{5}i\right) \frac{b^k + (\bar{b}/2)^k}{2^k + 1} \end{aligned}$$

The centroid of the convex hull around the segments can be calculated from the triangles making up the hull, as per the area HA (211). The centroid of a triangle is the mean of its three vertices and those centroids are weighted by triangle area.

$$\begin{aligned} HG_k &= \frac{HGtotal_k}{HA_k} \qquad k \ge 1 \qquad \text{convex hull centroid} \qquad (230) \\ &= \tfrac{2}{3} + \tfrac{1}{3}i,\; \tfrac{4}{9} + \tfrac{7}{9}i,\; -\tfrac{4}{9} + \tfrac{29}{27}i,\; -\tfrac{37}{23} + \tfrac{40}{69}i,\; -\tfrac{191}{84} - \tfrac{41}{42}i,\; \ldots \end{aligned}$$

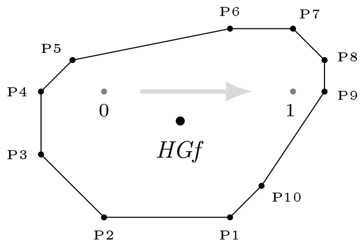
$$HGtotal_k = \sum \text{(triangle centroid)}\,.\,\text{(triangle area)}$$

$$= \begin{cases} \tfrac{1}{3} + \tfrac{1}{6}i & k{=}1 \\ b^k \left(\begin{array}{l} \quad \left(\tfrac{305}{648} - \tfrac{59}{324}i\right) 2^k \\ +\left[\tfrac{-79}{108}+\tfrac{25}{72}i,\ \tfrac{-199}{216}+\tfrac{35}{108}i,\ \tfrac{-83}{108}+\tfrac{11}{72}i,\ \tfrac{-251}{216}+\tfrac{23}{54}i\right].2^{\left\lfloor\frac{k}{2}\right\rfloor} \\ +\left[\tfrac{1}{108}-\tfrac{4}{27}i,\ \tfrac{-1}{108},\ \tfrac{7}{27}+\tfrac{11}{108}i,\ \tfrac{61}{216}-\tfrac{1}{8}i\right] \end{array} \right) & k{\ge}2 \\ \quad +\tfrac{1}{162}\begin{bmatrix} 7{+}4i, & 12{-}2i, & 4{-}7i, & -2{-}12i, \\ -7{-}4i, & -12{+}2i, & -4{+}7i, & 2{+}12i \end{bmatrix} & \end{cases}$$

$$= \tfrac{2+i}{6},\ \tfrac{4+7i}{6},\ \tfrac{-12+29i}{6},\ \tfrac{-111+40i}{6},\ \tfrac{-382-164i}{6},\ \ldots$$

The hull centroid scaled relative to the endpoint b^k is the coefficients of the high power terms in *HGtotal* and *HA*

$$\frac{1}{b^k}\frac{HGtotal_k}{HA_k} \to HGf = \frac{\frac{305}{648}-\frac{59}{324}i}{\frac{7}{6}} = \frac{305}{756} - \frac{59}{378}i \qquad (231)$$
$$= 0.403439\ldots - 0.156084\ldots i$$



convex hull
centroid limit

Real part $\frac{305}{756} = 0.403\ldots$ is close to the segments centroid $\operatorname{Re} Gf = \frac{2}{5} = 0.4$, but the imaginary part $-\frac{59}{378} = -0.156\ldots$ is higher than $\operatorname{Im} Gf = -\frac{1}{5} = -0.2$. Roughly speaking, there is more empty space filled by the hull above (left of the curve), moving the hull centroid up.

The centroid of the XY convex hull can be calculated too by removing the indent parts of theorem 45 from the full hull. The resulting expression is a lot of terms. The limits are

$$XYhullGf = \frac{97401 - 43463i}{254696} = 0.382420\ldots - 0.170646\ldots i \qquad (232)$$
$$DXhullGf = \frac{391569 - 173071i}{966952} = 0.404951\ldots - 0.178986\ldots i$$

8.1 Boundary Centroid

As per theorem 14, left boundary squares are symmetric in 180° rotation so their centroid is their midpoint

$$GLQ_k = \tfrac{1}{2}b^{\max(1,k)} \qquad (233)$$

The same is not true of the left boundary segments, in general. Their centroid differs from the midpoint by a small amount.

Theorem 48. *The centroid of the left boundary segments is*

$$GL_k = \frac{GLtotal_k}{L_k} \qquad\qquad \textit{centroid}$$

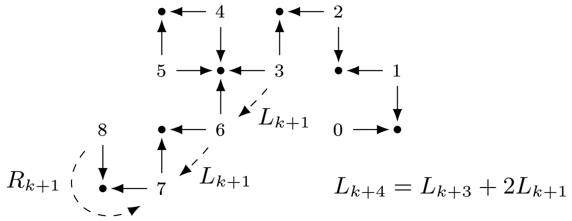
$$GLtotal_k = \begin{cases} \tfrac{1}{2} & k{=}0 \\ \left(\dfrac{1}{2}L_k - \dfrac{3{+}i}{5}k + \dfrac{31{-}33i}{50}\right)b^k - \dfrac{2{+}i}{10}\,\bar{b}^k + \dfrac{2{-}6i}{25}(-1)^k & k \ge 1 \end{cases}$$
$$= \tfrac{1}{2},\ \tfrac{3}{2}{+}\tfrac{1}{2}i,\ 2{+}3i,\ -3{+}9i,\ -16{+}6i,\ -36{-}24i,\ -14{-}122i, \dots$$

Generating function

$$\begin{aligned} gGLtotal(x) = {} & -\tfrac{1}{2}{+}i + \tfrac{1}{25}\frac{2{-}6i}{1+x} - \tfrac{1}{10}\frac{2{+}i}{1-\bar{b}x} + \tfrac{1}{2}gL(bx) \\ & + \tfrac{1}{50}\frac{61{-}23i}{1-bx} - \tfrac{1}{5}\frac{3{+}i}{(1-bx)^2} \end{aligned} \tag{234}$$

$GLtotal_k$ is the sum of the midpoints of the left boundary segments. GL_k divides by L_k for the mean which is the centroid. The term $\frac{1}{2}b^kL_k$ in $GLtotal_k$ is the midpoint between the curve ends which is the boundary squares centroid. The other terms are displacements from there.

Proof. Take the left boundary in the breakdown of figure 14.



The L_{k+1} parts are rotated 180° so $-GLtotal_{k+1}$. They are at positions $-1{+}i$ and -2 so that offset weighted by their segment count L_{k+1}.

7 to 8 is the end portion 3 to 4 of L_{k+3}, shifted down $(-2{-}2i)b^k$ to form L_{k+4}. This section is a right boundary R_{k+1} so the offset is weighted by that length.

$$\begin{aligned} GLtotal_{k+4} = {} & GLtotal_{k+3} \\ & - GLtotal_{k+1} + (-1{+}i)\,b^k\,L_{k+1} \\ & - GLtotal_{k+1} + (-2\quad)\,b^k\,L_{k+1} \\ & + (-2{-}2i)\,b^k\,R_{k+1} \end{aligned} \tag{235}$$

It's convenient to combine L and R with $GLtotal$ using the generating functions. The b^k factor is incorporated as $gL(bx)$ and $gR(bx)$. The first term for the recurrence is at x^4 so the lower terms in each function are subtracted to leave the initial values $GLtotal_0$ to $GLtotal_3$ unchanged.

$$\begin{aligned} gGLtotal(x) = {} & \quad x\,(\,gGLtotal(x) - GLtotal_0 - GLtotal_1x - GLtotal_2x^2) \\ & - 2x^3\,(\,gGLtotal(x) - GLtotal_0) \\ & + (-3{+}i)\,x^3\,(gL(bx) - 1) \\ & + (-3{-}i)\,x^3\,(gR(bx) - 1) \end{aligned}$$

$$+\ GLtotal_0 + GLtotal_1 x + GLtotal_2 x^2 + GLtotal_3 x^3$$

$$= \frac{\begin{pmatrix} \frac{1}{2} - \frac{1}{2}i\,x + (-\frac{1}{2}+\frac{1}{2}i)x^2 + x^3 + (7-7i)x^4 \\ + (-8-4i)x^5 + 8x^6 + (16-16i)x^7 \end{pmatrix}}{(1+x)(1-\bar{b}x)(1-bx)^2(1-bx-2b^3x^3)}$$

Splitting *gGLtotal* into partial fractions gives (234). From it the roots -1, b, $\bar{b}$ are powers in $GLtotal_k$. $1/(1-bx)^2$ gives power term kb^k.

The partial fraction for $1-bx-2b^3x^3$ has numerator $1+b^2x^2$ which is per numerator $1+x^2$ in $gL(x)$ at (101), resulting in $\frac{1}{2}gL(bx)$ for *gGLtotal* and $\frac{1}{2}b^kL_k$ for *GLtotal*.

$$\frac{1+b^2x^2}{1-bx-2b^3x^3} = \frac{gL(bx)+1}{2} \qquad \square$$

The $GLtotal_{k+1}$ terms in (235) give $1-x+2x^3$ for the generating function denominator. This is not the dragon cubic. Rather it factorizes

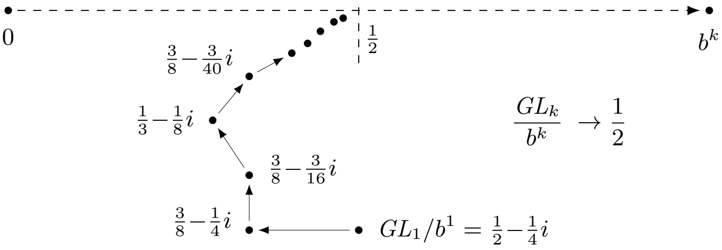
$$1-x+2x^3 \;=\; (1-x)(1-bx)(1-\bar{b}x)$$

There is already a $1-bx$ term from $gR(bx)$ and this cubic divided in is a second one, hence $(1-bx)^2$.

Scaling by b^k for a fractal of unit length gives limit $\frac{1}{2}$ the same as the boundary squares centroid.

$$\frac{GL_k}{b^k} = \frac{1}{2} + \left(\text{terms in } \frac{1}{L_k} \text{ and } \frac{1}{b^k}\right) \to \frac{1}{2} \quad \text{as } k \to \infty$$

For k=0, the curve is a single line segment and its centre is the limit $\frac{1}{2}$. For $k \geq 1$ the limit is approached from below.



Theorem 49. *The centroid of the right boundary squares is*

$$GRQ_k = \frac{GRQtotal_k}{RQ_k} \qquad \textit{squares centroid}$$

$$GRQtotal_k = \begin{cases} \frac{1}{2}-\frac{1}{2}i & \textit{if } k{=}0 \\ \dfrac{b^k}{82}\begin{pmatrix} (70-15i)\,RQ_k \\ +(\ 3+14i)\,RQ_{k+1} \\ -(17+11i)\,RQ_{k+2} \end{pmatrix} + \dfrac{57-21i}{82} & \textit{if } k \geq 1 \end{cases}$$

$$= \tfrac{1-i}{2},\ 2,\ \tfrac{5+3i}{2},\ \tfrac{1+11i}{2},\ \tfrac{-23+19i}{2},\ \tfrac{-71-5i}{2},\ \ldots$$

$GRQtotal_k$ is the sum of the midpoints of the right boundary squares. GRQ_k divides by RQ_k for the mean which is the centroid.

Proof. Repeated unfolding of section 3.2 for right boundary gives $GRQtotal_k$ as a sum of the left boundary centroid GLQ_k weighted by the number of left boundary squares LQ_k. The sum is taken as empty when $k{=}0$.

$$GRQtotal_k = \tfrac{1}{2}{-}\tfrac{1}{2}i \;+\; \sum_{j=0}^{k-1} LQ_j \Big(b^{j+1} - i.GLQ_j \Big)$$

$GLQ_k = \frac{1}{2}b^k$ for $k \ge 1$ from (233) gives $(1{+}\frac{1}{2}i)b^j$ for the sum. An offset is applied for the fact $GLQ_0 \ne \frac{1}{2}b^0$. Including that unconditionally restricts to $k \ge 1$.

$$\begin{aligned} GRQtotal_k &= (\tfrac{1}{2}{-}\tfrac{1}{2}i) \;-\; i(GLQ_0 - \tfrac{1}{2}b^0) \;+\; (1{+}\tfrac{1}{2}i)\sum_{j=0}^{k-1} LQ_j b^j && k \ge 1 \\ &= (1{-}\tfrac{1}{2}i) \;+\; (1{+}\tfrac{1}{2}i)\sum_{j=0}^{k-1} LQ_j b^j && k \ge 1 \qquad (236) \end{aligned}$$

It's convenient to go by the generating functions to express the sum in terms of RQ_k. The constant term $-\frac{1}{2}$ adjusts to include $GRQtotal_0$.

$$\begin{aligned} gGRQtotal(x) &= -\tfrac{1}{2} \;+\; (1{-}\tfrac{1}{2}i)\frac{1}{1{-}x} \;+\; (1{+}\tfrac{1}{2}i)\,x\,\frac{1}{1-x}\,gLQ(bx) \\ &= -\tfrac{1}{2} \;+\; \tfrac{57-21i}{82}\,\frac{1}{1{-}x} \;+\; \tfrac{1}{82}\,\frac{(25{-}20i) + (62{+}16i)x + (62{+}16i)x^2}{1 - bx - 2b^3x^3} \qquad (237) \end{aligned}$$

The cubic part is expressed in terms of $gRQ(x)$ and generating functions of RQ_{k+1} and RQ_{k+2}

$$\begin{aligned} gRQ(x) &= && = (1 + \;\; x + \;\; x^2)/(1{-}x{-}2x^3) \\ gRQ_{k+1}(x) &= \tfrac{1}{x}(gRQ(x) - RQ_0) && = (2 + \;\; x + 2x^2)/(1{-}x{-}2x^3) \\ gRQ_{k+2}(x) &= \tfrac{1}{x^2}(gRQ(x) - RQ_0 - RQ_1 x) && = (3 + 2x + 4x^2)/(1{-}x{-}2x^3) \end{aligned}$$

Some linear algebra for the coefficients in the numerators gives

$$gGRQtotal(x) = -\frac{1}{2} \;+\; \frac{57{-}21i}{82}.\frac{1}{1-x} \;+\; \frac{1}{82}\begin{pmatrix} \;(\;\;70{-}15i)\,gRQ(bx) \\ +(\;\;\;3{+}14i)\,gRQ_{k+1}(bx) \\ +({-}17{-}11i)\,gRQ_{k+2}(bx) \end{pmatrix} \qquad \square$$

$gGRQtotal$ form (237) has denominator $1{-}bx{-}2b^3x^3$ which is a cubic recurrence for $GRQtotal$,

$$GRQtotal_k = b\,GRQtotal_{k-1} + 2b^3\,GRQtotal_{k-3} \;+\tfrac{3}{2}{-}\tfrac{9}{2}i \qquad k \ge 4$$

The centroid of the right boundary segments is similarly

$$GR_k = \frac{GRtotal_k}{R_k} \qquad\qquad \text{segments centroid}$$

$$
GRtotal_k = \begin{cases} \tfrac{1}{2} & \text{if } k{=}0 \\ \left.\begin{array}{l} \dfrac{b^k}{82}\begin{pmatrix} (70{-}15i)\,R_k \\ +\,(\ 3{+}14i)\,R_{k+1} \\ -\,(17{+}11i)\,R_{k+2} \end{pmatrix} \\ \quad + \left(\dfrac{3{+}i}{5}\,k + \dfrac{-111{+}1573i}{2050}\right) b^k - \dfrac{2{+}i}{10}\,\overline{b}^k \\ \quad + \dfrac{3{+}i}{25}(-1)^k + \dfrac{57{-}21i}{41} \end{array}\right\} & \text{if } k \geq 1 \end{cases}
$$

$$
= \tfrac{1}{2},\ \tfrac{3+i}{2},\ 2{+}3i,\ -3{+}9i,\ -26{+}12i,\ -68{-}20i,\ \ldots
$$

$GRtotal_k$ is the sum of the midpoints of the right boundary segments. GR_k divides by R_k for the mean which is the centroid.

$gR(x)$ has the same cubic part as $gRQ(x)$ and gives the same coefficients on R as for RQ.

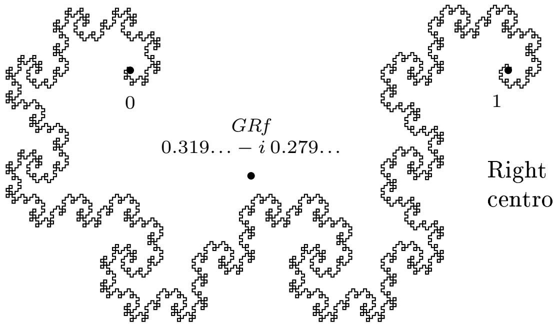
Scaled by b^k to a unit length for the dragon curve fractal, the right boundary squares centroid GRQ_k/b^k has ratios RQ_{k+1}/RQ_k and RQ_{k+2}/RQ_k. Since RQ grows as a power of the cubic root r these approach r and r^2. Similarly GR the right boundary segments centroid.

$$
\frac{GRQ_k}{b^k} \to \frac{1}{82}\begin{pmatrix} (\ \ 70{-}15i) \\ (\ \ \ 3{+}14i)\ RQ_{k+1}/RQ_k \\ (-17{-}11i)\ RQ_{k+2}/RQ_k \end{pmatrix} \qquad \text{squares limit}
$$

$$
\to GRf = \frac{70 + 3r - 17r^2}{82} + \frac{-15 + 14r - 11r^2}{82}\,i \tag{238}
$$

$$
= 0.319629\ldots - i\,0.279118\ldots
$$

$$
\frac{GR_k}{b^k} \to GRf \qquad \text{segments limit the same}
$$



Right boundary centroid limit

The centroid of all boundary squares is the sum of the left and right centroids, then weighted by the total squares.

$$
GBQ_k = \frac{GBQtotal_k}{BQ_k} \qquad \text{boundary squares centroid}
$$

$$
GBQtotal_k = GLQtotal_k + GRQtotal_k
$$

$$GBQtotal_k = \begin{cases} 1 & \text{if } k \ge 1 \\ \dfrac{b^k}{164} \begin{pmatrix} (88{+}28i)\, BQ_k \\ -(63+7i)\, BQ_{k+1} \\ +(29{-}15i)\, BQ_{k+2} \end{pmatrix} + \dfrac{57{-}21i}{82} & \text{if } k \ge 1 \end{cases}$$

$$= 1,\ \tfrac{5}{2}{+}\tfrac{1}{2}i,\ \tfrac{5}{2}{+}\tfrac{7}{2}i,\ -\tfrac{7}{2}{+}\tfrac{19}{2}i,\ -\tfrac{47}{2}{+}\tfrac{19}{2}i,\ -\tfrac{111}{2}{-}\tfrac{45}{2}i, \ldots$$

And similarly all boundary segments B

$$GB_k = \frac{GBtotal_k}{B_k} \qquad \text{boundary segments centroid}$$

$$GBtotal_k = GLtotal_k + GRtotal_k$$

$$GBtotal_k = \begin{cases} 1 & \text{if } k{=}0 \\ \dfrac{b^k}{164} \begin{pmatrix} (88{+}28i)\, B_k \\ -(63+7i)\, B_{k+1} \\ +(29{-}15i)\, B_{k+2} \end{pmatrix} + \dfrac{-29{+}97i}{205} b^k & \text{if } k \ge 1 \\ -\ \dfrac{2{+}i}{5}\,\overline{b}^k + \dfrac{1{-}i}{5}(-1)^k + \dfrac{57{-}21i}{41} & \end{cases}$$

$$= 1,\ 3{+}i,\ 4{+}6i,\ -6{+}18i,\ -42{+}18i,\ -104{-}44i,\ \ldots$$

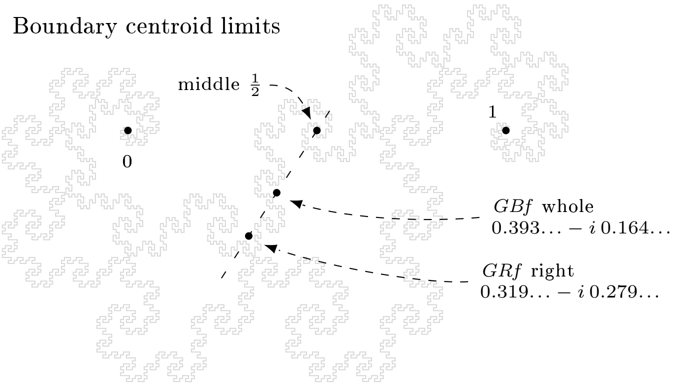
Terms $\pm\frac{3+i}{5}kb^k$ from $GLtotal$ and $GRtotal$ cancel out in $GBtotal$.

Scaled by b^k to a unit length for the dragon curve fractal, the whole boundary centroid GBQ_k/b^k limit is the b^kBQ terms. The limit is the same for the boundary segments.

$$\frac{GBQ_k}{b^k} \to GBf = \frac{88 - 63r + 29r^2}{164} + \frac{28 - 7r - 15r^2}{164}\, i \tag{239}$$

$$= 0.393625\ldots -\ i\, 0.164611\ldots$$

$$\frac{GB_k}{b^k} \to GBf \qquad \text{segments limit the same}$$



Boundary centroid limits

The left boundary centroid is the curve middle $\frac{1}{2}$. The whole boundary limit is between the left and right limits in proportion to the right and left boundary lengths as from page 59.

$$GBf = \frac{1}{r}GRf + \frac{2}{r^3}\cdot\frac{1}{2}$$

8.2 Join Area Centroid

Theorem 50. *The centroid of the join area squares when curve $k \geq 3$ unfolds is*

$$\begin{aligned} GJA_k &= \frac{GJAtotal_k}{JA_k} \qquad\qquad \textit{join centroid, } k \geq 3 \\ &= \tfrac{-5+3i}{2},\ -4-i,\ \tfrac{-21-29i}{6},\ \tfrac{4-23i}{3},\ \tfrac{195-133i}{22},\ \tfrac{136+25i}{9}, \ldots \qquad k{\geq}3 \\ GJAtotal_k &= \frac{b^k}{82}\begin{pmatrix} (57+20i)\ JA_k \\ -(\ 4+\ 5i)\ JA_{k+1} \\ +(\ 9+\ \ i)\ JA_{k+2} \end{pmatrix} + \frac{32-i}{410}\,b^k + \frac{1-3i}{20}(-1)^k + \frac{-21-57i}{164} \\ &= \tfrac{-5+3i}{2},\ -8-2i,\ \tfrac{-21-29i}{2},\ 8-46i,\ \tfrac{195-133i}{2},\ 272+50i, \ldots \qquad k{\geq}3 \end{aligned}$$

$GJAtotal_k$ is the sum of the midpoints of the join area squares. GJA_k divides by JA_k for the mean which is the centroid.

Proof. From theorem 29 the left boundary squares are two join areas and a square in between. The left boundary squares centroid GLQ_k (233) weighted by the number of squares LQ_k is the total of those midpoints.

$$GLQtotal_k = LQ_k.GLQ_k \tag{240}$$

Join area JA_{k+1} at the start of LQ is a 180° rotation of the normal join direction. This is clearest in the diagram of theorem 30, with the first JA_k growing to JA_{k+1} by the two middle parts. The offset b^{k+1} here to move to the origin is weighted by the number of squares JA_{k+1}.

$$GLQtotal_k = \Big(b^{k+1}JA_{k+1} - GJAtotal_{k+1}\Big) + JEQ_k + GJAtotal_k \tag{241}$$

It's convenient to use the generating functions to combine the $b^k JA$ powers.

$$gGJAtotal(x) = \frac{1}{1-x}\Big(-x\,gGLQtotal(x) + x\,gJEQ(x) + gJA(bx)\Big)$$

where

$$\begin{aligned} gGLQtotal(x) &= \tfrac{1}{2}i + \tfrac{1}{2}gLQ(bx) && \text{per (233) and (240)} \\ gJEQ(x) &= \frac{(3+i)/5}{1-bx} - \frac{(1-3i)/10}{1+x} && \text{per (167)} \end{aligned}$$

Expanding $gGJAtotal(x)$ and some linear algebra to express its cubic in terms of $gJA(x)$ gives the theorem. ☐

Breakdown (241) also gives $GJAtotal$ as a cumulative sum.

$$GJAtotal_{k+1} = GJAtotal_k \quad + JEQ_k + b^{k+1}JA_{k+1} - GLQtotal_k \tag{242}$$

$$GJAtotal_k = \sum_{j=2}^{k-1} JEQ_j + b^{j+1}JA_{j+1} - \tfrac{1}{2}b^j LQ_j \qquad k \geq 3 \tag{243}$$

Repeated expansion of (242) goes down to $GJAtotal_3$ with sum starting j=3. Noting that $GJAtotal_3 = JEQ_2 + b^3 JA_3 - GLQtotal_2$ is the j=2 term means the sum can start at j=2.

The LQ part of (243) is similar to that in the right boundary squares $GRQtotal$ at (236). The JA term can become LQ by an identity from the recurrences $JA_{k+1} = \frac{1}{4}\big(LQ_{k+1} + 2LQ_k - 2LQ_{k-1} - 2\big)$. This then gives $GJAtotal$ as three $GRQtotal$ terms if desired.

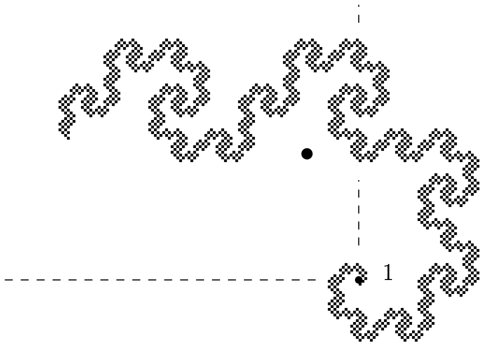
$$GJAtotal_k = \frac{2-i}{10}\begin{pmatrix} GRQtotal_{k+1} + 2i\,GRQtotal_k - 4i\,GRQtotal_{k-1} \\ -\,b^{k+1} \;+\; \tfrac{1}{2}\bar{b}\,(-1)^k \end{pmatrix} \qquad k{\ge}3$$

The join area centroid can be scaled by b^k for the curve a unit length. This has ratios JA_{k+1}/JA_k and JA_{k+2}/JA_k and since JA grows as a power of the cubic root r those ratios approach r and r^2.

$$\frac{GJA_k}{b^k} \to \frac{1}{82}\begin{pmatrix} (57{+}20i) \\ -(\ 4{+}\ 5i)\ JA_{k+1}/JA_k \\ +(\ 9{+}11i)\ JA_{k+2}/JA_k \end{pmatrix}$$

$$\to GJAf = \frac{57{-}4r{+}9r^2}{82} + \frac{20{-}5r{+}r^2}{82}i \quad = \frac{59}{84{+}10r{-}13r^2} + \frac{1}{1{+}4r}i$$

$$= 0.92797\ldots\ +\ 0.17557\ldots i$$



Join area
centroid limit

$GJAf$
$= 0.927\ldots + i\,0.175\ldots$

The equivalent to breakdown (241) for the limit is start join expanded by factor b and weight factor r. The weighted mean is the left centroid limit $\frac{1}{2}$.

$$\frac{r.b\big(1{-}GJAf\big)\ +\ GJAf}{r+1} = \frac{1}{2} \qquad \text{giving } GJAf = \frac{\frac{1}{2}(1+r) - r.b}{-r.b+1}$$

9 Moment of Inertia

The mass moment of inertia $I = \sum mr^2$ of a rigid body rotating around a given axis is the ratio of torque to angular acceleration, similar to the way ordinary mass is the ratio of force to linear acceleration.

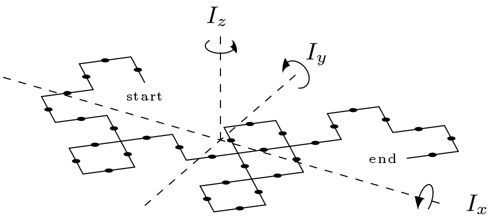


Figure 41:
moment of inertia,
axes through centroid,
x axis parallel to curve
start to end

Theorem 51. *Consider the dragon curve to have a unit mass in the middle of each line segment. With the x axis parallel to the endpoints and all axes through the centroid, the moment of inertia tensor is*

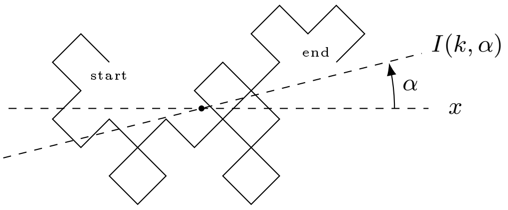
$$\begin{pmatrix} I_x & -I_{xy} & 0 \\ -I_{xy} & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \qquad \begin{matrix} I_x = \sum y^2 & I_{xy} = \sum xy \\ I_y = \sum x^2 & I_z = \sum x^2{+}y^2 \end{matrix}$$

where

$$\begin{aligned} I_x(k) &= \begin{cases} 0 & \textit{if } k = 0 \\ \frac{3}{50}4^k \ - \frac{1}{200}[29, 23, 21, 27].2^k \ - [\frac{1}{25}, \frac{1}{100}] & \textit{if } k \ge 1 \end{cases} \\ &= 0,\, 0,\, \tfrac{1}{2},\, \tfrac{11}{4},\, 13,\, \tfrac{231}{4},\, 239,\, \tfrac{3863}{4},\, 3895,\, \tfrac{62679}{4},\, \ldots \\ I_y(k) &= \begin{cases} 0 & \textit{if } k = 0 \\ \frac{7}{50}4^k \ - \frac{1}{200}[1, 27, 49, 23].2^k \ - [\frac{1}{100}, \frac{1}{25}] & \textit{if } k \ge 1 \end{cases} \\ &= 0,\, \tfrac{1}{4},\, \tfrac{5}{4},\, 8,\, \tfrac{143}{4},\, 139,\, \tfrac{2231}{4},\, 2279,\, \tfrac{36695}{4},\, 36631,\, \ldots \\ I_{xy}(k) &= \tfrac{3}{100}4^k \ + \tfrac{1}{100}[-1, -7, 1, 7].2^k \ - \tfrac{1}{50}(-1)^k \\ &= 0,\ 0,\ \tfrac{1}{2},\ \tfrac{5}{2},\ \tfrac{15}{2},\ \tfrac{57}{2},\ \tfrac{247}{2},\ \tfrac{1001}{2},\ \ldots \\ I_z(k) &= \tfrac{1}{5}4^k \ - \tfrac{1}{20}[3, 5, 7, 5].2^k \ - \tfrac{1}{20} \\ &= 0,\, \tfrac{1}{4},\, \tfrac{7}{4},\, \tfrac{43}{4},\, \tfrac{195}{4},\, \tfrac{787}{4},\, \tfrac{3187}{4},\, \tfrac{12979}{4},\, \ldots \end{aligned}$$

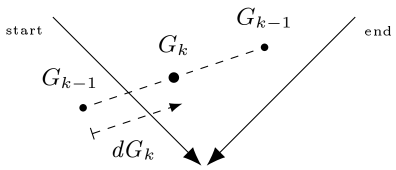
I_x and I_y are the moments of inertia rotating about the x or y axes as in figure 41. They can be combined with I_{xy} in the usual way for inertia about an axis in the x,y plane at angle α from the x axis.

$$I(k,\alpha) \ = \ I_x(k).\cos^2\alpha \ - \ 2I_{xy}(k).\cos\alpha\sin\alpha \ + \ I_y(k).\sin^2\alpha$$



Proof. For $k = 0$, there is a single point mass which has inertia 0.

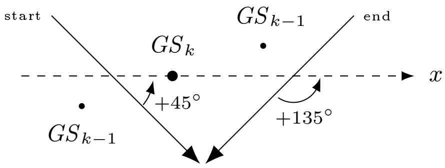
For $k \ge 1$, the inertia is calculated by rotations and the parallel axis theorem from the 2 copies of level $k{-}1$.



dG is the increment from the $k{-}1$ centroid to the k centroid, with the endpoints of k aligned to the x axis. The centroid of the midpoints is the same as the centroid of the segments GS from section 8.

$$\begin{aligned} dG_k &= (GS_k - GS_{k-1})\,.\overline{\omega}_8^k \qquad\qquad \text{centroid increment, } k \ge 1 \\ &= (\tfrac{3}{10}{+}\tfrac{1}{10}i)(\sqrt{2})^k \;+\; (\tfrac{1}{5}{-}\tfrac{1}{10}i)(\tfrac{-i}{\sqrt{2}})^k \end{aligned}$$

The first sub-curve has the x axis at +45° relative to that copy.



The axes are turned by a matrix of rotation in the usual way

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{rotate axes by } +45^\circ \qquad (244)$$

The second sub-curve is axes at +135°. The centroids of those sub-curves are shifted by the parallel axis theorem as a point masses $\pm sG$. Inertia of a point mass is

$$\mathit{Ipoint}(z) = \begin{pmatrix} y^2 & -xy & 0 \\ -xy & x^2 & 0 \\ 0 & 0 & x^2{+}y^2 \end{pmatrix} \qquad \text{for } z = x{+}iy$$

So total

$$\begin{aligned} I(k) = {} & \quad R^{-1}\,.\,I(k{-}1)\,.\,R && \text{first, } +45^\circ && (245) \\ & + R^{-3}\,.\,I(k{-}1)\,.\,R^3 && \text{second, } +135^\circ \\ & + 2^k\,\mathit{Ipoint}(dG_k) && \text{centroid offsets} \end{aligned}$$

It's convenient to scale dG by factor $\sqrt{2}^k$ to incorporate the mass part 2^k, and further factor 2 for integer real and imaginary parts.

$$\begin{aligned} sG_k &= 2\sqrt{2}^k\,dG_k \qquad\qquad \text{scaled centroid increment, } k \ge 1 \qquad (246) \\ &= (\tfrac{3}{5}{+}\tfrac{1}{5}i)\,2^k \;+\; (\tfrac{2}{5}{-}\tfrac{1}{5}i)(-i)^k \end{aligned}$$

$$\begin{aligned} \operatorname{Re} sG_k &= \tfrac{3}{5}2^k + \tfrac{1}{5}\,[2, -1, -2, 1] \\ &= \text{binary } 1001\ldots1001 \quad \text{(zero or more repeat 1001)} \\ &\qquad \text{then 1, 10, 101, or 1010} \quad \text{for } k \text{ bits} \end{aligned}$$

$$\begin{aligned}
&= 1, 1, 2, 5, 10, 19, 38, 77, 154, 307, 614, \ldots && \text{A016029}\\
\operatorname{Im} sG_k &= \tfrac{1}{5}2^k + \tfrac{1}{5}[-1, -2, 1, 2]\\
&= 2^{k-3} + \operatorname{Re} sG_{k-3} \qquad \text{for } k \ge 3\\
&= -2^{k-1} \operatorname{Im} \mathit{GSF}_{k-1} \qquad \text{for } k \ge 1,\ \mathit{GSF} \text{ from (228)}\\
&= \text{binary } 1100 \ldots 1100 \qquad \text{(zero or more repeat 1100)}\\
&\qquad \text{then 11, 110, 1101, or 11010} \quad \text{for } k{-}2 \text{ bits},\ k \ge 4\\
&= 0, 0, 1, 2,\ \ 3,\ \ 6, 13, 26,\ \ 51, 102, 205, \ldots && {\scriptstyle k\ge 2}\ \text{A007910}
\end{aligned}$$

Multiplying through in (245) is mutual recurrences

$$I_x(k) = I_x(k{-}1) \ + I_y(k{-}1) \ + \tfrac{1}{4} \operatorname{Im}^2 sG_k \tag{247}$$

$$I_y(k) = I_x(k{-}1) \ + I_y(k{-}1) \ + \tfrac{1}{4} \operatorname{Re}^2 sG_k \tag{248}$$

$$I_{xy}(k) = \tfrac{1}{4}(\operatorname{Re} sG_k)(\operatorname{Im} sG_k)$$

$$I_z(k) = 2 I_z(k{-}1) \ + \ \tfrac{1}{4} |sG_k|^2$$

Repeatedly expanding I_z is a descending powers-of-2 sum down to sG_1.

$$I_z(k) = \sum_{j=1}^{k} 2^{k-j} \tfrac{1}{4} |sG_k|^2 \qquad \text{since } I_z(0) = 0$$

$I_z = I_x{+}I_y$ is true of any plane figure so (247),(248) are then previous $I_z(k{-}1)$ with squares of the sG parts.

$$I_x(k) = I_z(k{-}1) \ + \tfrac{1}{4} \operatorname{Im}^2 sG_k \tag{249}$$

$$I_y(k) = I_z(k{-}1) \ + \tfrac{1}{4} \operatorname{Re}^2 sG_k \qquad \square$$

An inertia matrix is real and symmetric so can be diagonalized with a suitable matrix of rotation turning to its eigenvectors which are the principal axes. The physical significance of this is that rotation about those axes is perfectly balanced with no torque exerted on the mounting points.

In the usual way for a 2×2 matrix, the eigenvectors are in direction d where

$$d^2 = \big(I_x(k) - I_y(k)\big) \ - \ 2I_{xy}(k)\, i \tag{250}$$

$$\alpha = \tfrac{1}{2} \arctan \frac{-2 I_{xy}(k)}{I_x(k) - I_y(k)} \ + (0 \,\text{or}\, 90^\circ)$$

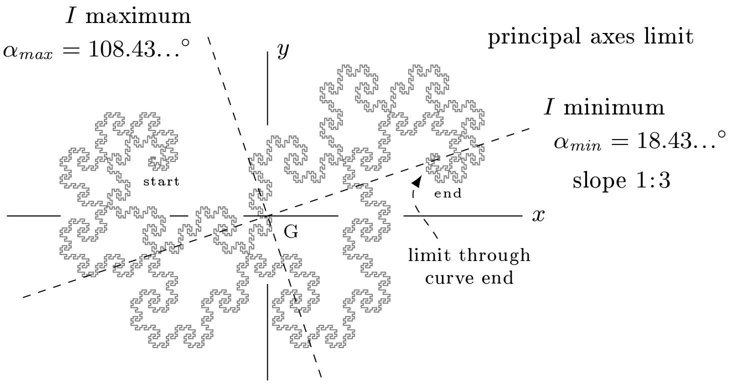
$$= \tfrac{1}{2} \arctan \frac{2\,(\operatorname{Re} sG_k).(\operatorname{Im} sG_k)}{(\operatorname{Re} sG_k)^2 - (\operatorname{Im} sG_k)^2} \tag{251}$$

$$= \arctan \frac{\operatorname{Im} sG_k}{\operatorname{Re} sG_k} \qquad \text{half-angle, or complex square root}$$

$$\to \alpha_{min} = \arctan \tfrac{1}{3} \tag{252}$$

$$\alpha_{min} \qquad\qquad\quad = \ \ 18.434948^\circ \ldots \qquad \text{radians A105531}$$

$$\alpha_{max} = \alpha_{max} + 90^\circ \ = 108.434948^\circ \ldots$$



Roughly speaking, the minimum inertia is where the curve is closest to the axis and the maximum is where the curve is furthest from the axis, as measured by mr^2.

In the diagram it can be seen the α_{min} limit passes through the curve endpoint. The centroid scaled to a unit length is at $Gf = \frac{2}{5} - \frac{1}{5}i$ from (229), so the α_{min} slope $3+i$ goes up to the curve end.

The endpoint is not touched for finite k, except for k=0 where inertia is 0 in all directions. For k=1, 2, have $I_x(k) = 0$ so the minimum inertia is parallel to the x axis through the centroid, but the centroid is below the curve start to end. For $k \geq 2$, the sG slope up from centroid GSF gives an intersection on the horizontal at

$$GSF_k - \frac{\operatorname{Im} GSF_k}{\operatorname{Im} sG_k} sG_k \;=\; 1 + \tfrac{1}{10}\frac{[1, 3, -1, -3] - \frac{1}{2^k}}{\operatorname{Im} sG_k} \quad \neq 1 \qquad k \geq 2$$

$\operatorname{Im} sG_k \to 2^k$ so this intersection approaches 1 but is never equal to 1. Term $[1, 3, -1, -3]$ is bigger than $1/2^k$ for $k \geq 2$ and shows the intersection is alternately just after and just before the endpoint in a period-4 pattern.

In $I_x - I_y$ for the principal axes at (250), the $I_z(k-1)$ terms of (249) cancel to leave just sG parts. (251) is the usual tan double-angle formula and complex squaring $sG_k^2 = -4d^2$ (for $k \geq 1$). The real and imaginary parts of sG are integers so they are also the usual "p, q" parameterization of a Pythagorean triple (217). $\operatorname{Re} sG$ and $\operatorname{Im} sG$ have no common factor, as seen by one Euclidean GCD step to write multiples of them giving ± 1, so they are a primitive Pythagorean triple for $-4d^2$.

$$\operatorname{Re} sG_k - 3 \operatorname{Im} sG_k = [1, 1, -1, -1] \qquad \text{so } \gcd\bigl(\operatorname{Re} sG_k, \operatorname{Im} sG_k\bigr) = 1$$

The first non-zero $\operatorname{Im} sG$ is $sG_2 = 2+i$ giving $-4d = 3+4i$ which is triple $3, 4, 5$.

For the curve scaled to unit length and unit mass, the inertia limit is the coefficients of 4^k in $I_x(k)$ etc. The axes can be rotated by the $\alpha_{\min}$ limit to principal axes.

$$\frac{I(k)}{4^k} \to If = \begin{pmatrix} \frac{3}{50} & -\frac{3}{100} & 0 \\ -\frac{3}{100} & \frac{7}{50} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix}, \qquad \begin{matrix}\text{principal} \\ \text{axes}\end{matrix} \begin{pmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{3}{20} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} \qquad (253)$$

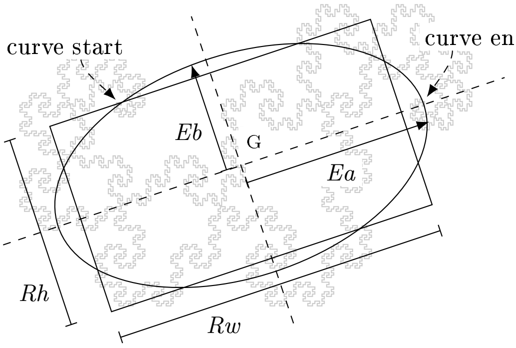
The area of the curve scaled this way is $\frac{1}{2}$ (section 4), so density $mass = 2.area$. A rectangle $w \times h$ of density 1 has inertia $IR_x = \frac{1}{12}wh^3, IR_y = \frac{1}{12}w^3h$. So a rectangle of density 2 with same inertia as the curve limit has width and height along the principal axes

$$\begin{aligned} Rh &= \sqrt[8]{\tfrac{3}{100}} &&= 0.645119\ldots && \text{height} \\ Rw &= \sqrt{3}.Rh &&= 1.117379\ldots && \text{width} \end{aligned}$$

Or an ellipse of radii a, b is inertia $\frac{\pi}{4}ab^3$, $\frac{\pi}{4}a^3b$ so likewise with density 2 and same inertia as the curve limit,

$$\begin{aligned} Eb &= \frac{1}{\sqrt[4]{10\pi\sqrt{3}}} &&= 0.368190\ldots && \text{smaller radius} \\ Ea &= \sqrt{3}.Eb &&= 0.637724\ldots && \text{larger radius} \end{aligned}$$

The ratio of width to height in both cases is $\sqrt{3}$ per ratio 3 in the I_x, I_y principal axes inertia (253).



The area of the curve is $\frac{1}{2}$. The area of the rectangle and ellipse are both bigger than this. Roughly speaking, their area and hence mass are nearer the axes so more is needed to give the same inertia as the spread-out dragon curve.

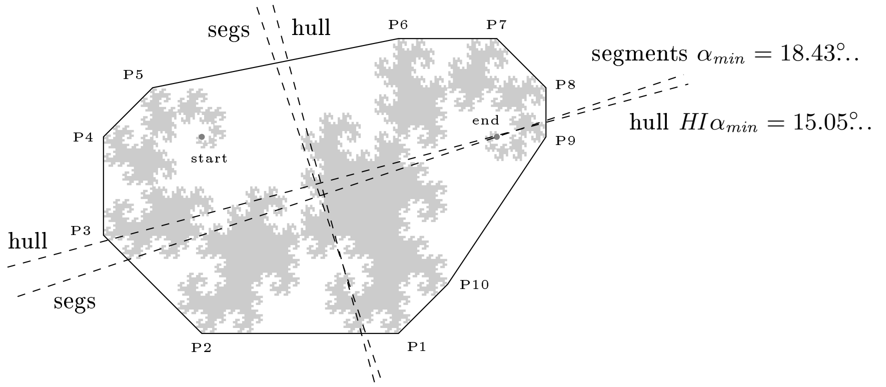
$$\begin{aligned} Rarea &= Rh.Rw = \sqrt[4]{\tfrac{27}{100}} &&= 0.720843\ldots \;> \tfrac{1}{2} \\ Earea &= \pi.Ea.Eb = \sqrt[4]{\tfrac{3\pi^2}{100}} &&= 0.737658\ldots \end{aligned}$$

Both rectangle and ellipse pass close to the curve start, and the ellipse is close to the curve end, but not exactly. In coordinates with G as the origin and rotated $-\alpha_{min}$ to minimum axis horizontal, curve start is at $(-1+i)/\sqrt{10}$ but rectangle $\frac{1}{2}Rh > 1/\sqrt{10}$ so the start is inside the rectangle. Or for the ellipse, curve start location has $((1/\sqrt{10})/Ea)^2 + ((1/\sqrt{10})/Eb)^2 < 1$ so inside the ellipse. Curve end is on the principal axis at $2/\sqrt{10} < Ea$ so it too is just inside the ellipse.

The inertia of the convex hull (section 7) can be compared to the inertia of the curve it surrounds. The hull inertia is calculated from its polygon. For the limit with curve unit length, density $mass = 2.area$ the same as the curve, and axes through the hull centroid HGf,

$$HI_x = \tfrac{9497}{61236} \qquad HI_y = \tfrac{40963}{122472} \qquad HI_{xy} = \tfrac{12743}{244944} \qquad \text{hull inertia}$$

$$HI\alpha_{min} = \tfrac{1}{2} \arctan \tfrac{12743}{21969} = 15.057822^\circ \ldots$$



The hull principal axis angle $HI\alpha_{min}$ is a little smaller, ie. nearer the x axis, than the curve principal axis, but the hull centroid HGf is a little higher.

The hull axis passes close to the curve endpoint but not through it (the way the curve axis does). The hull limit arctan fraction 12743/21969 is not a Pythagorean triple (ie. $12743^2 + 21969^2$ not a perfect square) so the resulting slope is irrational and from rational HGf it cannot pass through 1. At 1, the imaginary part is

$$(\operatorname{Im} HGf) + (1 - \operatorname{Re} HGf) \tan HI\alpha_{min} = 0.004408\ldots$$

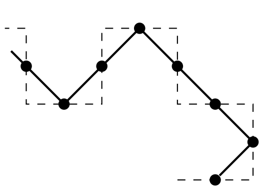
The hull axis is also close to the P3 hull vertex at $-\frac{1}{3} - \frac{1}{3}i$, but again passes the vertical there not at $-\frac{1}{3}i$ but a little below,

$$(\operatorname{Im} HGf) - (\tfrac{1}{3} + \operatorname{Re} HGf) \tan HI\alpha_{min} - (-\tfrac{1}{3}i) = -0.0209660\ldots$$

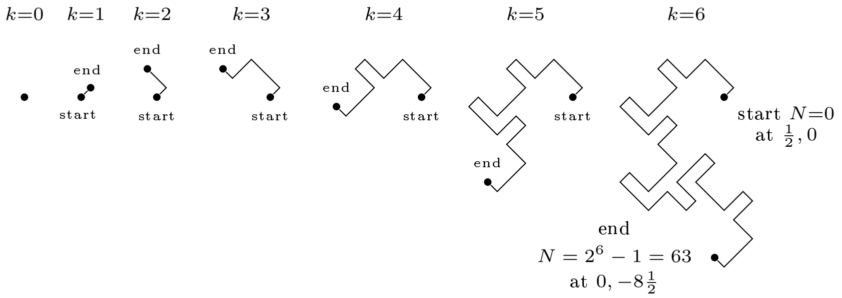
10 Midpoint Curve

A midpoint curve can be made by connecting the midpoints of each dragon curve line segment.

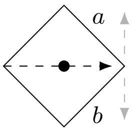
$$midpoint(n) = \tfrac{1}{2}(point(n) + point(n{+}1)) \tag{254}$$



Midpoint curve
with dragon curve
shown dashed

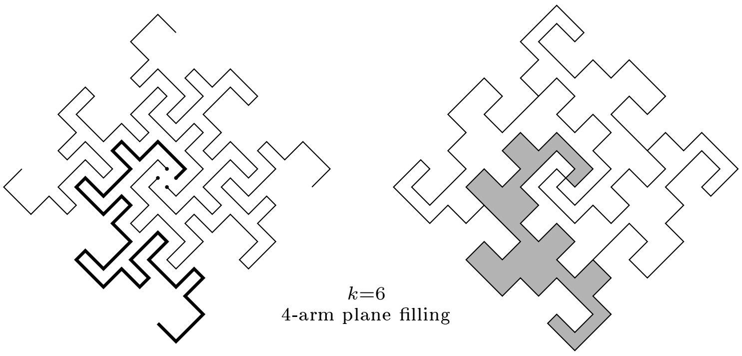


The dragon segments can be thought of as having a diamond shape around each midpoint per Knuth[32].



The dragon curve always turns ±90° so the next diamond touches either side a or b shown. The dragon curve does not repeat any segment so the diamonds are disjoint. When 4 dragon curves fill the plane (theorem 2) the diamonds are a tiling of the plane.

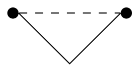
The midpoint curve passes through the touching side of the diamonds and therefore does not cross or overlap itself.



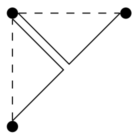
Riddle[46] uses squares adjacent to pairs of segments to show to show the area of the fractal is $\frac{1}{2}$. Such a square is equivalent to the midpoint diamonds here since at the next expansion the segment becomes an L shape which is two sides of the diamond. In level k, each diamond is area $\frac{1}{2}$. There are 2^k segments with midpoints and the curve length start to end is $1/\sqrt{2}^k$. Scaled down to length 1 this is total area exactly $\frac{1}{2}$ for every k and such areas tile the plane without overlap.

The boundary length of the diamonds can be calculated from how they extend from a dragon boundary segment into the adjacent square.

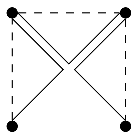
1-side boundary square
2 diamond boundary sides



2-side boundary square
2 diamond boundary sides



3-side boundary square
2 diamond boundary sides



Each 1-side, 2-side or 3-side boundary square becomes 2 diamond sides, so

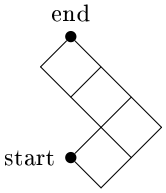
$$\begin{aligned} MB_k &= 2BQ_k && \text{midpoint diamonds boundary} \\ &= 4, 6, 10, 18, 30, 50, 86, 146, 246, 418, \ldots && \text{A203175} \end{aligned}$$

The non-boundary sides of the diamonds touch in pairs. The total dragon segments are 2^k so total 4.2^k diamond sides. The inside sides are the difference from the boundary sides. Counting each inside pair as 1,

$$\begin{aligned} MI_k &= \tfrac{1}{2}\left(4.2^k - MB_k\right) && \text{midpoint inside touches} \\ &= AL_{k+3} \\ &= 0, 1, 3, 7, 17, 39, 85, 183, 389, 815, \ldots && k{\geq}1 \ \text{A003478} \end{aligned}$$

The total segments to make the blocks of the diamonds is the boundary plus the insides. This is equivalent to the 4.2^k total less one for each inside to unduplicate those.

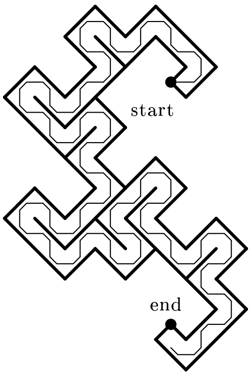
$$\begin{aligned} MS_k &= MB_k + MI_k && \text{block segments} \\ &= 4.2^k - MI_k \\ &= dP_{k+2} && \text{from (179)} \end{aligned} \tag{255}$$



k=2

$MB_2 = 10$ segments boundary
$MI_2 = \ \ 3$ segments inside
$MS_2 = 13$ segments total

Knuth [31] notes that the diamonds can be treated as walls of a labyrinth which the dragon curve bounces off like a light beam or a ball.



k=6

$MW_6 \ = 108$ wall length
$MWI_6 = \ \ 22$ inside wall length

The total wall length to make level k is the block segments MS_k less 1 for each diamond side the dragon curve passes through. There are 2^k dragon segments so 2^k-1 sides passed through.

$$\begin{aligned} MW_k &= MS_k - (2^k{-}1) && \text{wall length} \\ &= 4, 6, 10, 18, 32, 58, 108, 202, 380, 722, \ldots \end{aligned}$$

The wall includes the boundary MB_k. The inside wall is

$$\begin{aligned} MWI_k &= MW_k - MB_k && \text{inner wall length} \\ &= 2\,A_k \quad \text{area} \\ &= 0, 0, 0, 0, 2, 8, 22, 56, 134, 304, \ldots && k{\ge}4 \quad 2{\times}\text{A003230} \end{aligned}$$

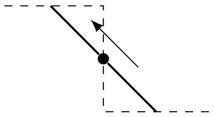
$MWI_k = 2\,A_k$ area can be seen from expanding the various recurrences. Or geometrically 2 segments of inner wall occur at a double visited point, and doubles $D_k =$ area A_k.

Or alternatively, inner walls occur in unit squares enclosed by the dragon curve. The dragon curve doesn't cross itself so those squares are a tree structure starting from a boundary square. The number of inner wall segments in a square is the number of squares connected to it, so the degree of the node. Total degree is $2{\times}(\text{nodes}-1)$. The -1 is the boundary square, giving $2A_k$.

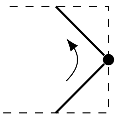
See section 15 for more on area as trees.

10.1 Midpoint Turn Sequence

The dragon curve always turns ±90° so the midpoint curve goes by diagonals. At each midpoint the midpoint curve can turn $+90°, 0°$ or $-90°$ according to the dragon curve $turn(n)$ (section 1.2) before and after that midpoint.

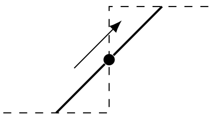


dragon −1, +1



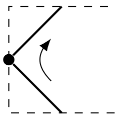
dragon +1, +1

$Mturn = +1$
left



dragon +1, −1

$Mturn = 0$
straight



dragon −1, −1

$Mturn = -1$
right

Counting the first midpoint as $n{=}0$, the first midpoint curve turn is at $n{=}1$. The dragon curve vertices before and after midpoint n are $turn(n)$ and $turn(n{+}1)$. The midpoint turn sequence is then

$$\begin{aligned} Mturn(n) &= \begin{cases} turn(n) & \text{if } turn(n) = turn(n{+}1) \\ 0 & \text{if } turn(n) = -turn(n{+}1) \end{cases} && n \ge 1 \\ &= \tfrac{1}{2}\, sturn(n) && \text{from (17)} \end{aligned}$$

$$
\begin{aligned}
&= 1 - \mathit{sBitAboveLowest}(n) \qquad \text{from (19)} \\
&= 1, 0, 0, 1, 0, -1, 0, 1, 1, 0, -1, 0, 0, -1, 0, 1, 1, \ldots \qquad \text{abs} = \text{A090678}
\end{aligned}
$$

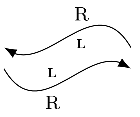
A generating function for *Mturn* follows from *gsBitAboveLowest*(x) from (20), or in terms of the paperfolding *gTurnLpred*.

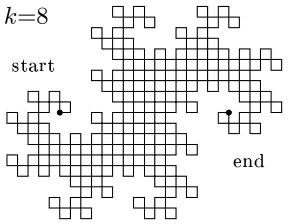
$$
\begin{aligned}
\mathit{gMturn}(x) &= \frac{x}{1-x} - \mathit{gsBitAboveLowest}(x) \\
&= \left(1 + \frac{1}{x}\right) \mathit{gTurnLpred}(x) \; - \; \frac{1}{1-x}
\end{aligned}
$$

A midpoint turn constant can be formed in the style of the paperfolding constant (8), although *Mturn* has terms $-1, 0, 1$ rather than bits 0,1.

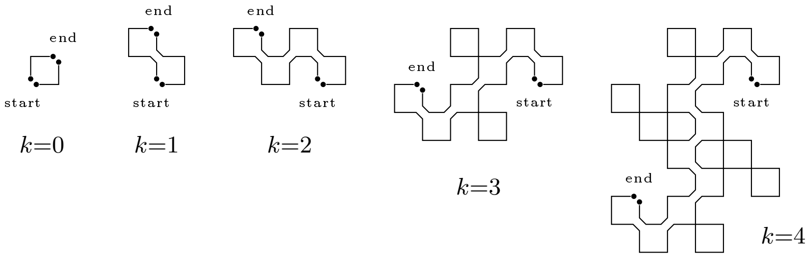
$$
\begin{aligned}
\mathit{gMturn}\left(\tfrac{1}{2}\right) &= \sum_{n=1}^{\infty} \frac{\mathit{Mturn}(n)}{2^n} \qquad = 3\,\mathit{PaperConst} - 2 \\
&= 0.552208564\ldots
\end{aligned}
$$

11 Twindragon

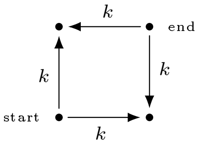




When a dragon curve and its 180° rotation are placed with their left sides touching, start to end and end to start, they mesh perfectly. The result is the twindragon. In the following diagram, corners are chamfered off to show where the two component dragons meet.



The curves mesh perfectly because the two left sides are a square per figure 26.

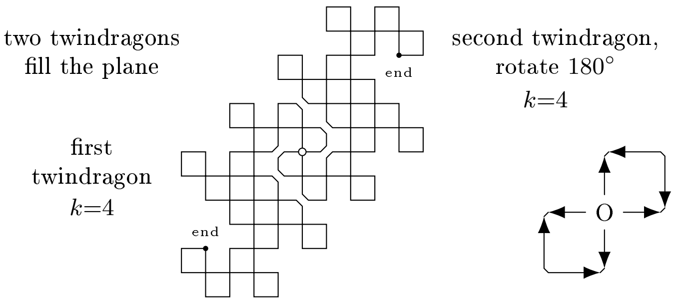


Twindragon
as four k sides

The twindragon levels are numbered here starting from $k{=}0$ for the unit square, so twindragon k is four sides of k. The two dragon curves placed start to end are level $k{+}1$ each.

Another possibility is to start $k{=}0$ as two line segments which are dragon curves $k{=}0$. Mandelbrot [34] does this. But that would be two line segments overlapping which doesn't otherwise occur, and $k{=}0$ as square has the attraction of corresponding to complex base $i{\pm}1$ and generalizing to other self-similar curves on a square grid (the θ-loop of Dekking [14]) where an initial pair of curves may not expand to a square.

Two twindragons at 180° angles at the origin fill the plane. This is since they have 4 dragon curves at the origin and those curves fill the plane (theorem 2). In the following diagram, corners where the two twindragons touch are chamfered off to show how they curl around each other.



Theorem 52. *The boundary length and the area enclosed by the twindragon are*

$$\begin{aligned}
TB_k &= 2R_{k+1} = 2B_k && \textit{boundary length}\\
&= 4, 8, 16, 32, 56, 96, 168, 288, 488, 832, 1416, \ldots && \text{2×A227036}\\
TBQ_k &= 2RQ_{k+1} = 2BQ_k && \textit{boundary squares}\\
&= 4, 6, 10, 18, 30, 50, 86, 146, 246, 418, 710, \ldots && n{\ge}3 \ \text{A203175}\\
TA_k &= 2A_{k+1} + LQ_{k+1} && \textit{area}\\
&= AL_{k+3} + 1\\
&= 1, 2, 4, 8, 18, 40, 86, 184, 390, 816, 1694, \ldots\\
gTA(x) &= \frac{1 - 2x + x^2 - 2x^3 + 4x^4}{(1{-}x)(1{-}2x)(1{-}x - 2x^3)}
\end{aligned}$$

Proof. The boundary is simply the two right sides which are the outside of the square of dragons.

The area is the two curves plus LQ_{k+1} squares on the boundary where they meet. From the recurrences the total is also $AL_{k+3} + 1$.

Or the area within the square is 2^k and then further $2AR_{k+1}$ which is the right-side enclosed area outside the square.

$$TA_k = 2^k + 2\,AR_{k+1}$$

As a square of four sides there are two outside lefts and two outside rights, giving $2\,A_k$.

$$\begin{aligned} TA_k &= 2^k + 2AL_k + 2AR_k \qquad (256) \\ &= 2^k + 2A_k \qquad \square \end{aligned}$$

The boundary and area are related as per lemma 1

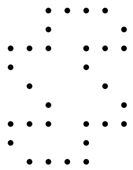
$$4\,TA_k + TB_k = 2.2^{k+2}$$

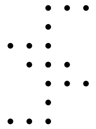
Riddle[46] calculates the boundary (and area) of the twindragon fractal by taking it as two dragon right sides which are made from two joins $RQ_k = JA_{k+2} + 1 + JA_k$ (theorem 29). They then expand into three joins by $JA_k = JA_{k-1} + 1 + 2JA_{k-3}$ (theorem 28). Each +1 is negligible for the fractal.

Theorem 53. *The number of single-visited and double-visited points in the twindragon are*

$$\begin{aligned} TD_k &= TA_k - 1 && \textit{double-visited} \\ &= AL_{k+3} \\ TS_k &= 2^{k+2} - 2\,TD_k && \textit{single-visited} \qquad (257) \\ &= 2S_{k+1} - 2LQ_k && \qquad (258) \\ &= LQ_{k+3} \;\; = 2BQ_k \end{aligned}$$

k=4
single-visited
points
$TS_4 = 30$





k=4
double-visited
points
$TD_4 = 17$

Proof. The double-visited points can be calculated from the area as per lemma 2. In the twindragon, the starting point is revisited and lemma 2 would count that as a double-visit. Here it counts as single-visited, so for example twindragon k=0 has $TD_0 = 0$ no double-visited and k=1 has $TD_1 = 1$ double-visited. Hence $TD_k = TA_k - 1$.

The single points can be calculated by difference from TD as (257) or by lemma 2 from TB_k. Or they can be calculated from the singles in the two dragon curves, less the points where the curves touch and so become doubles. There are LQ_{k+1} many squares in common and between squares two singles touch to becomes a double. At the two endpoints the start and end singles become one single each, so (258).

$$TS_k = 2S_{k+1} - 2\big(LQ_k - 1\big) - 2 \qquad \square$$

Total distinct points visited by the twindragon is the sum

$$\begin{aligned} TP_k &= TS_k + TD_k \\ &= 2P_{k+1} - LQ_{k+1} - 1 \\ &= 2^{k+1} + BQ_k \\ &= MS_k = dP_{k+2} \quad \text{from (255),(179)} \end{aligned}$$

11.1 Complex Base $i+1$

Some Gaussian integers can be written in base $b = i+1$ using digits 0 and 1. Those digits can be taken as bits of an n in binary.

$$\begin{aligned} \mathit{PlusPoint}(n) &= b^{k-1}d_{k-1} + \cdots + b^2 d_2 \ + b d_1 \ + d_0 \\ \text{where } n &= \text{binary } d_{k-1}\ldots d_2 d_1 d_0, \quad \text{low bit } d_0 \\ &= 0,\ 1,\ 1{+}i,\ 2{+}i,\ 2i,\ 1{+}2i,\ 1{+}3i,\ 2{+}3i, \ldots \end{aligned}$$

– Re A290885, Im A290884

Each such point can be taken as a unit square centred on the point. The layout of those squares corresponds to the layout of the unit squares inside the twindragon. The twindragon is rotated −135° for this so it is diamonds rather than squares. In the following diagram, the vertices of the twindragon are chamfered off to better see the inside and outside of the curve.

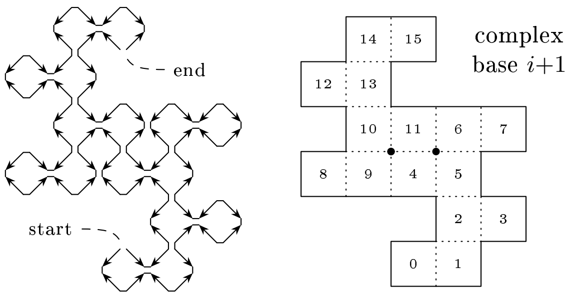
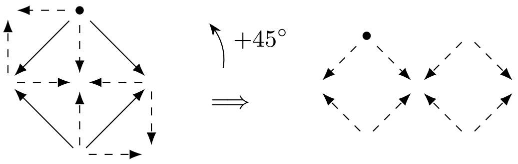


Figure 42

This correspondence works since the segments of a twindragon diamond expand



The area inside the diamond becomes two squares and then rotate +45° to keep the twindragon first segment in a fixed direction. This is the same as the complex base expansion where its $bz + d$ rotates by factor $b = i+1$ and expands horizontally with new bit $d = 0 \, \text{or} \, 1$.

A complex base point at z is twindragon unit square with lower left corner

$$\mathit{PlusToDragon}(z) \ = \ z.(-1{+}i) \tag{259}$$

For the twindragon as a fractal, this correspondence means the points as dragon curves are the same set as the points of complex base, down to the factor $-1{+}i$. The complex base form is often used for the fractal, with descending fractional powers b^{-k}.

The twindragon vertices correspond to the sides of the complex base unit squares. The outer sides of the twindragon segments correspond to the corners of those unit squares.

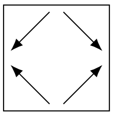
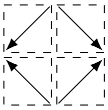


Figure 43: twindragon segments in complex base unit square

Single-visited twindragon vertices are sides of the complex base which are on the boundary of its shape. Double-visited twindragon vertices are sides of the complex base which are inside, ie. not on the boundary. (TS_k and TD_k above.)

The twindragon has some diamonds enclosed by the outside of the curve. These are right-side area $2AR_{k+1}$ as from theorem 52. The segments of such an outside enclosure are 4 corners of the complex base, so is a corner which is fully enclosed. In the k=4 sample above, there are two such outside enclosed diamonds and two enclosed complex base corners (the middle two dots).

The square of figure 43 enclosing a twindragon diamond can also be thought of as four squares, one on each twindragon segment giving 2×2 blocks. These small squares correspond to the diamonds on the segments for the midpoint curve section 10.



The midpoint curve walls per Knuth, from section 10 here, can be applied to the twindragon. The result is two midpoint labyrinths back-to-back and with the endpoints opened to have a twindragon cycle all the way around.

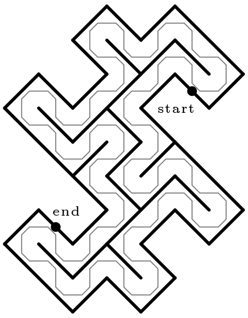


Figure 44:
Twindragon k=4 walls
$TW_4 = 94$ wall length

As per the midpoint curve, the boundary wall is 2 for each boundary square and the non-boundary wall is 2 for each double-visited point.

$$\begin{aligned} TW_k &= 2\left(TBQ_k + TD_k\right) \qquad \text{twindragon wall} \\ &= 4.2^k + 2BQ_k \\ &= 8, 14, 26, 50, 94, 178, 342, 658, 1270, \ldots \end{aligned}$$

The inside walls connected to the boundary correspond to the enclosed area $2AR_{k+1}$ on the outside of the curve. The inside walls not connected to the boundary are a tree structure (see twindragon area tree section 15.3). This is the 2^k squares on the inside of the twindragon and they have $2^k - 1$ connections between them.

$$TW_k = 2\left(TBQ_k + 2AR_{k+1} + 2^k - 1\right)$$

Gilbert[22] draws the twindragon within the complex base shape by putting even numbered twindragon segments through the middle of complex base unit squares,

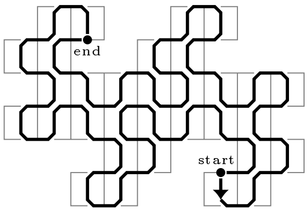
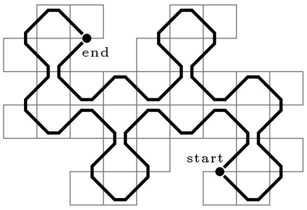


Figure 45: k=5 base i+1, and twindragon $k-1$ turned −90°

This is the same as the diamonds, but one twindragon level smaller. On expansion, its segments spread out and up or down to make the diamonds of figure 42.

Going to 2 expansion levels smaller is also possible. In this case, each twindragon segment corresponds to an i+1 unit square. The centre of the i+1 square is on the right of even number segments and on the left of odd number segments. That is the side of expansion, giving figure 45.

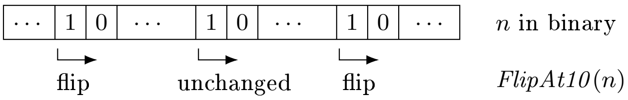


k=5 base i+1 and twindragon $k-2$ turned −45°

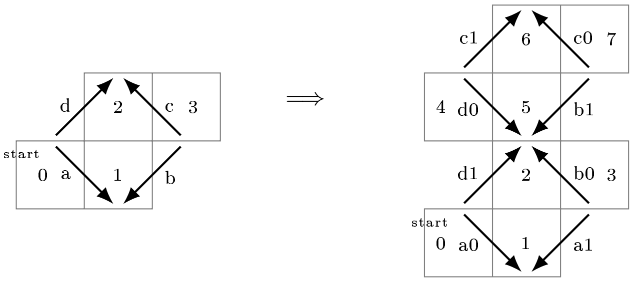
Theorem 54. *The base* i+1 *unit square number corresponding to dragon curve segment* n *is*

$$\begin{aligned} \mathit{NdragonToPlus}(n) &= \mathit{FlipAt10}(2n) \qquad (260)\\ &= 0,1,3,5,\ 7,6,11,13,\ 15,14,12,10,\ 23,22,27,29,\ldots\\ \mathit{FlipAt10}(n) &= \textit{at each } 10 \textit{ pair in } n\textit{, output flip it and bits below}\\ &= 0,1,1,3,3,2,5,7,7,6,6,4,11,10,13,15,15,\ldots\\ \mathit{binary} &= 0,1,1,11,11,10,101,111,111,110,\ldots \end{aligned}$$

FlipAt10 is cumulative, so at an even number of 10 pairs there is no change.

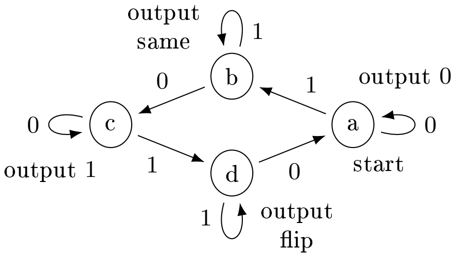


Proof. A set of 4 dragon segments correspond to i+1 unit squares and expand as follows



Number the segments a to d as 0 to 3 which is the curve going anti-clockwise around. So 2 bits of segment number is 2 bits of $i{+}1$ unit square number.

Expansion brings in a new low bit, giving segments and squares numbered 0 to 7. The result is two copies of the 0 to 3 arrangement. The segments in squares 0 to 3 are a 0-bit output for the square number. The segments in squares 4 to 7 are a 1-bit output for the square number. The expanded segments become new segment types,



The state machine begins in a, representing a dragon curve of certain level and two halves of base $i{+}1$. Each bit of segment number n is a transition and output. The output is at bit position 2 places higher. After all bits of n, the state gives the low 2 bits of the output, being the $i{-}1$ square numbers 0,1,3,2 for states a, b, c, d.

The output from a is always 0 so the high 1-bit of n gives a 0-bit at 2 places above. The output is therefore at most one bit more than n. The final 0,1,3,2 can be handled by putting a further two 0-bits through the state machine if preferred.

Working through the possible bits of n shows the output is to bit flip at and below each 10 pair of $2n$. ☐

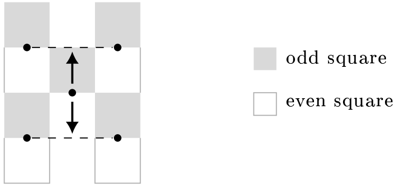
NdragonToPlus is for a single dragon (first half of twindragon), so its first 2^k segments cover half of the 2^{k+1} squares of base $i{+}1$. The other half, by symmetry, are a reversal back from the end $2^{k+1}-1$, which is bit flip $k{+}1$ bits.

In the 2-segment correspondence of figure 45, the segment direction corresponds to base $i{+}1$ odd or even. In an odd square, the curve goes up. In an even square, the curve goes down. (That is with $-90°$ rotation of the twindragon, so these are left and right segments respectively.)

$$2\,\mathit{NdragonToPlus}(n) \equiv \mathit{dir}(2n) \mod 4$$

In terms of the bit flips (260), this is since each 10 is the low end of a run of 1-bits, so that the net flip for the low bit of *NdragonToPlus* is the number of such runs, which is per *dir* at (43). The state machine has the same structure as *dir* from figure 9.

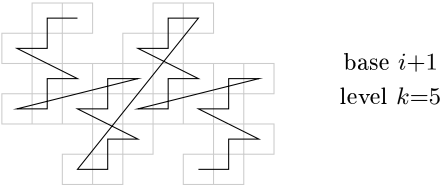
The geometric interpretation is simply that the twindragon starts with an odd square above and even square below, then since it always turns ±90° two segments go to the same type of location.



NdragonToPlus can be reversed by undoing the flips low to high. Given $q = \mathit{NdragonToPlus}(n)$, the unflips are to give $2n$. The unflip at the low of q must turn its least significant bit into a 0-bit for $2n$. Then continuing upwards each 10 output bit pair means flip the bits above there (and not further acting on that 10).

11.1.1 Complex Base $i{+}1$ Jumps

The point numbers n of base $i{+}1$ do not go in the twindragon curve sequence but take jumps across replication levels.



The end of level k is $n{=}2^k{-}1$ and the first of the next level is $n{=}2^k$. Writing those in bits as b powers gives jump

$$\begin{aligned}
\mathit{PlusJump}_k &= \mathit{PlusPoint}(2^k) - \mathit{PlusPoint}(2^k{-}1) \\
&= b^k - \sum_{j=0}^{k-1} b^j \qquad = b^{k+1} - i \\
&= 1,\ i,\ -2{+}i,\ -4{-}i,\ -4{-}5i, \dots \qquad\qquad \text{Re} = \text{A146559} \\
|\mathit{PlusJump}_k| &= \left|b^{k+1} - i\right| \\
&= \sqrt{2^{k+1} + [-2,-2,-2,0,2,2,2,0].2^{\lceil k/2 \rceil} + 1} \\
&= 1,\ 1,\ \sqrt{5},\ \sqrt{17},\ \sqrt{41},\ 9,\ \sqrt{145},\ \sqrt{257},\ \sqrt{481},\ 31, \dots
\end{aligned}$$

When $k \equiv 1 \bmod 4$, the power b^{k+1} is vertical the same as offset i giving integer $|\mathit{PlusJump}_k|$, and otherwise it is irrational. When $k \equiv 3 \bmod 4$, $|\text{Im}| = 1$ and the next square is further than that. When $k \equiv 0, 2 \bmod 4$, using a difference of squares with power of 2 product like (215) is

$$\begin{aligned}
x = |\text{Re}| = 2^d, \quad y = |\text{Im}| = 2^d{\pm}1, \quad c = |\mathit{PlusJump}_k| \\
x^2 = (c+y)(c-y) = 2^{e+f} \quad \text{with } e{+}f = 2d
\end{aligned}$$

$$\begin{aligned} c+y &= 2^e & \qquad 2c &= 2^e + 2^f \\ c-y &= 2^f & \qquad 2y &= 2^e - 2^f \end{aligned}$$

When $y{=}2^d+1$ the only solution is $y{=}3$ to have a single bit run for $2^e{-}2^f$, but $y{=}3$ does not occur. When $y{=}2^d-1$ would need $e{=}d{+}1, f{=}1$ for that bit run, which is then $e{+}f = d{+}2$ and for $e{+}f = 2d$ must have $d{=}2$, which is $k{=}3,4$ which are not $y{=}2^d-1$.

The total distance of all jumps within level k is, with the sums taken as empty and so 0 when $k{=}0$,

$$\begin{aligned} \mathit{PlusDist}_k &= \sum_{n=1}^{2^k-1} \bigl|\mathit{PlusPoint}(n) - \mathit{PlusPoint}(n{-}1)\bigr| \\ &= 2\,\mathit{PlusDist}_{k-1} + |\mathit{PlusJump}_{k-1}| \\ &= \sum_{j=0}^{k-1} 2^j\,|\mathit{PlusJump}_{k-1-j}| \\ &= 0,\ 1,\ 3,\ 6{+}\sqrt{5},\ 12{+}2\sqrt{5}{+}\sqrt{17}, \dots \end{aligned}$$

There are a total $2^k{-}1$ jumps in level k. Many are small and some are big. The small ones are greater in number by enough for a mean jump to converge

$$\begin{aligned} \mathit{PlusMean}_k = \frac{\mathit{PlusDist}_k}{2^k-1} &= \frac{2^k}{2^k{-}1} \sum_{j=1}^{k} \left|\frac{1}{b^j} + \frac{i}{2^j}\right| \\ &= \frac{2^k}{2^k{-}1} \sum_{j=1}^{k} \left|\frac{1}{\sqrt{2}^j} - \frac{i}{(2\omega_8)^j}\right| && (261) \\ &= \frac{2^k}{2^k{-}1} \sum_{j=1}^{k} \frac{1}{\sqrt{2}^j} \sqrt{1{+}\frac{[0,-2,-2,-2,\,0,2,2,2]}{2^{\lceil j/2\rceil}}{+}\frac{1}{2^j}} && (262) \\ &\to 1.932993\dots \quad \text{as } k{\to}\infty \end{aligned}$$

(261) puts an ω_8^j though to reach a real $1/\sqrt{2}^j$ and a rotating offset from it. That offset is a factor at (262) making terms a little bigger or smaller according to $j \bmod 8$. The maximum factor is $\sqrt{41/32}$ at $j{=}5$ where the periodic part $2^{\lceil j/2\rceil}$ is first positive, so the sum is monotonic and bounded above.

Negatives in the periodic part $2^{\lceil j/2\rceil}$ are before the positives so are bigger. Roughly speaking, this is why the sum is smaller than if the factors were omitted $\sum_{j=1}^{\infty} 1/\sqrt{2}^j = 1{+}\sqrt{2} = 2.414213\dots$ (A014176).

The mean is simpler if jumps are measured instead by

$$\mathit{Manhattan}(z) = |\operatorname{Re} z| + |\operatorname{Im} z|$$

This is $\geq$ the geometric length (by triangle inequality) so the total distance and mean are bigger.

$$\begin{aligned} \mathit{Manhattan}(\mathit{PlusJump}_k) &= 2.2^{\lfloor k/2\rfloor} + [-1,-1,-1,1,\ 1,1,1,1] \\ &= 1,1,3,5,9,9,17,17,31,31,\dots \end{aligned}$$

$$\begin{aligned}
\mathit{PlusDistManhattan}_k &= \sum_{j=0}^{k-1} 2^j \,\mathit{Manhattan}\big(\mathit{PlusJump}_{k-1-j}\big)\\
&= \tfrac{572}{255}2^k - [3,4].2^{\lfloor k/2\rfloor} + \tfrac{1}{255}[193, 131, 7, -241, -227, -199, -143, -31]\\
&= 0, 1, 3, 9, 23, 55, 119, 255, 527, 1085, \ldots
\end{aligned}$$

$$\frac{\mathit{PlusDistManhattan}_k}{2^k-1} \to \frac{572}{255} = 2.243137\ldots \qquad \text{mean Manhattan jump}$$

11.2 Complex Base $i{-}1$

Complex base $i{-}1$ of Khmelnik [29] and Penney [40] is similar to base $i{+}1$.

$$\begin{aligned}
\mathit{MinusPoint}(n) &= (i{-}1)^{k-1}d_{k-1} + \cdots + (i{-}1)^2 d_2 + (i{-}1)d_1 + d_0 \qquad (263)\\
\text{where } n &= \text{ binary } d_{k-1}\ldots d_2 d_1 d_0, \quad \text{low bit } d_0\\
&= 0,\ 1,\ -1{+}i,\ i,\ -2i,\ 1{-}2i,\ -1{-}i,\ -i, \ldots
\end{aligned}$$

Re A318438, Im A318439

The inverse is the point number for a location z,

$$\mathit{MinusUnpoint}(z) = n$$

This can be found by bits low to high. The powers and digits in (263) mean low bit d_0 is determined by $z \equiv 0 \text{ or } 1 \bmod i{-}1$, which means z even or odd ($x{+}yi$ with $x{+}y$ even or odd). Then subtract d_0 from z, divide by $i{-}1$, and repeat. There are no rotations or sign changes to apply at successive bits.

The layout of points 0 to 2^k-1 has the same shape as base $i{+}1$ but in mirror image and offset.

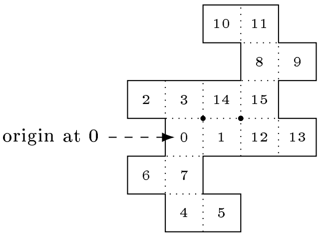
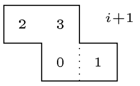
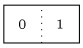
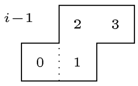


Figure 46: complex base $i{-}1$

This can be seen from bases $i{+}1$ and $i{-}1$ starting as the same 2×1 brick and thereafter each 2×1 brick expands to a mirror image.







Or algebraically going termwise

$$\begin{aligned}
& d_0 + (i{+}1)d_1 \quad + (i{+}1)^2 d_2 + (i{+}1)^3 d_3 \quad + \cdots\\
& \quad - (i{+}1) \qquad\qquad\quad - (i{+}1)^3 \qquad - \cdots \qquad (264)\\
&= \operatorname{conj}\big(d_0 + (i{-}1)(1{-}d_1) + (i{-}1)^2 d_2 + (i{-}1)^3(1{-}d_3) + \cdots\big)
\end{aligned}$$

$$\mathit{PlusPoint}(n) - \mathit{PlusOffsetV}_k = \overline{\mathit{MinusPoint}(\mathit{FlipOddBits}_k(n))}$$

conj is complex conjugate. The even powers of $i{+}1$ and $i{-}1$ are unchanged by this. At the odd powers it's necessary to negate to $-d$ for $-(\overline{i{-}1}) = i{+}1$. This negation is accomplished by a bit flip $0{\leftrightarrow}1$ to $1{-}d$ and then subtract an offset for the $1.(i{+}1)$. That offset is the terms shown at (264).

The conj doesn't change the origin of the $i{-}1$ part and the bit flips only change the order of the points. So the terms subtracted are the location of the $i{-}1$ origin in the $i{+}1$ shape, ready to mirror vertically there.

$$\begin{aligned} \mathit{PlusOffsetV}(k) &= b + b^3 + b^5 + \cdots + b^{odd<k} \qquad\qquad b = i{+}1 \\ &= \tfrac{-1+3i}{5}\left(1 - (2i)^{\lfloor k/2 \rfloor}\right) \\ &= 0,\ 0,\ 1{+}i,\ 1{+}i,\ -1{+}3i,\ -1{+}3i,\ -5{-}i,\ -5{-}i,\ 3{-}9i,\ 3{-}9i,\ \ldots \end{aligned}$$

It's also possible to mirror horizontally by $-\operatorname{conj}$. This is equivalent to a 180° rotation of the vertical mirror. The shape is symmetric in 180° rotation but the $i{+}1$ origin is not the middle so the offset is different.

$$\begin{array}{rllll} & 1.d_0 & +\ (i{+}1)d_1 & +\ (i{+}1)^2 d_2 & +\ (i{+}1)^3 d_3 + \cdots \\ & -1 & & -\ (i{+}1)^2 & -\ \cdots \\ = -\operatorname{conj}\big(& 1.(1{-}d_0) & +\ (i{-}1)d_1 & +\ (i{-}1)^2(1{-}d_2) & +\ (i{-}1)^3 d_3 + \cdots \big) \end{array}$$

$$\mathit{PlusPoint}(n) - \mathit{PlusOffsetH}_k \ = \ -\overline{\mathit{MinusPoint}(\mathit{FlipEvenBits}_k(n))}$$

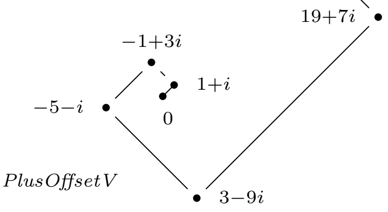
This time the digits at odd terms are unchanged by $-\operatorname{conj}$ and even terms are bit flipped and the offset subtracted. The offset is the location of the $i{-}1$ origin within the $i{+}1$ shape, ready to mirror horizontally there.

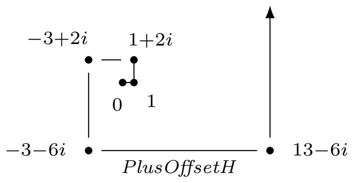
$$\begin{aligned} \mathit{PlusOffsetH}(k) &= 1 + b^2 + b^4 + \cdots + b^{even<k} \\ &= \tfrac{1+2i}{5}\left(1 - (2i)^{\lceil k/2 \rceil}\right) \\ &= 0,\ 1,\ 1,\ 1{+}2i,\ 1{+}2i,\ -3{+}2i,\ -3{+}2i,\ -3{-}6i,\ -3{-}6i,\ \ldots \end{aligned}$$

4-repeats Re = A014985 Im = 2×A014985

In $k{=}4$ sample figure 46, these ends are points $n{=}10$ at the top and $n{=}5$ at the bottom which are respectively $\mathit{PlusOffsetV}(4) = -1{+}3i$ and $\mathit{PlusOffsetH}(4) = 1{+}2i$.

In both, offsets the effect of power $(2i)^h$ is to spiral around anti-clockwise.





The two offsets are alternating powers of b. The one with the smaller high term is nearest the origin (smaller in magnitude). This corresponds to the join end square from theorem 31. This is on the left boundary of the two component dragons and is the 3-overlap square of figure 28.

$$\mathit{PlusOffsetJ}_k = b^{k-2} + b^{k-4} + b^{k-6} + \cdots b^{0 \text{ or } 1}$$

$$\begin{aligned}
&= (-\tfrac{1}{5}-\tfrac{2}{5})\,b^k + [\tfrac{1}{5}+\tfrac{2}{5}i,\ -\tfrac{1}{5}+\tfrac{3}{5}i] \\
&= \min\big(\mathit{PlusOffsetH}_k, \mathit{PlusOffsetV}_k\big) \qquad \text{by magnitude} \\
&= \begin{cases} \mathit{PlusOffsetH}_k & \text{if } k \text{ even} \\ \mathit{PlusOffsetV}_k & \text{if } k \text{ odd} \end{cases}
\end{aligned}$$

$$= 0,\ 0,\ 1,\ 1{+}i,\ 1{+}2i,\ -1{+}3i,\ -3{+}2i,\ -5{-}i,\ -3{-}6i,\ \dots \qquad \text{Re=A077950} \quad \text{Im=A077870}$$

$$\mathit{PlusToDragon}(\mathit{PlusOffsetJ}_k) = \mathit{JEC}_k$$

The bigger of the two offsets is the next *PlusOffsetJ*, being one higher b power.

$$\mathit{PlusOffsetJ}_{k+1} = \text{max high power}\big(\mathit{PlusOffsetH}_k, \mathit{PlusOffsetV}_k\big)$$

Penney[40] characterizes n on the real axis of base $i{-}1$ by considering what combinations of the powers $(i{-}1)^k$ can sum to Im=0, and gives the following x to n conversion rule. Base -4 is since $(i{-}1)^4 = -4$ has Im=0.

$$\begin{aligned}
\mathit{MinusUnpoint}(x) &= \text{write } x \text{ in base } -4 \text{ with digits 0 to 3,} \\
&\quad \text{change digits to bits } 0000, 0001, 1100, 1101 \text{ respectively} \qquad (265) \\
&= 0, 1, 12, 13, 464, 465, 476, 477, 448, 449, \dots \qquad x \ge 0 \qquad \text{A066321} \\
&= \ 29, 28, 17, 16, 205, 204, 193, 192, 221, 220, \dots \quad x < 0 \qquad \text{A256441}
\end{aligned}$$

At (265), digits 0 to 3 are replaced by bit strings whose number of 1-bits are also 0 to 3 respectively, hence

$$\begin{aligned}
\mathit{CountOneBits}\big(\mathit{MinusUnpoint}(x)\big) &= \ \text{sum base } -4 \text{ digits of } x \\
&= 0, 1, 2, 3, 4, 5, 6, 7, 3, 4, \dots \qquad x \ge 0 \qquad \text{A066323} \\
&= \quad 4, 3, 2, 1, 5, 4, 3, 2, 6, \dots \qquad x < 0
\end{aligned}$$

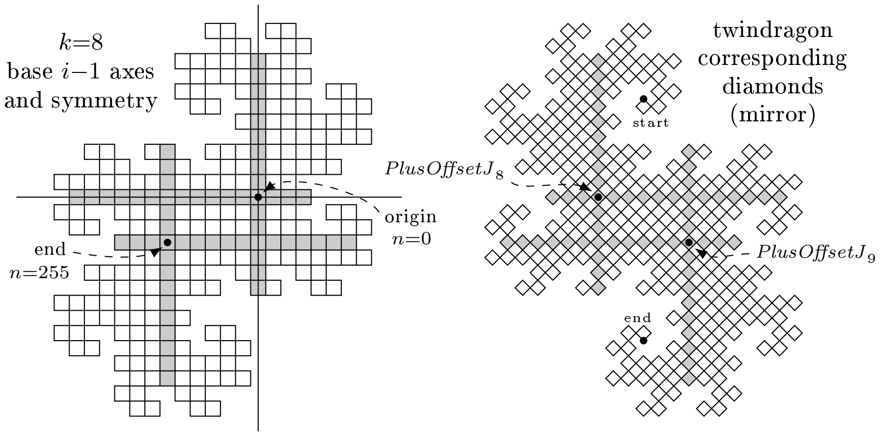
Penney gives a more complicated rule for the imaginary axis. Write yi in base $8i$ with digits $-6i$ to i at even positions and -4 to 3 at odd positions (least significant digit as position 0) then substitute those digits for certain 6-bit replacements to form n.

Another approach (as for instance in demo code by Jörg Arndt) is to re-use the x axis form, since $(i{-}1)^2 = -2i$ so that

$$\begin{aligned}
\mathit{MinusPoint}(4n) &= -2i\,\mathit{MinusPoint}(n) \\
\mathit{MinusUnpoint}(yi) &= \tfrac{1}{4}\,\mathit{MinusUnpoint}(2y)
\end{aligned}$$

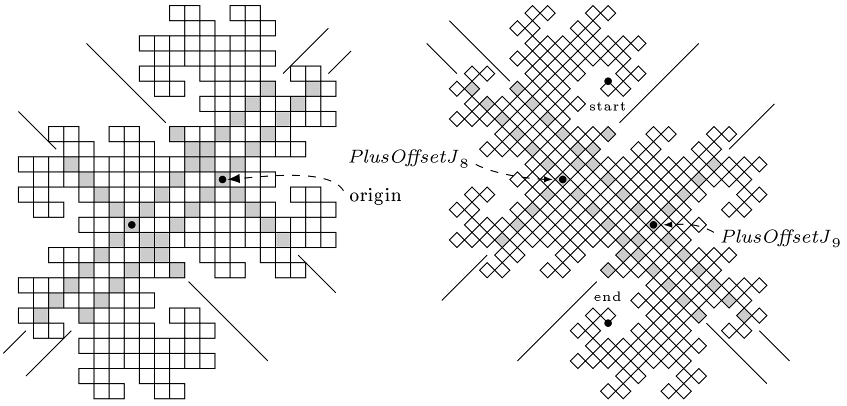
Within a level k, on both axes, these representations have the high digit restricted but are a contiguous part of the respective axis. In iterations of the twindragon, these are contiguous diagonals of unit squares inside the curve. Reckoned from the start of the twindragon, these diagonals are through the two *PlusOffset* locations.

The following diagram shows complex base $i{-}1$ and twindragon. The twindragon is rotated $-135°$ to correspond to base $i{+}1$ which in turn corresponds to base $i{-}1$ in horizontal mirror image.



When each complex base point expands to two points by multiply $i-1$ and add 0 or 1, the x, y axes become the diagonals of the next higher level. So in each level the diagonals are contiguous lines too.

In iterations of the twindragon, these diagonals are contiguous verticals and horizontals of unit squares inside the curve, which is every second square.



The twindragon endpoint b^{k+1} is straight or diagonal according as k is odd or even. The respective contiguous lines straight or diagonal can be taken aligned to the endpoint. Akiyama and Scheicher [1] show that axes aligned to the endpoint are contiguous in the complex base shape fractal.

The straight lines intersect to make a rectangle. The diagonal lines intersect to make a rectangle at 45°.

The twindragon has all the inside unit square within these rectangles, since if there were any holes the adjacent dragon curves could not get in and out again between the contiguous squares for plane filling.

It's convenient to take the straight and diagonal rectangles alternately so that the side is aligned to the twindragon ends. In this form, the area of the aligned rectangle in the complex base shape is

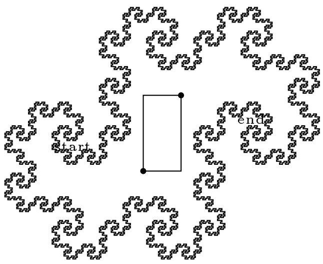
$$\begin{aligned} \mathit{AlignedRect}_k &= |\mathrm{Re}\times\mathrm{Im}| \text{ of } \ \omega_8^k.(\mathit{PlusOffsetJ}_{k+1} - \mathit{PlusOffsetJ}_k) \\ &= \tfrac{2}{25}2^k + \tfrac{1}{25}[-4,7,-3,-1,4,-7,3,1].2^{\lfloor k/2 \rfloor} + \tfrac{1}{50}[4,3,-4,-3] \\ &= 0, \tfrac{1}{2}, 0, \tfrac{1}{2}, 2, \tfrac{3}{2}, 6, \tfrac{21}{2}, 18, \tfrac{91}{2}, 78, \tfrac{325}{2}, \ldots \end{aligned}$$

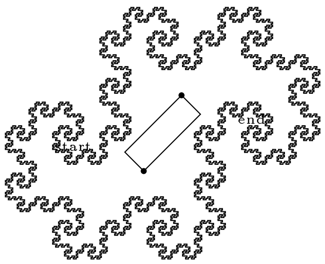
Or rectangle which is at 45° to the twindragon endpoints, so the opposite diagonal/straight,

$$\begin{aligned} \mathit{OppositeRect}_k &= |\mathrm{Re}\times\mathrm{Im}|\text{ of }\ \omega_8^{k+1}.(\mathit{PlusOffsetJ}_{k+1} - \mathit{PlusOffsetJ}_k) \\ &= \tfrac{3}{50}2^k + \tfrac{1}{25}[-3,-1,4,-7,3,1,-4,7].2^{\lfloor k/2\rfloor} + \tfrac{1}{50}[3,-4,-3,4] \\ &= 0, 0, \tfrac{1}{2}, 0, \tfrac{3}{2}, 2, \tfrac{5}{2}, 10, \tfrac{27}{2}, 30, \tfrac{133}{2}, 114, \ldots \end{aligned}$$

Limits for the area of the rectangles as a fraction of the 2^k complex base area are the high coefficients

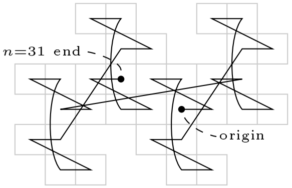
$$\frac{\mathit{AlignedRect}_k}{2^k} \to \frac{2}{25} \qquad \frac{\mathit{OppositeRect}_k}{2^k} \to \frac{3}{50}$$





11.2.1 Complex Base $i-1$ Jumps

The point numbers of base $i-1$ do not go in the twindragon curve sequence but take jumps between replication levels.



base $i-1$
level k=5

A similar calculation to *PlusMean* from section 11.1.1 can be made. The limit for the mean in base $i-1$ is bigger than in $i+1$.

$$\begin{aligned} \mathit{MinusJump}_k &= \mathit{MinusPoint}(2^k) - \mathit{MinusPoint}(2^k-1) \\ &= (i-1)^k - \sum_{j=0}^{k-1}(i-1)^j \quad = -\tfrac{1}{5}\Big((3+4i)(i-1)^{k+1} + 2+i\Big) \\ &= 1,\ -2+i,\ -3i,\ 2+3i,\ -6-i, \ldots \qquad \tfrac{1}{2}\text{ Re } k\geq 2\text{, and Im } k\geq 0\text{ A137426} \end{aligned}$$

$$\begin{aligned} |\mathit{MinusJump}_k| &= \sqrt{2^{k+1} + \tfrac{1}{5}[-6,2,2,-4,6,-2,-2,4].2^{\lceil k/2\rceil} + \tfrac{1}{5}} \\ &= \surd\ 1, 5, 9, 13, 37, 61, 125, 269, 493, 1037, \ldots \end{aligned}$$

$$\begin{aligned} \mathit{MinusDist}_k &= \sum_{n=1}^{2^k-1} \Big|\mathit{MinusPoint}(n) - \mathit{MinusPoint}(n-1)\Big| \\ &= \sum_{j=0}^{k-1} 2^j \left|\mathit{MinusJump}_{k-1-j}\right| \end{aligned}$$

$$= 0,\ 1,\ 2{+}\sqrt{5},\ 7{+}2\sqrt{5},\ 14{+}4\sqrt{5}{+}\sqrt{13}, \ldots$$

$$\frac{\mathit{MinusDist}_k}{2^k{-}1} = \frac{2^k}{2^k-1} \sum_{j=1}^{k} \left| \frac{1}{(i{-}1)^j} + \frac{\frac{2}{5}{+}\frac{1}{5}i}{2^j} \right|$$
$$= \frac{2^k}{2^k-1} \sum_{j=1}^{k} \frac{1}{\sqrt{2}^j} \sqrt{1 + \tfrac{1}{5}\,\frac{[4,-6,2,2,\ -4,6,-2,-2]}{2^{\lceil j/2 \rceil}} + \tfrac{1}{5}\,\frac{1}{2^j}}$$
$$\to 2.273168\ldots \quad \text{as } k{\to}\infty$$

MinusJump is an integer length at $|\mathit{MinusJump}_0| = 1$ unit step right, and $|\mathit{MinusJump}_2| = 3$ down. But otherwise $|\mathit{MinusJump}_k|$ is irrational since working through the powers shows $k \geq 3$ has $\mathrm{Re} \equiv 2 \bmod 4$ and $\mathrm{Im} \equiv 3 \bmod 4$, but a Pythagorean triple must have one leg a multiple of 4, as from (217).

Jumps by Manhattan distance are

$$\mathit{Manhattan}(\mathit{MinusJump}_k) = \tfrac{1}{5}\,[8,7]\,2^{\lceil k/2 \rceil} + \tfrac{1}{5}\,[-3{,}1,-1,-3,\,3,-1{,}1{,}3]$$
$$= 1, 3, 3, 5, 7, 11, 13, 23, 25, 45, \ldots$$

$$\mathit{MinusDistManhattan}_k = \sum_{j=0}^{k-1} 2^j\ \mathit{Manhattan}\big(\mathit{MinusJump}_{k-1-j}\big)$$
$$= \tfrac{46}{17}\, 2^k - \tfrac{1}{5}\,[15, 22].2^{\lfloor k/2 \rfloor} + \tfrac{1}{85}\,[25, -1, 15, 13, -25, 1, -15, -13]$$
$$= 0, 1, 5, 13, 31, 69, 149, 311, 645, 1315, \ldots$$

$$\frac{\mathit{MinusDistManhattan}_k}{2^k - 1} \to \frac{46}{17} = 2.705882\ldots$$

11.2.2 Complex Base $i{-}1$ Boundary

Gilbert[21] calculates the boundary length of the $i{-}1$ shape (and any base $i{-}n$) by taking the boundary in 3 sections a,b,c which are the lengths of touching with the 6 adjacent shapes when tiling the plane. The shape is symmetric in 180° rotation so there are 3 join types for the 6 neighbours.

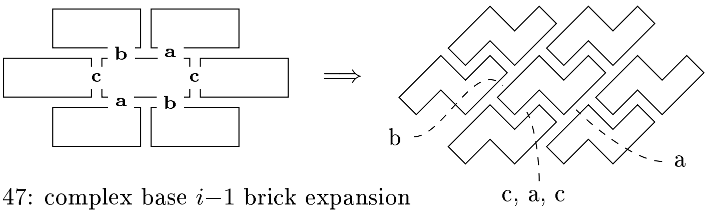


Figure 47: complex base $i{-}1$ brick expansion

For the initial 2×1 brick at k=1, c is an end and a and b are the long side. On further expansion, these parts become jagged and spiralling but remain as the boundary sections which touch the respective expanded neighbour.

In figure 47, the expanded c join becomes a b join, so length c_k is previous level b_{k-1}. Similarly b becomes an a. The a join becomes two c and one a. So mutual recurrences

$$\begin{aligned} a_k &= a_{k-1} \quad + 2c_{k-1} && \text{starting} \\ b_k &= a_{k-1} && a_0 = b_0 = 1 \end{aligned} \tag{266}$$

$$\begin{aligned} c_k = &\quad b_{k-1} & c_0 = 0 \\ \mathit{ComplexBoundary}_k = &\ TS_k = 2(a_k + b_k + c_k) & \end{aligned} \tag{267}$$

The start for a k=1 single 2×1 brick is $a_1 = b_1 = c_1 = 1$. This can be reversed one step if desired to begin $a_0 = b_0 = 1$ and $c_0 = 0$.

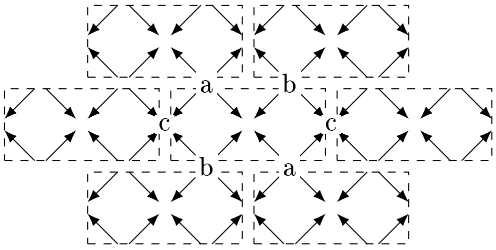
As described above, the complex base boundary segments correspond to twindragon single-visited points. So $\mathit{ComplexBoundary}_k$ by a,b,c (267) is TS_k. The individual a,b,c are how many of those single points touch the respective adjacent twindragon when twindragons tile the plane.

Theorem 55. *The lengths of complex base touching from Gilbert correspond to dragon curve joins*

$$\begin{aligned} a_k &= 2JA_{k+1} + 1 &&= dJA_{k+3} \\ b_k &= 2JA_k \quad\ + 1 &&= dJA_{k+2} \\ c_k &= 2JA_{k-1} + 1 &&= dJA_{k+1} \end{aligned}$$

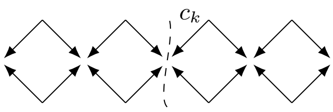
Proof. a, b, c are the dragon curve recurrence and from a starting 1,1,3 is seen to be dJA. This is not just a numerical correspondence, the shapes correspond to the joins too.

The following diagram shows twindragons level k with one expansion so each line is a dragon curve of level k−1. (When they are unit segments the bricks are twindragons of level k=1.)



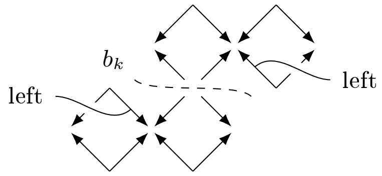
Notice a and b are the opposite way around here than in base i−1 because the twindragon is base i+1 so mirror image.

Side c boundary between horizontal adjacent twindragons is



Above the midpoint is two dragons k−1 pointing towards the centre. They enclose the join area JA_{k-1}. Likewise below the midpoint. The number of vertices touching is the middle point plus 1 more for each unit square of join area. So $c_k = 2JA_{k-1} + 1$ and which is dJA_{k+1} per theorem 28.

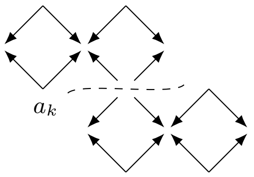
Side b boundary between its adjacent twindragons is



Going left from the middle is two dragons level $k{-}1$ pointing away from the centre point. There are three sides but the one marked "left" is a left side and so does not touch the opposite side. Likewise going right from the middle.

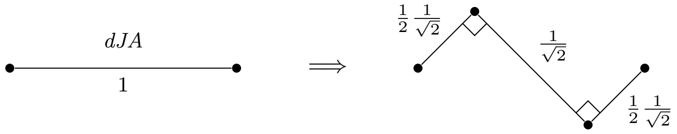
Per figure 27 these away-pointing sides are a one higher join enclosure, so $b_k = 2JA_{k-1+1} + 1 = c_{k+1}$.

Side a boundary between its adjacent twindragons is



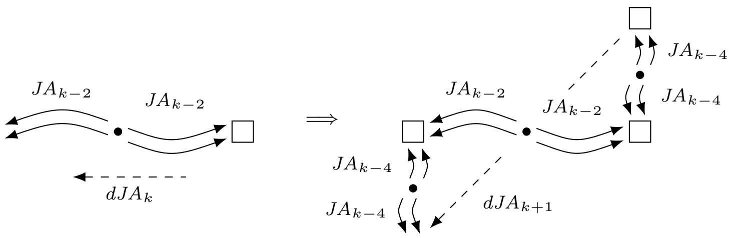
Going left or right from the middle is two dragons level $k{-}1$ pointing away and an additional side. This is a join one level higher again than configuration b, per the expansion in figure 27. So $a_k = 2JA_{k-1+2} + 1 = b_{k+1}$. □

For the twindragon fractal, these three boundary parts differ only by successive scale factor $1/\sqrt{2}$. Mandelbrot [34] gives them as a "twindragon skin" which can be generated by a line replaced by a right-angle zig-zag.



Each new line is replaced by further zig-zag, infinitely. These replacements are by rotation and scaling. There are no alternating sides the way there are for the dragon curve segments.

This replacement corresponds to theorem 28 where dJA_k is two outward pointing JA_{k-2}. Those joins each grow by a new dJA_{k-2} at each end, which is a new pair of JA_{k-4}.



dJA has a unit square at its start and no square at its end. For the fractal, this extra square goes to zero and so can be ignored. In finite iterations, the expansion maintains an extra square at the start and not at the end.

The initial lines are positioned around the twindragon square according to the join end limit $\frac{3}{5}+\frac{1}{5}i$ (the left of the component dragon verticals), as in the following diagram. The shape is a 6-sided polygon, with long side $l = 2\sqrt{\frac{2}{5}} = 1.264911\ldots$.

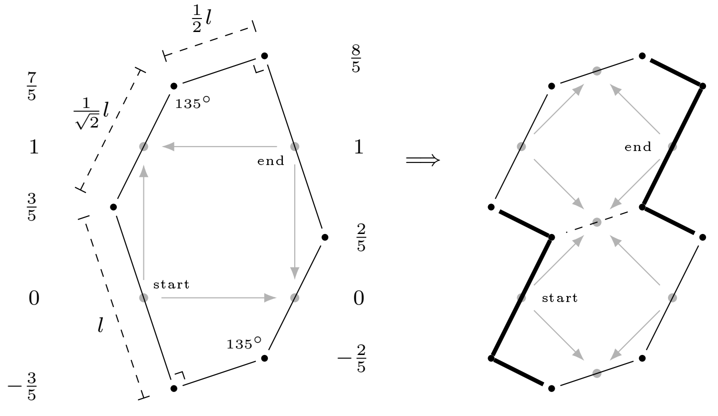
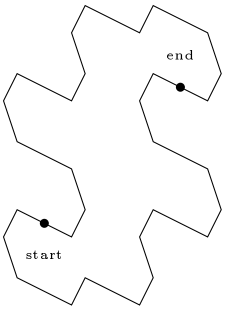


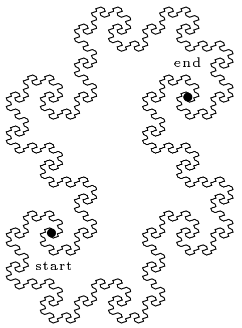
Figure 48

For the next expansion level, the longest lines are replaced by the zig-zag. Notice this maintains the $\frac{3}{5}+\frac{1}{5}i$ positioning relative to the new squares. The dashed line across the middle is where the shape would be divided to make two hexagons around the two squares if taken separately.

The line lengths begin in the ratio $l : \frac{1}{\sqrt{2}}l : \frac{1}{2}l$. The replacement rolls l down to a $\frac{1}{\sqrt{2}}l$ and two of a new smallest $\frac{1}{2\sqrt{2}}l$.



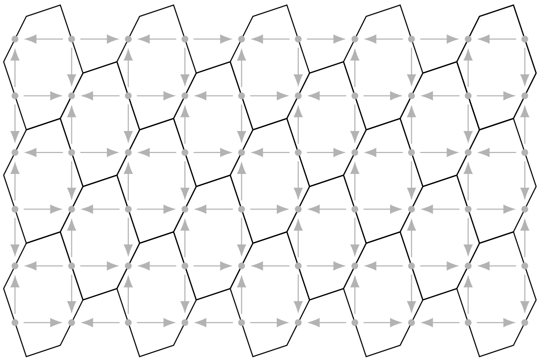
3
expansions



10
expansions

See section 18.7 on computer graphics using these expansions.

In a tiling of the plane, the squares in between the twindragons are divided in the manner of the 3-overlap case from figure 28, but here turned 90° since on odd squares. The actual connecting lines curl and spiral but these are the points where they meet.



The area of the initial polygon in figure 48 is $l.\frac{3}{2}l - (\frac{1}{2}l)^2 = 2$. The zig-zag replacement in that figure leaves area unchanged each time since it extends outwards the same as it takes away inwards. In the tiling, this is each twindragon identified with a net 2 unit squares (by a little shift upwards for example).

The tile is an irregular hexagon. It and its tiling is "type 1" in Reinhardt's classification[45] of the hexagonal tilings. That type requires one pair of opposite sides be parallel and equal length. Here all three pairs of sides are equal and parallel, for a skew of the regular hexagon.

11.3 Complex Base Unit Sides

A "bottom up" variation on Gilbert's boundary length calculation can be made by labelling the unit sides of each 2×1 brick as a', b', c' then counting how many sides are on the boundary of the complex base shape.

The same expansion as figure 47 applies, but interpreted as c sides give that many b sides in the next higher level. The b sides give that many a in the next level. And the a sides each give two c and one a in the next level. The total of these on the boundary of the next level is then

$$\begin{aligned} a'_k &= a'_{k-1} + b'_{k-1} && \text{starting } a'_1 = b'_1 = c'_1 = 1 \\ b'_k &= \qquad\qquad c'_{k-1} \\ c'_k &= 2\,a'_{k-1} \\ a'_k &= RQ_{k-1} \quad b'_k = LQ_{k-1} \quad c'_k = LQ_k && k \geq 1 \end{aligned} \tag{268}$$

These mutual recurrences are a matrix transpose of the previous (266). The total is the twindragon single-visited points (267) again.

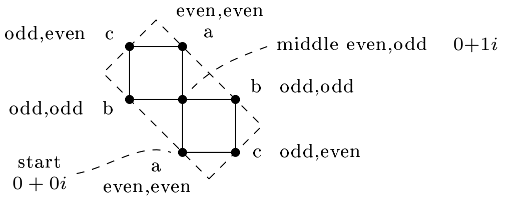
$$\mathit{ComplexBoundary}_k = \mathit{TS}_k = 2\,(a'_k + b'_k + c'_k)$$

$2c'_k$ is the number of vertical boundary sides. $2(a'_k + b'_k)$ is the number of horizontal boundary sides. In the initial few levels, these are also the height and width of the shape respectively. But soon the shape curls around so the boundary in each direction is greater than the extents. Horizontally this happens first at k=3. Vertically this happens first at k=5.

The recurrences can be reversed one step to start from $a'_0 = b'_0 = \frac{1}{2}$ and $c'_0 = 1$. The total $2(a'_0 + b'_0) = 2$ is the two horizontal sides of the unit square which is complex base shape k=0. But the separate fractions $a'_0 = b'_0 = \frac{1}{2}$ don't make sense as counts of unit sides.

The three types of unit sides correspond to twindragon single-visited points $x+iy$ with x and y parity in the following combinations,

$$\begin{aligned} a'_k &= \tfrac{1}{2} \text{ twindragon single points } x \text{ even and } y \text{ even} \\ b'_k &= \tfrac{1}{2} \text{ twindragon single points } x \text{ odd and } y \text{ odd} \\ c'_k &= \tfrac{1}{2} \text{ twindragon single points } x \text{ odd and } y \text{ even} \end{aligned}$$



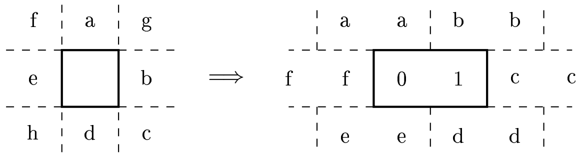
Notice a and b are the opposite way around than in base $i-1$ because the twindragon is base $i+1$ so mirror image.

These bricks repeat at locations $x + 2p, y + 2q$, an even offset in both coordinates, so the parity is maintained as the twindragon expands.

The middle point is x even and y odd. Those even,odd points are always double-visited. (Double-visited points of other parity occur too.)

The a,b diagonals which are $x+y$ even are the complex base horizontals. The twindragon horizontals are points a,c for even rows or points b alone for odd rows.

The n point numbers of complex base shape $i+1$ which have a given side on the boundary can be determined from the way each point expands. Label the sides and diagonals around a square as a---h. When such an n expands to $2n$ and $2n+1$ the squares around those are



$i+1$
expansion,
new low bit

Notice diagonal squares g and h are not around either of the new 0 or 1 so just the six a---f determine the neighbours of the new squares.

Take bits of n low to high by considering where each side of an expanded square came from. For example the top side is a. In the expansion, the 1-bit square has b on its top side. This means that the top side of the 1-bit square is determined by what was at the b position of the n from the level above. The top side of the 0-bit square comes from a, unchanged. Applying this to each of a---h around the 0 and 1 gives a state machine

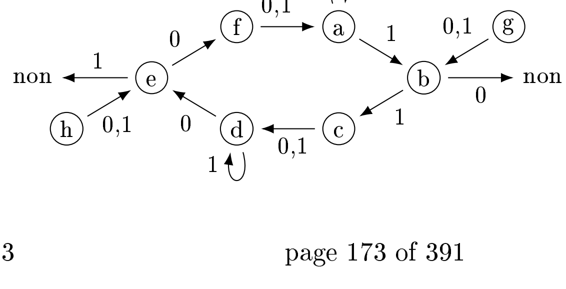


Figure 49:
$i+1$ boundary
squares,
bits of n
low to high

Write n using k many bits, padded at the high end with 0s if necessary. To test a given side of n, start at that state and follow the bits of n low to high. If they reach "non" then the start side is not on the boundary (it has a neighbouring square on that side).

Those n with top side a boundary have the following bit pattern. Each x bit can be either 0 or 1.

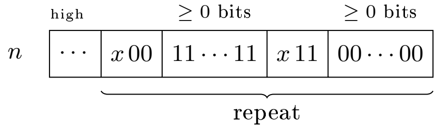


Figure 50:
base i+1
a side boundary

The low end is a run of ≥ 0 many 0-bits which stay in a. Then $x11$ goes to d where a run of 0-bits and $x00$ back to a (each going low to high).

The number of sides of each type can be counted from the bit patterns. This is more complicated than taking a recurrence but gives a combinatorial interpretation to the side counts (similar to the boundary segment combinations in section 3.4).

Suppose there are t many of the $x00$ and $x11$ triplets and all fall entirely within the k bits of n (so final state either a or d). The possible locations for them is a binomial coefficient. Each triplet is 3 bits so wherever one is located there are 2 bits unavailable to others, so $k{-}2t$ choose t.

If the high end of n is one or two bits of a further $x00$ or $x11$ then all the other such triplets are within $k{-}1$ or $k{-}2$ bits respectively. Those one or two bits of high triplet don't include the x so there are no extra choices in them. So total a sides

$$as_k = \sum_{h=0}^{2} \sum_{t=0}^{k-h-2t \geq t} 2^t \binom{k-h-2t}{t}$$

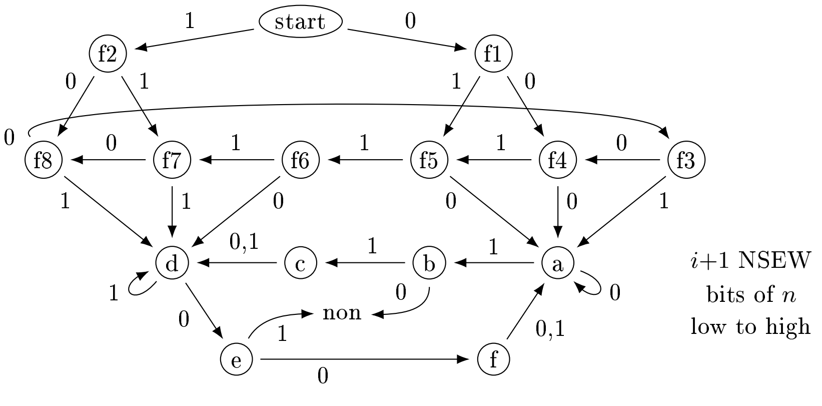
The count a'_k from (268) is sides on 2×1 bricks so is one k later than these unit square sides.

$$as_k = \tfrac{1}{2}\, a'_{k+1} \quad = RQ_k$$

In the state transitions, it can be noted a run of high 0-bits from anywhere except e never reaches "non". Unit sides of level k ending in those states are not enclosed by any higher k.

Another boundary measure can be made on the complex base shape by considering unit squares with at least one boundary side. Enclosed squares are those with neighbours in all 4 directions NSEW.

A square which is enclosed or not can be determined by testing each side a, b, d, e. Those tests can be combined into a simultaneous state machine by some usual DFA manipulations.



The loop a---f is the same as the single side above, but now ‘‘non’’ means all 4 sides NSEW non-boundary, so a 4-side enclosed square.

Loop f3--f8 skips a low run $\ldots 111000\,111000$. The start and f1,f2 transitions enter that run at one of 4 positions. The boundary squares can be counted from this by supposing a low run of $l \geq 3$ bits. The lowest two bits of l are arbitrary so 4 combinations. The high bit of l goes to a or d and in both cases is count as_{k-l} in the remainder. When $l = k$ the high bit need not go to a or d but can stay in the f3--f8 loop which is 1 further combination. (For k=3 the formula and 2^k cases are equal.)

$$\begin{aligned} \mathit{FourB}_k &= \begin{cases} 2^k & \text{if } k \leq 3 \\ 4\left(1 + \displaystyle\sum_{l=3}^{k} as_{k-l}\right) & \text{if } k \geq 3 \end{cases} \qquad \begin{array}{c}\text{NSEW}\\ \text{boundary}\\ \text{squares}\end{array} \\ &= R_k \end{aligned} \tag{269}$$

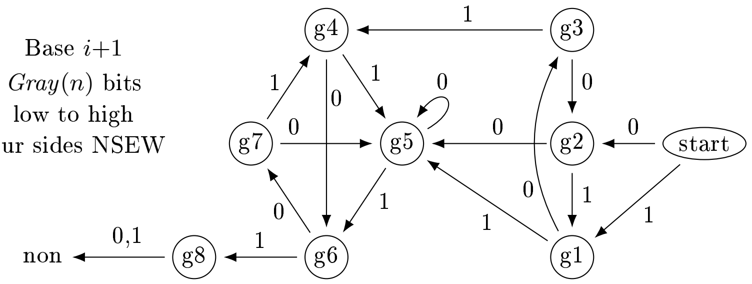
The same result is found by some linear algebra on the state machine transitions. All except ‘‘non’’ are on the boundary. The non-boundary are enclosed squares

$$\mathit{FourA}_k = 2^k - \mathit{FourB}_k = 4\,AR_k$$

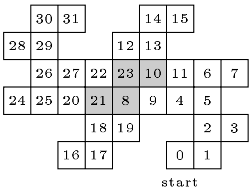
A similar state machine applies for base $i-1$, again being a bit flip 0↔1 of every second bit.

The state machine can be simplified a little by considering the Gray code of n, taken low to high still. As from (45), the Gray code is a shift and XOR which means the bit difference to the next higher bit. For base $i-1$, the equivalent Gray code state machine is identical except all bit transitions shown are 0↔1 opposite.

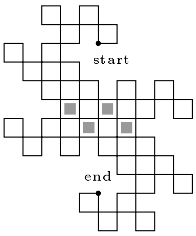
Base $i+1$
Gray(n) bits
low to high
four sides NSEW



The correspondence between complex base unit squares and twindragon diamonds (section 11.1) means these NSEW enclosed squares are diamonds with double-visited points at all four corners. In the following diagram, there are 4 such for k=5. (There are other squares in the twindragon with four double-visited corners, but they are at odd positions so outside the curve.)

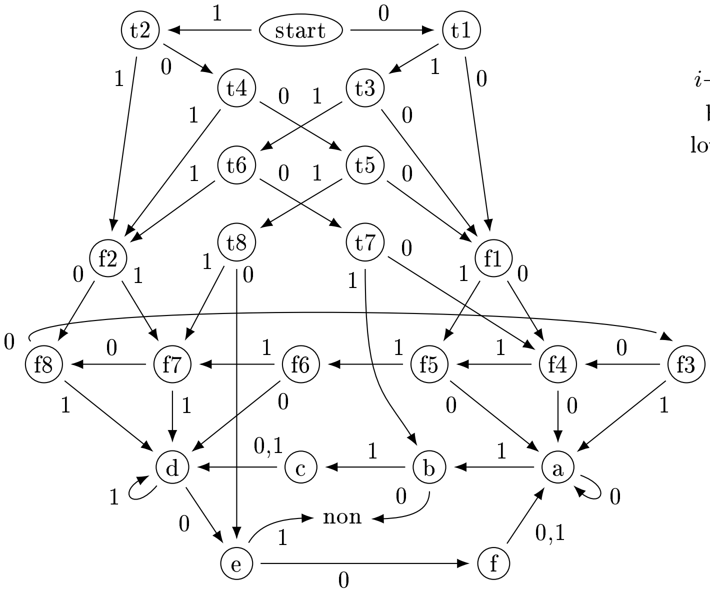


Base $i+1$ NSEW enclosed squares
k=5 $\quad FourA_5 = 4$
$n = 8, 10, 21, 23$



Twindragon k=5
interior (even) squares
where all 4 corners are
double-visited points

Another boundary measure can be made by considering squares with any side or corner on the boundary, so any of eight a---h boundary. A simultaneous DFA for these 8 tests is



$i+1$ Eight,
bits of n
low to high

States a---f and f1--f8 are the same as four sides but additional ‘‘t’’ states start in different ways.

Some linear algebra on this state machine gives a count of boundary squares or of 8-enclosed squares (those reaching ‘‘non’’). The boundary count can be written using R similar to *FourB* at (269). This shows an additional $\frac{1}{2}R_{k-1}-4$ squares with a corner on the boundary for $k \ge 4$.

$$EightB_k = \begin{cases} 2^k & \text{if } k \le 3 \\ R_k + \frac{1}{2}R_{k-1} - 4 & \text{if } k \ge 4 \end{cases} \qquad \text{boundary} \tag{270}$$

$$= \text{same recurrence as } R_k \text{ for } k \ge 8\text{, different initial values}$$

$$= 1, 2, 4, 8, 16, 32, 58, 104, 182, 312, 534, \ldots$$

$$gEightB(x) = \tfrac{11}{2} + 4x + 3x^2 + 2x^3 \; - 7\frac{1}{1-x} \; + \tfrac{1}{2}\frac{5+5x+6x^2}{1-x-2x^3} \tag{271}$$

$$EightA_k = 2^k - EightB_k \qquad \text{enclosed}$$

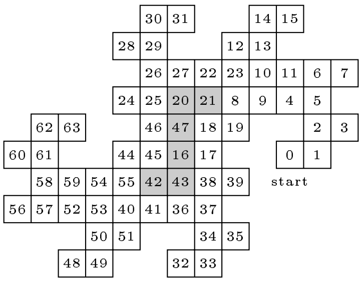
$$= \text{same recurrence as } A_k \text{ for } k \ge 9\text{, different initial values}$$

$$= 0, 0, 0, 0, 0, 0, 6, 24, 74, 200, 490, \ldots$$

$$gEightA(x) = x^6 \frac{6+8x^2}{(1-x)\,(1-2x)\,(1-x-2x^3)}$$

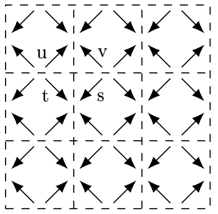
The recurrence for *EightB* is the same as right boundary R (108), but starting $16, 32, 58, 104$ then terms k=8 onwards by the recurrence. In the generating function (271), the low constant terms adjust for k=0 through k=3 not following the recurrence.

The recurrence for *EightA* is the same as area A (140), but starting $0, 0, 6, 24, 74$ then terms k=9 onwards by the recurrence.



base i+1 level k=6
8-side enclosed
$EightA_6 = 6$
$n = 16, 20, 21, 42, 43, 47$

In the twindragon, these 8-enclosed squares are diamonds where all 4 sides of the diamond are enclosed (not on the boundary). In the following diagram, the centre diamond has all four sides enclosed. s is one of those sides.

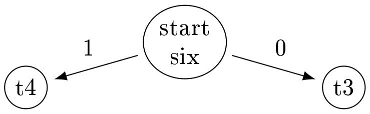


Twindragon 8-enclosed

s is non-boundary because enclosed by t, u, v. All segments of the twindragon occur as diamonds so the only way for those t, u, v to be present is their respective diamonds which are the complex base neighbours NSEW plus diagonals. The NSEW neighbours give double-visited points and the sides t, v. The diagonal neighbours give also side u.

See theorem 83 in section 13.1.1 for *EightB* as twindragon double-visited boundary points.

A yet further boundary variation can be made by considering which squares have any of six a---f boundary. After one further expansion (one greater k), each square with such a boundary or not becomes a 2×1 block with a side on the boundary or fully enclosed. The state machine for six a---f is contained within the 8-side above but start into states t3,t4 (with t1,t2 then not reached).



Some linear algebra on this variation gives a count of 6-side boundary or enclosed. The boundary can again be written using R.

$$\begin{aligned}
\mathit{SixB}_k &= \begin{cases} 2^k & \text{if } k \le 1 \\ \tfrac{1}{2}R_k + 2R_{k-2} & \text{if } k \ge 2 \end{cases} \qquad \text{boundary} \qquad (272)\\
&= \text{same recurrence as } R_k \text{ for } k \ge 7\text{, different initial values}\\
&= 1, 2, 4, 8, 16, 30, 56, 98, 168, 290, 496, 842, \dots\\
g\mathit{SixB}(x) &= 3{+}3x{+}2x^2 \;-\; 5\frac{1}{1{-}x} \;+\; \frac{3+x+3x^2}{1{-}x{-}2x^3}\\
\mathit{SixA}_k &= 2^k - \mathit{SixB}_k \qquad\qquad \text{enclosed}\\
&= \text{same recurrence as } A_k \text{ for } k \ge 8\text{, different initial values}\\
&= 0, 0, 0, 0, 0, 2, 8, 30, 88, 222, 528, 1206, \dots\\
g\mathit{SixA}(x) &= x^5 \frac{2+8x^2}{(1{-}x)\,(1{-}2x)\,(1{-}x{-}2x^3)}
\end{aligned}$$

SixB and *EightB* are greater than *FourB*. Limits for the ratios follow from their R formulas.

$$\begin{aligned}
\frac{\mathit{SixB}_k}{\mathit{FourB}_k} &= \frac{\frac{1}{2}R_k + 2R_{k-2}}{R_k} \;\to \frac{1}{2}{+}\frac{2}{r^2} \;= 1 + \frac{5}{2r^5}{+}\frac{2}{r^9} \;= 1.195620\dots\\
\frac{\mathit{EightB}}{\mathit{FourB}} &= \frac{R_k + \frac{1}{2}R_{k-1} - 4}{R_k} \text{ for } k{\ge}4 \;\to 1 + \frac{1}{2r} \;= 1.294877\dots\\
\frac{\mathit{EightB}_k}{\mathit{SixB}_k} &\to \frac{1 + 1/(2r)}{\frac{1}{2} + 2/r^2} \;= 1 + \frac{8}{4r^5 + r^7} \;= 1.083016\dots
\end{aligned}$$

11.4 Complex Base $i{-}1$ Arithmetic

An attraction of base $i{-}1$ is that all Gaussian integers are represented uniquely. Khmelnik[29] considers complex number arithmetic in base $i{-}1$ (and other $i{-}n$), including addition. A special case of this is to add an increment $dz = 1, i, -1, -i$.

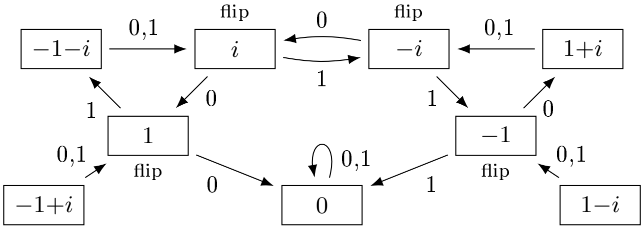


Figure 51: *MinusInc* states, bits of n low to high

$$MinusInc(n, dz) = MinusUnpoint(MinusPoint(n) + dz)$$

$dz{=}1$	1,	12,	3,	14,	5,	56,	7,	58,	…
$dz{=}i$	3,	14,	25,	116,	7,	58,	29,	0,	…
$dz{=}{-1}$	29,	0,	31,	2,	233,	4,	235,	6,	…
$dz{=}{-i}$	7,	58,	29,	0,	51,	62,	233,	4,	…

The state names are carry dz to apply above. The starting state is the desired increment.

$$z' = z + dz \tag{273}$$

The n' for location z' is sought from the bits of the n which is z. The low bit of n and the dz carry determine the low bit of n',

$$bit' \equiv bit + dz \mod i{-}1 \tag{274}$$

Mod $i{-}1$ is the same as mod $i{+}1$. For $z = x{+}iy$, it is 0 or 1 according as $x{+}y$ is even or odd respectively. When dz is odd, output bit' is an $0{\leftrightarrow}1$ flip of the input bit of n. Those dz states are marked “flip”.

New carry $next\,dz$ follows by requiring (273) on the higher bits of n. The division has its numerator always a multiple of $i{-}1$ due to (274).

$$next\,dz = (dz + bit - bit')/(i{-}1)$$

MinusInc gives a boundary side test. If an n reaches state 0 within k many bits then this means its neighbouring square is also in level k and hence n is non-boundary in that direction. So figure 51 is the $i{-}1$ equivalent of figure 49 for $i{+}1$, with 0 here corresponding to “non” there.

As from section 11.2, bases $i{+}1$ and $i{-}1$ differ by flipping every second bit. The equivalent to $i{-}1$ bit patterns figure 50 for “a” boundary is side $dz{=}i$ here with every second bit flipped $0{\leftrightarrow}1$ (starting from the lowest). This means 0-bit and 1-bit runs become alternating bits. An odd length run changes between starting 0 or 1, so between the two lines in the following,

	high		≥ 0 bits alternating		≥ 0 bits alternating	
n	$\cdots$	$x\,10$	$01\cdots01$	$x\,10$	$01\cdots01$	base $i{-}1$
		$x\,01$	$10\cdots10$	$x\,01$	$1\cdots01$	a side boundary
		repeat				

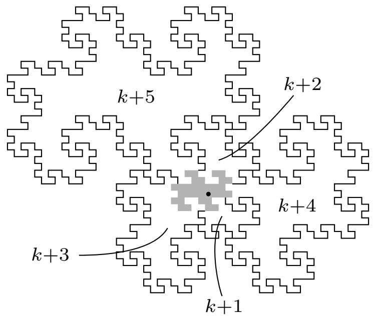
In figure 51, high 0-bits above n eventually reach state 0 from anywhere else. The longest run starts from $1{-}i$ where 6 zeros go to state 0. But that state is not the destination of any transition, so its 6-longer occurs only for $n{=}0$ of no bits in $k{=}0$, incrementing to $n = 58 = \text{binary}\,111010$ at $z = 1{-}i$.

n=2 10	n=3 11	n=12 1100
n=29 11101	n=0 0	n=1 1
n=6 110	n=7 111	n=58 111010

6 bits

base $i-1$
around n=0

State -1 is reached by various bit combinations of non-zero n and requires 5 high zeros to go to state 0. The geometric interpretation of 5 bits that any level k shape is entirely enclosed within a shape 5 levels higher. The following diagram illustrates each additional area in k+1, k+2, etc.



$k = 5$ base $i-1$
surrounding higher
levels enclosing

Khmelnik[29] also considers negation $-z$ and gives a table (table 4) of carries which are states. This is a pattern of bit flips for n.

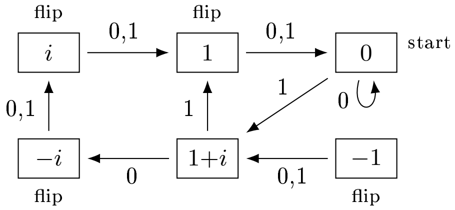


Figure 52:
MinusNeg states,
bits of n low to high

The state names are again carry dz. The initial state is dz=0. The transitions have factor -1 for the negation.

$$z' = (-1).z + dz$$

$$bit' \equiv (-1).bit + dz \bmod i-1 \tag{275}$$

$$next\, dz = (dz + (-1).bit - bit')/(i-1) \tag{276}$$

The bit flip rule is the same, since $-1 \equiv 1 \bmod i-1$ for (275). State -1 shown in figure 52 does not arise for negating n, but is used below for the *MinusNeg* boundary. The result is bit flips in a pattern,

$$\begin{aligned} MinusNeg(n) &= MinusUnpoint(-MinusPoint(n)) \\ &= \text{low to high, flip 3 bits above 01, flip 1 bit above 11} \\ &= 0, 29, 58, 7, 116, 25, 14, 3, 232, 21, 50, 239, \ldots \qquad \text{A340669} \\ \text{binary} &= 0,\ 11101,\ 111010,\ 111,\ 1110100,\ 11001,\ 1110, \ldots \end{aligned}$$

...	FFF 01 F 11	0...0	FFF 01 F 11	0...0	n binary, flip bits F

From the low end, the lowest 1 is the lowest of either $FFF01$ or $F11$. The F bits are arbitrary and are flipped $0 \leftrightarrow 1$. Then above these flipped bits, the next 1-bit is located and treated likewise, etc. 0-bits are considered at the high end of n where match or to flip. When they flip, the bit length of $\mathit{MinusNeg}(n)$ is greater than the bit length of n.

The possible flips occurring are runs of 1-bits of length 1 or 3 with 2 unflipped bits below each,

$$\begin{aligned}
\mathit{MinusNegXpred}(c) &= \begin{cases} 1 & \text{if } c = \text{BITXOR}(n, \mathit{MinusNeg}(n)) \text{ for some } n \\ 0 & \text{otherwise} \end{cases} \\
&= \text{all 1-bits are in runs 100 or 11100} \\
&= 1,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,1,\ldots \\
=1 \text{ at } c &= 0,4,8,16,28,32,36,56,64,68,72,112,\ldots \\
&= \text{binary } 0,100,1000,10000,11100,100000,100100,\ldots
\end{aligned}$$

The number of distinct *MinusNegXpred* within k bits is the ways to take bit runs length 3 or 5, and 0-bit of length 1. So a composition (partition with order) of integer k into terms $1,3,5$. This count grows as power of 1.570147... (A293506) which is the largest root of $x^5-x^4-x^2-1$.

$$\begin{aligned}
\mathit{MinusNegXdistinct}_k &= \sum_{c=0}^{2^k-1} \mathit{MinusNegXpred}(c) \\
&= \text{number of compositions of } k \text{ into parts 1,3,5} \\
&= \mathit{MinusNegXdistinct}_{k-1} + \mathit{MinusNegXdistinct}_{k-3} \qquad k \ge 5 \\
&\quad + \mathit{MinusNegXdistinct}_{k-5} \\
&= 1,1,1,2,3,5,8,12,19,30,47,74,116,182,\ldots \qquad \text{A060961} \\
\mathit{gMinusNegXdistinct}(x) &= \frac{1}{1-x-x^3-x^5}
\end{aligned}$$

Total number of bits flipped for all n in level k (including when they flip to exceed k) is

$$\begin{aligned}
\mathit{MinusNegXbits}_k &= \sum_{n=0}^{2^k-1} \text{count bits flipped } n \text{ to } \mathit{MinusNeg}(n) \\
&= \mathit{MinusNegXbits}_{k-1} + 2\left(\ 2^{k-3} + \mathit{MinusNegXbits}_{k-3}\right) \qquad k \ge 5 \quad (277) \\
&\qquad + 8\left(3.2^{k-5} + \mathit{MinusNegXbits}_{k-5}\right) \\
&= \left(\tfrac{2}{5}k + \tfrac{18}{25}\right) 2^k + \tfrac{1}{75}[-29,32,-3,-26,29,-32,3,26]\, 2^{\lfloor k/2 \rfloor} \qquad (278) \\
&\quad - \tfrac{2}{3}\,\mathrm{Im}\,\left(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{7}i\right)^{k+2} \qquad (279) \\
&= 0,3,7,15,37,83,201,459,1001,2211,4817,\ldots \\
\mathit{gMinusNegXbits}(x) &= \frac{3x-2x^2}{(1-2x)^2\,(1+2x+2x^2)\,(1-x+2x^2)}
\end{aligned}$$

$$= \tfrac{8}{25}\frac{1}{1-2x} + \tfrac{2}{5}\frac{1}{(1-2x)^2} - \tfrac{1}{3}\frac{1-2x}{1-x+2x^2} - \tfrac{1}{75}\frac{29+26x}{1+2x+2x^2}$$

Recurrence (277) follows from the bit patterns. Term $k{-}1$ is a low 0-bit. Term $k{-}3$ is a low $F11$ with 2 choices for bit F each of which is 1 bit flipped added to each of the 2^{k-3} many n above. Term $k{-}5$ is low $FFF01$ with 8 choices for FFF each of which is 3 bits flipped added to each of the 2^{k-5} many n above.

Powers (278) are by some recurrence or generating function manipulations. Term (279) grows as $\left|\frac{1}{2}+\frac{1}{2}\sqrt{7}i\right| = \sqrt{2}$. This is the same growth as the half power $2^{\lfloor k/2 \rfloor}$, but not periodic (because $2/\sqrt{7} \,.\, \mathrm{Im}\,(\frac{1}{2}+\frac{1}{2}\sqrt{7}i)^k \equiv 1 \bmod 2$ for $k{\ge}1$, so never returns to the direction of $k{=}0$).

$\mathit{MinusNegXbits}_k$ is over 2^k many n of k bits each. After flipping they are up to $k{+}3$ bits. In either case, the limit for mean proportion of bits flipped for $\mathit{MinusNeg}(n)$ is the coefficient of the $k.2^k$ term,

$$\frac{\mathit{MinusNegXbits}_k}{k.2^k} \to \frac{2}{5} \qquad \text{proportion of bits flipped}$$

For n within k bits, some $\mathit{MinusNeg}(n)$ remain within k bits. If base $i{-}1$ is used for complex integer arithmetic on a binary computer then these n are negatable without overflowing a k-bit word.

$$\mathit{MinusNegPred}_k(n) = \begin{cases} 1 & \text{if } n < 2^k \text{ and } \mathit{MinusNeg}(n) < 2^k \\ 0 & \text{otherwise} \end{cases}$$
$$= \mathit{MinusNeg} \text{ state machine } k \text{ bits, start 0, end 0}$$

In the *MinusNeg* state machine figure 52, high 0-bits above k in n lead to one or more bit-flips from any state except 0. So to be negatable within k bits means must be in state 0 after k bits.

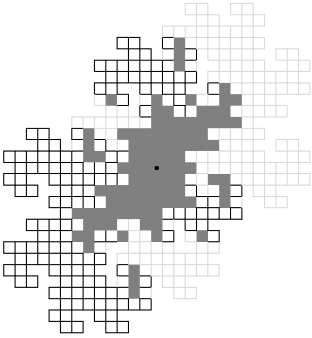
The number of these negatables within k bits is

$$\mathit{MinusNegA}_k = \sum_{n=0}^{2^k-1} \mathit{MinusNegPred}_k(n) \qquad \text{negatables}$$
$$= \mathit{MinusNegA}_{k-1} + 2\,\mathit{MinusNegA}_{k-3} + 8\,\mathit{MinusNegA}_{k-5} \qquad k{\ge}5 \quad (280)$$
$$= \tfrac{2}{5}\,2^k + \tfrac{2}{15}\,[2,-1,-1,3,\ -2,1,1,-3]\,2^{\lfloor k/2 \rfloor} \quad (281)$$
$$+ \tfrac{2}{3}\,\mathrm{Im}\,(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{7}i)^{k+1} \quad (282)$$
$$= 1,1,1,3,5,15,29,47,101,199,413,847,\ldots \qquad \text{A340670}$$

$$\mathit{gMinusNegA}(x) = \frac{1}{1-x-2x^3-8x^5}$$
$$= \tfrac{2}{5}\frac{1}{1-2x} + \tfrac{1}{3}\frac{1}{1-x+2x^2} + \tfrac{2}{15}\frac{2+3x}{1+2x+2x^2}$$

Recurrence (280) is how the bit flip patterns may sum to k. For a 0 bit, there are the $k{-}1$ bits of further combinations above. For $F11$ the F bit is arbitrary, so 2× the further $k{-}3$ combinations above it. For $FFF01$ the 3 F bits are arbitrary, so 8× the $k{-}5$ further combinations. These are compositions of k into parts 1,3,5 with each part 3 having 2 types and each part 5 having 8 types. Powers (281) follow by recurrence or generating function manipulations. The $\sqrt{7}$ part at (282) is like (279).

The geometric interpretation of *MinusNeg* in k bits is the area of intersection of base $i{-}1$ shape and a copy of it rotated 180° about the origin.



$k{=}8$ base $i{-}1$
intersection with
180° rotation
$MinusNegA_8 = 101$
$MinusNegB_8 = 136$

The boundary of this shape can be characterized by combining *MinusNeg* and *MinusInc* bit patterns. A neighbour $MinusPoint(n) + dz$ is negatable when

$$MinusNegNeighbourPred_k(n, dz) = MinusNegPred_k(MinusInc(n, dz))$$

$$\begin{aligned} = \quad & MinusInc \text{ state machine, start } dz\text{, end } 0 && (283)\\ \text{and } & MinusNeg \text{ state machine, start } -dz\text{, end } 0 && (284) \end{aligned}$$

Here *Inc* (283) asks for a neighbour within k bits, and *Neg* (284) asks that it is negatable, ie. that $-(z{+}dz)$ is within k bits, hence starting $-dz$ in the *MinusNeg* state machine.

A boundary square side in the *MinusNeg* shape is a negatable n with its neighbour not negatable. Boundary length is then

$$MinusNegB_k = \sum_{n=0}^{2^k-1} \sum_{dz=1,\,i,\,-1,-i} \begin{array}{c} MinusNegPred_k(n) \\ \text{and not} \\ MinusNegNeighbourPred_k(n, dz) \end{array} \qquad (285)$$

$$= 4, 4, 4, 8, 12, 36, 56, 80, 136, 228, 460, 784, \ldots$$

$$gMinusNegB(x) = \frac{1}{1 - x + 2x^2} + \tfrac{4}{3}\frac{2 + x + 2x^2}{1 - x - 2x^3} \qquad (286)$$

$$- \frac{1 + x}{1 - x^2 - 2x^3} + \tfrac{2}{3}\frac{2 + 2x + x^2 + 4x^3}{1 + x + x^2 + x^3 + 2x^4} \qquad (287)$$

Some state machine manipulations combine the conditions in (285). There are 34 states for $dz{=}{\pm}1$ and 53 states for $dz{=}{\pm}i$. Mutual recurrences on those states give a 12 term recurrence and generating function (286) for *MinusNegB* count.

The negated shape is symmetric in 180° rotation, so side counts for $dz = 1$ are the same as for $dz = -1$. Likewise $dz = i$ the same count as $dz = -i$.

The quadratic term at (286) is the $\frac{1}{2}{+}\frac{1}{2}\sqrt{7}i$ power like *MinusNegA* and grows as $\sqrt{2}$. The cubic at (286) is $\frac{2}{3}ComplexBoundary_k$. The cubic at (287) is different and grows as a power of 1.521…. The quartic at (287) grows as a power of 1.251…. Of these, *ComplexBoundary* grows fastest and limit ratio of negatable boundary to whole boundary

$$\frac{\mathit{MinusNegB}_k}{\mathit{ComplexBoundary}_k} \to \frac{2}{3}$$

The negatable shape comprises various separate pieces. With more state machine conditions they can be counted.

Some pieces are single unit squares. They are either completely isolated or are hanging squares touching another piece at a corner. Both grow as the 1.521… cubic. The only corner touches between pieces are hanging squares, and each hanging square touches only one other piece.

The total number of pieces also grows as the 1.521 cubic (with or without hanging squares counted separately).

Multiplication by i is similar to negation. In the dz transition (276), factor -1 becomes i.

$$\begin{aligned} \mathit{MinusRot}(n) &= \mathit{MinusUnpoint}\,(i.\mathit{MinusPoint}(n)) \\ &= 0, 3, 6, 29, 12, 15, 58, 1, 24, 27, 30, 933, \ldots \end{aligned}$$

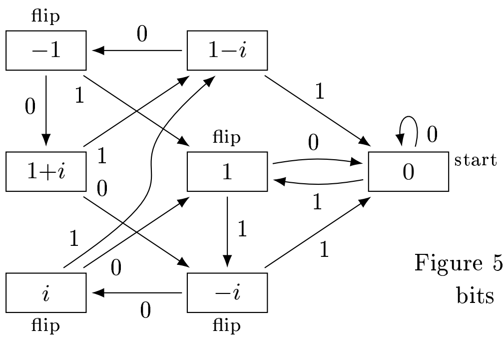
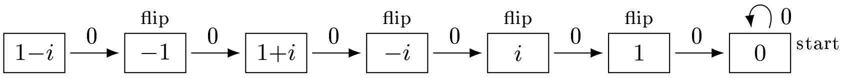


Figure 53: *MinusRot* states, bits of n low to high

Some $\mathit{MinusRot}(n)$ require more bits than n. High 0 bits above the top of n eventually reach state 0 from any other state. They do so along a chain



State 1 is a flip of its 0-bit, so is always a 1-bit above n, and then state 0 is 0s unchanged.

The most is from state $1{-}i$ where 6 zeros become various bits before the 1-bit from state 1 and reaching state 0. This first occurs for $n = 11 =$ binary 1011 which is $\mathit{MinusRot}(11) = 933 =$ binary 1110100101.

Those n ending in state 0 have no higher bits and are "rotatable" by i in complex base $i{-}1$. The count of these follows from the transitions as mutual recurrences on counts in each state,

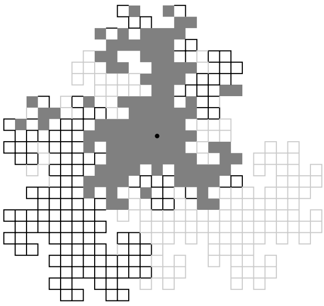
$$\begin{aligned} \mathit{MinusRotPred}_k(n) &= \begin{cases} 1 & \text{if } n < 2^k \text{ and } \mathit{MinusRot}(n) < 2^k \\ 0 & \text{otherwise} \end{cases} \qquad (288) \\ &= \mathit{MinusRot} \text{ state machine } k \text{ bits, start 0, end 0} \end{aligned}$$

$$\mathit{MinusRotA}_k = \sum_{n=0}^{2^k-1} \mathit{MinusRotPred}(n)$$

$$= \begin{cases} 1 & \text{if } k{=}0 \\ \frac{9}{20}\,2^k + \frac{1}{10}\,[1,-1,0,1,-1,1,0,-1]\,2^{\lfloor k/2 \rfloor} + \frac{1}{5} & \text{if } k \ge 1 \\ \quad = \frac{9}{20}\left|(i{-}1)^k + \frac{1}{9}\right|^2 + \frac{7}{36} & \end{cases}$$

$$= 1, 1, 2, 4, 7, 15, 29, 57, 117, 229, 461, 925, \ldots$$

The geometric interpretation of *MinusRotA* is the area of intersection of base $i{-}1$ shape and a copy of it rotated +90° about the origin. Its limit 9/20 is a little bigger than *MinusNegA* at 2/5.



$k = 8$ base $i{-}1$
intersection with
+90° rotation
$\mathit{MinusRotA}_8 = 117$
$\mathit{MinusRotB}_8 = 168$

The boundary length is

$$\mathit{MinusRotNeighbourPred}_k(n, \mathit{dz}) = \mathit{MinusRotPred}_k(\mathit{MinusInc}(n, \mathit{dz}))$$

$$\mathit{MinusRotB}_k = \sum_{n=0}^{2^k-1} \sum_{\mathit{dz}=1,\,i,\,-1,-i} \begin{array}{c} \mathit{MinusRotPred}_k(n) \\ \text{and not} \\ \mathit{MinusRotNeighbourPred}_k(n, \mathit{dz}) \end{array}$$

$$= 4, 4, 6, 12, 18, 32, 58, 94, 168, 294, 484, 852, \ldots$$

$$\mathit{gMinusRotB}(x) = 1 + \frac{1}{1{-}x} + \tfrac{1}{17}\,\frac{7{-}5x{+}6x^2{+}3x^3}{1 - x^4} \tag{289}$$

$$+ \tfrac{1}{7}\,\frac{4 + 3x - 2x^2}{1 + x + x^2 - x^3} - \frac{2 + x + 2x^2}{1 - 2x^3 - x^4} \tag{290}$$

$$+ \tfrac{1}{119}\,\frac{359 + 169x + 348x^2}{1 - x - 2x^3} \tag{291}$$

At (289), term $1{-}x^4$ is periodic constants. At (290), the cubic grows as a power of 1.356… and the quartic grows as a power of 1.395…. (291) is the dragon cubic and its r is greater than those other powers so it dominates eventually.

With some linear algebra to write the dragon cubic part in terms of the whole *ComplexBoundary*, limit fraction of whole complex base boundary is

$$\frac{\mathit{MinusRotB}_k}{\mathit{ComplexBoundary}_k} \to \frac{201 - 6r - 5r^2}{238} = 0.741389\ldots$$

Some pieces are single unit squares. Similar to *MinusNeg*, they are either completely isolated or are hanging squares touching another piece at a corner. Both grow as the 1.395… quartic. The only corner touches between pieces are hanging squares, and each hanging square touches only one other piece. The total number of pieces grows as the 1.395 quartic (with or without hanging squares counted separately).

Multiplication by $-i$ is the same state machine structure as figure 53, except that at odd states $1, i, -1, -i$ the destinations for bits 0,1 are swapped. The intersection shape is the same as *MinusRot* but turned $-90°$.

$$\begin{aligned} \mathit{MinusUnrot}(n) &= \mathit{MinusUnpoint}\big(-i.\mathit{MinusPoint}(n)\big) \\ &= \mathit{MinusNeg}\big(\mathit{MinusRot}(n)\big) \\ &= 0, 7, 14, 1, 28, 235, 2, 29, 56, 63, 470, 57, \ldots \end{aligned}$$

Doubling (multiplication by 2) is another special case of Khmelnik's addition, and is related to rotation.

$$\begin{aligned} \mathit{MinusDouble}(n) &= \mathit{MinusUnpoint}\big(2\,\mathit{MinusPoint}(n)\big) \\ &= 4\,\mathit{MinusRot}(n) \qquad (292) \\ &= 0, 12, 24, 116, 48, 60, 232, 4, 96, 108, 120, 3732, \ldots \end{aligned}$$

(292) follows from two low 0-bits in n are two expansion factors of $i{-}1$, which is $(i{-}1)^2 = -2i$,

$$\begin{aligned} (i{-}1)^2\,\mathit{MinusPoint}(n) &= \mathit{MinusPoint}(4n) \\ 2\,\mathit{MinusPoint}(n) &= i\,\mathit{MinusPoint}(4n) \end{aligned}$$

and *MinusUnpoint* of both sides

$$\mathit{MinusDouble}(n) = \mathit{MinusRot}(4n) \quad = 4\,\mathit{MinusRot}(n) \qquad (293)$$

At (293), factor 4 moves out of *MinusRot* since rotation can commence above any low 0s. In its state machine figure 53, low 0 bits are unflipped and stay in state 0.

A direct state machine for *MinusDouble* can be written out (figure 54). In dz transition (276), factor -1 becomes 2. Since $2 \equiv 0 \bmod b$, the output bit in each state is determined just by $dz \equiv 0\,\mathrm{or}\,1 \bmod b$, without reference to the input bit.

The final result is the same as *MinusRot* figure 53 with two low 0-bits output first. Start state 0, and the next state either 0 again or $-1{-}i$, are both are even so two low 0-bits output.

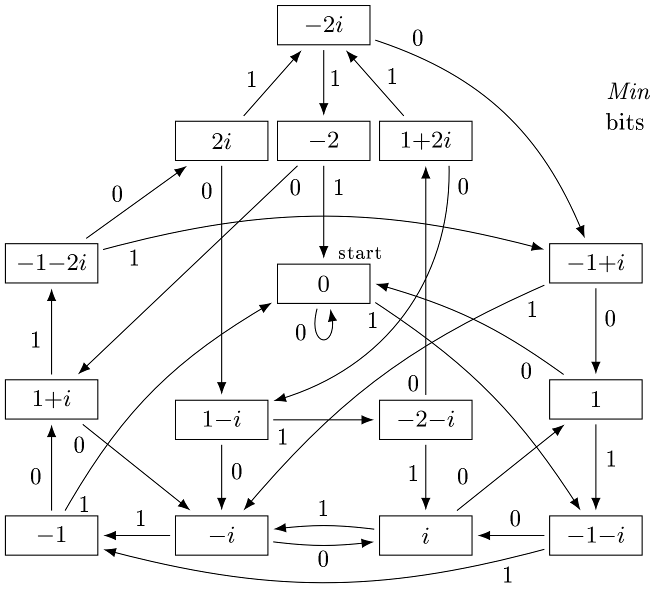


Figure 54: *MinusDouble* states, bits of n low to high

output $0, 1$ according to state $x+y \equiv 0, 1 \mod 2$

Doubling pushes some points outside the level k boundary. When that happens, $\mathit{MinusDouble}(n)$ requires more bits than n. High 0-bits above the top of n eventually reach state 0 (which is then all 0 output) from any other state. The only 0-bit transition to state 0 is from state 1, and its output is a 1-bit. So only those n in state 0 after k bits have $\mathit{MinusDouble}(n)$ within k many bits. The count of these follows from the states as mutual recurrences on counts of each state,

$$\mathit{MinusDoublePred}_k(n) = \begin{cases} 1 & \text{if } n < 2^k \text{ and } \mathit{MinusDouble}(n) < 2^k \\ 0 & \text{otherwise} \end{cases}$$

$$= \mathit{MinusDouble} \text{ state machine } k \text{ bits, start 0, end 0}$$

$$\mathit{MinusDoubleA}_k = \sum_{n=0}^{2^k-1} \mathit{MinusDoublePred}(n)$$

$$= \begin{cases} 1 & \text{if } k \le 2 \\ \frac{87}{400}\, 2^k + \frac{1}{100}\, [-1, 3, -2, 1, 1, -3, 2, -1]\, 2^{\lfloor k/2 \rfloor} & \text{if } k \ge 3 \\ \quad + \frac{1}{25}\, [12, 4, -2, 6] & \end{cases}$$

$$= 1, 1, 1, 2, 4, 7, 14, 28, 56, 112, 222, 446, \ldots$$

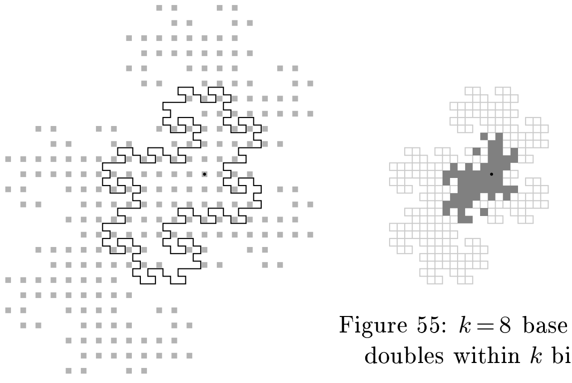


Figure 55: $k=8$ base $i{-}1$ doubles within k bits

$MinusDoubleA_8 = 56$
$MinusDoubleB_8 = 80$

Doubled points are x,y both even. 1/4 of points in a level k are multiples of 4, but the *MinusDoubleA* limit (coefficient of 2^k) is smaller than that at 87/400. In figure 55 at left, there are incomplete regions near the top and bottom of the k boundary, a total $2^8/4 - 56 = 8$ absent both-even points.

In terms of *MinusRot* at (292), doubling within k is *MinusRot* within $k{-}2$ which is a tighter restriction than *MinusRotPred* etc at (288).

Gilbert[19] notes that addition in base $i{-}1$ may need up to 8 more bits than the addends, and gives an example $n{=}11$ at $z = 2{+}3i$ with $MinusDouble(11) = 3732$ being 8 bits longer. This is the *MinusRot* first 6 bits longer and additional 2 low 0s.

The shape of the doublable points is shown at the right in figure 55. From the state machine and additional states for $dz{=}2$, the boundary length is

$$MinusDoubleB_k = 4, 4, 4, 6, 10, 16, 26, 46, 80, 136, 230, 394, \ldots$$

$$gMinusDoubleB(x) = \tfrac{5}{2} + x + x^2 + \frac{1}{1{-}x} + \frac{1}{1{+}x^2} + \tfrac{1}{17}\frac{2{+}x{+}9x^2{-}4x^3}{1 - x^4} \qquad (294)$$

$$+ \frac{-2 - x^2 + 3x^3}{1 - 2x^3 - x^4} + \tfrac{1}{34}\frac{47{+}19x{+}52x^2}{1 - x - 2x^3} \qquad (295)$$

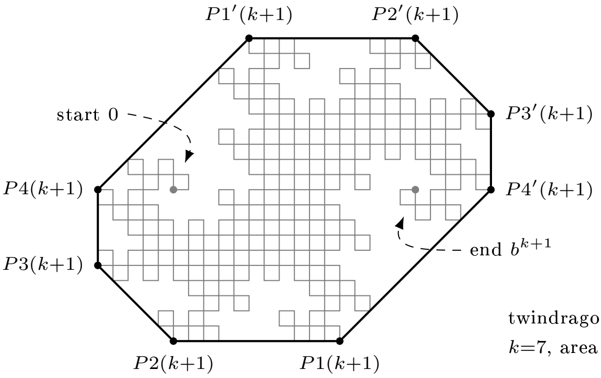
The terms at (294) are periodic constants. At (295), the quartic is like *gMinusRotB* at (290) and the cubic is the dragon cubic which dominates eventually. Writing that in terms of *ComplexBoundary* gives limit

$$\frac{MinusDoubleB_k}{ComplexBoundary_k} \to \frac{23 + 12r - 7r^2}{68} = 0.341493\ldots$$

Some doublable pieces are single unit squares. Similar to *MinusRot*, they are either completely isolated or are hanging squares touching another piece at a corner and both grow as the 1.395… quartic like *MinusRot*. The only corner touches between pieces are hanging squares, and each touches only one other piece. The total number of pieces grows as the 1.395 quartic (with or without hanging squares counted separately).

11.5 Twindragon Convex Hull

Benedek and Panzone[7] show the convex hull around the twindragon fractal is an 8-sided polygon. This shape is the convex hull around finite iterations k=3 onwards.

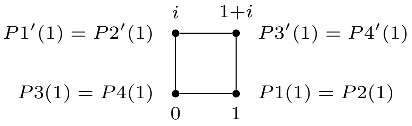


twindragon convex hull
k=7, area THA_7 = 395

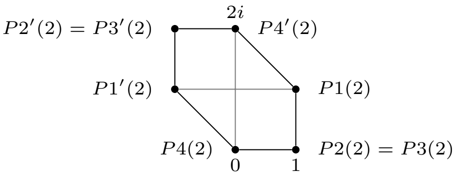
Theorem 56. *The convex hull around twindragon level k has vertices $P1$ to $P4$ from the dragon curve (theorem 37).*

$$\begin{aligned}
&P1(k{+}1),\ P2(k{+}1),\ P3(k{+}1),\ P4(k{+}1)\\
&P1'(k{+}1) = b^{k+1} - P1(k{+}1),\\
&P2'(k{+}1) = b^{k+1} - P2(k{+}1),\\
&P3'(k{+}1) = b^{k+1} - P3(k{+}1),\\
&P4'(k{+}1) = b^{k+1} - P4(k{+}1)
\end{aligned}$$

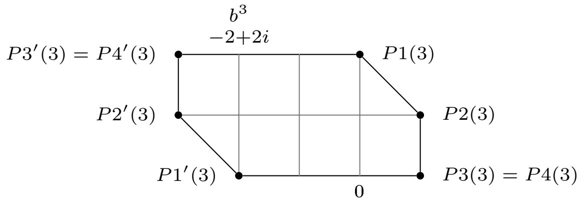
Proof. For k=0, $P1(k{+}1) = P2(k{+}1)$ and $P3(k{+}1) = P4(k{+}1)$ so there are just 4 distinct vertices making the unit square which is k=0.



For k=1, $P2(k{+}1) = P3(k{+}1)$ so there are just 6 distinct vertices.



For k=2, $P3(k{+}1) = P4(k{+}1)$ so there are just 6 distinct vertices.



For $k \geq 3$ the 8 vertices are distinct.

For k=0 through k=4, the twindragon hull is formed explicitly and is per the formulas.

For twindragon $k \geq 5$, the convex hull is formed from the convex hulls of the two component dragon curves level k+1. The second copy is points $P1'(k{+}1) = b^{k+1} - P1(k{+}1)$ etc.

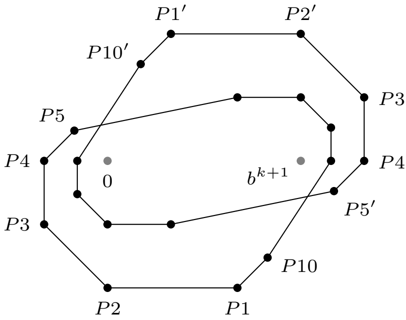


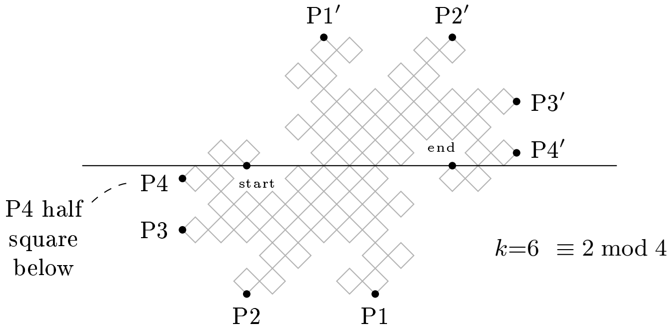
Figure 56: twindragon hull from two dragon k+1 hulls

Relative to the b^{k+1} endpoint, sides P4--P5 and P1--P10 are 45° diagonal per (200),(202). They are on the same diagonal by adapting (209) to see that $P10$ is 45° down from $P4'$,

$$P10(k{+}1) = P4'(k{+}1) \;-\; (\tfrac{1}{2}{+}\tfrac{1}{2}i)\, b^{k+1} \qquad \square$$

In figure 56, vertex $P4$ is close to the line from twindragon start to end. It is on the line when $k \equiv 0, 1, 3 \bmod 4$, but a half unit square below when $k \equiv 2$.

$$\operatorname{Im}\, P4(k{+}1).\overline{\omega}_8^{k+1} = \begin{cases} 0 & \text{if } k \equiv 0,1,3 \bmod 4 \\ -\frac{1}{2}\sqrt{2} & \text{if } k \equiv 2 \bmod 4 \end{cases}$$



Theorem 57. *The two points of twindragon level k furthest apart are* $P3(k{+}1)$ *and* $P3'(k{+}1)$ *from the convex hull. They are at a distance*

$$\begin{aligned} \mathit{THdiam}_k &= \sqrt[2]{\ \tfrac{1}{9}\Big(58.2^k - [56,64,28,56].2^{\left\lfloor \frac{k}{2}\right\rfloor} + [16,20,4,8]\Big)} \qquad (296)\\ &= \sqrt[2]{\ 2, 8, 20, 40, 80, 180, 388, 776, 1552, 3188, 6500, \ldots} \end{aligned}$$

THdiam_k *is always irrational.*

Proof. For the maximum distance, proceed as in theorem 39. Comparing factors of b^k, points P3--P3′ are furthest apart. Their distance is at least

$$\big|P3(k{+}1) - P3'(k{+}1)\big| \ \geq\ \big|(-1{-}\tfrac{7}{3}i)\,b^k\big| - 2\mathit{pmax} \ = \sqrt{\tfrac{58}{9}}\,\sqrt{2}^{\,k} - 2\mathit{pmax}$$

The second furthest by b^k factors is P3--P4′ and their distance (and the distance of any with smaller b^k factor too) is at most

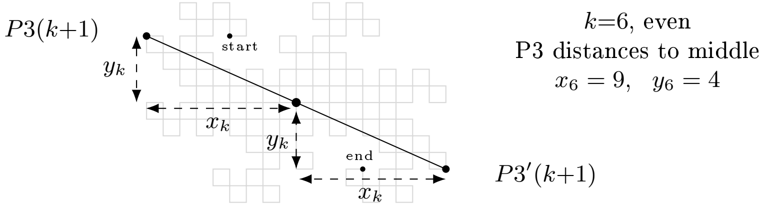
$$\big|P3(k{+}1) - P4'(k{+}1)\big| \ \leq\ \big|(-\tfrac{4}{3}{-}2i)\,b^k\big| + 2\mathit{pmax} \ = \sqrt{\tfrac{52}{9}}\,\sqrt{2}^{\,k} + 2\mathit{pmax}$$

For $k < 9$, it can be verified explicitly that P3--P3′ is the maximum among all vertices. For $k \geq 9$, the difference between the bounds is positive.

$$\Big(\sqrt{\tfrac{58}{9}} - \sqrt{\tfrac{52}{9}}\Big)\sqrt{2}^{\,k} \ >\ 4\mathit{pmax} \qquad \text{for } k \geq 9$$

As a remark, P3′--P4′ is vertical and with P3′ higher it is clear P3--P3′ is longer than P3--P4′. But this is not so clear for the third biggest P3--P2′ (distance $\sqrt{50/9}.\sqrt{2}^k$) and the general approach allowing *pmax* in any direction saves determining precisely which direction the offsets apply in each vertex pair.

To show THdiam_k is irrational, firstly for k even take the curve aligned to the segments with endpoint at −45°. P3 and P3′ are opposites across the middle. x_k and y_k are coordinate distances to the middle.



This alignment is rotation by factor $i^{-(k/2+1)}$ so

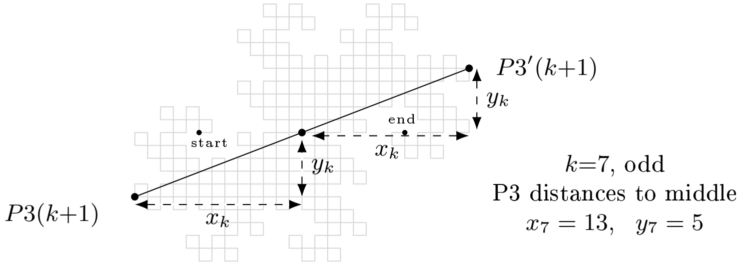
$$\begin{aligned} y_k &= \big|\mathrm{Im}\,\big(\tfrac{1}{2}b^{k+1} - P3(k{+}1)\big).i^{-(k/2+1)}\big| \ = \tfrac{1}{2}2^{k/2} \qquad k \geq 2 \text{ even}\\ (\tfrac{1}{2}\,&\mathit{THdiam}_k)^2 = x_k^2 + y_k^2 \end{aligned}$$

Similar to *Hdiam* from theorem 39, x_k and y_k are integers and y_k is a power-of-2 so if the hypotenuse $\frac{1}{2}\mathit{THdiam}_k$ is an integer then it must have at most 2 bits and THdiam_k^2 at most 3 bits.

The effect of $\frac{1}{9}$ in THdiam_k^2 formula (296) is a repeating bit pattern 111000. For $k < 16$, it can be verified explicitly that THdiam_k is irrational and for $k \geq 16$ going mod 256 shows too many 1s in the low 8 bits

$$(\tfrac{1}{2}\,\mathit{THdiam}_k)^2 \bmod 256 \equiv \begin{cases} 228 = 11100100 \text{ binary} & \text{if } k \equiv 0 \bmod 4 \\ \;\;57 = 111001 \quad\;\; \text{binary} & \text{if } k \equiv 2 \bmod 4 \end{cases} \qquad k \geq 16$$

For k odd, take the endpoints aligned horizontally



This alignment is rotation by factor $i^{-(k+1)/2}$ so

$$\begin{aligned} x_k &= \operatorname{Re}\, \big(\tfrac{1}{2}b^{k+1} - P3(k{+}1)\big).i^{-(k+1)/2} &&= \tfrac{1}{3}\Big(5.2^{(k-1)/2} - [2,1]_{(k-1)/2}\Big) \\ y_k &= \operatorname{Im}\, \big(\tfrac{1}{2}b^{k+1} - P3(k{+}1)\big).i^{-(k+1)/2} &&= \tfrac{1}{3}\Big(2^{(k+1)/2} - (-1)^{(k+1)/2}\Big) \end{aligned}$$

y_k even and odd $k{\geq}1$ A135318

$$(\tfrac{1}{2}\,\mathit{THdiam}_k)^2 = x_k^2 + y_k^2$$

x_k and y_k are integers. If THdiam_k is an integer then $\frac{1}{2}\mathit{THdiam}_k$ forms a Pythagorean triple with x_k and y_k. Any Pythagorean triple has at least one leg a multiple of 4 as from (217). But from the power forms neither x nor y are,

$$\left.\begin{aligned} x_k &\equiv 2 \bmod 4 \\ y_k &\equiv 3 \bmod 4 \end{aligned}\right\} \text{ when } k \equiv 1 \bmod 4 \text{ and } k \geq 5$$

$$\left.\begin{aligned} x_k &\equiv 1 \bmod 4 \\ y_k &\equiv 1 \bmod 4 \end{aligned}\right\} \text{ when } k \equiv 3 \bmod 4 \text{ and } k \geq 7 \qquad \square$$

The area of the twindragon hull is calculated by triangles as for *HA* from section 7. The area is an integer because the hull is symmetric in 180° rotation.

$$\begin{aligned} \mathit{THA}_k &= \tfrac{10}{3}2^k - [\tfrac{8}{3}, 4, \tfrac{10}{3}, 4].2^{\lfloor k/2 \rfloor} + \tfrac{1}{3} \qquad \text{area} \\ &= 1, 3, 7, 19, 43, 91, 187, 395, 811, 1643, \ldots \end{aligned}$$

For the curve scaled to b^{k+1} endpoint a unit length, the limits for *THdiam* and *THA* are given by the coefficients of their 2^k terms. Both limits are approached from below since the half-power terms in each are negative and exceed the constants.

$$\frac{\mathit{THdiam}_k}{\sqrt{2}^{k+1}} \to \frac{1}{3}\sqrt{29} = 1.795054\ldots \qquad \frac{\mathit{THA}_k}{2^{k+1}} \to \frac{5}{3} \tag{297}$$

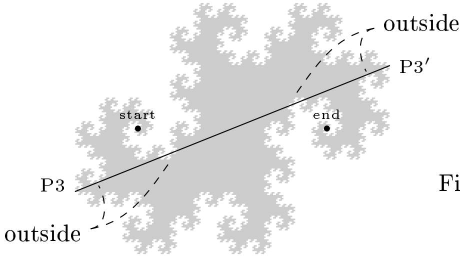


Figure 57

Much of line P3--P3′ is within the twindragon. But some of it goes outside. This can be seen by expanding once to k=1.

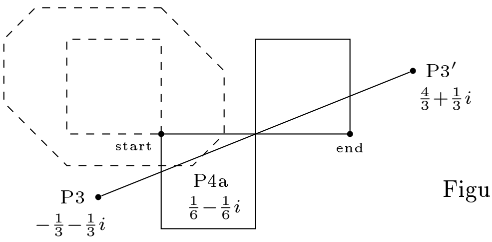
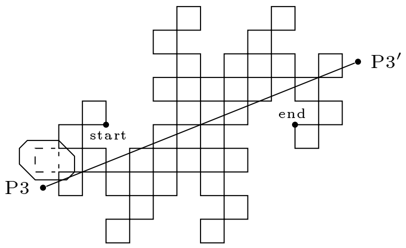


Figure 58

The dashed square is the first half of the absent other twindragon at curve start. The dashed convex hull around it has a vertex $P4a = -P4/b = \frac{1}{6}-\frac{1}{6}i$. At its $\operatorname{Re} P4a = \frac{1}{6}$, the line P3--P3′ is (using its slope) above and therefore some of the line goes outside. (The twindragon having no cut-points or holes which might allow the line might stay within and yet have some outside at P4a.)

$$\text{at } x{=}\tfrac{1}{6} \text{ line P3--P3}' \text{ Im} = -\tfrac{2}{15} \;>\; -\tfrac{1}{6} = \operatorname{Im} P4a$$

In figure 57 the second "outside" shown closer to P3 is the same, but expanded a further 4 times to k=5 sub-curves.



P3--P3′ is slope 5:2 and this takes it from the twindragon middle down to pass through the grid point above and to the right of P3. That point is then the same as the middle in figure 58. Repeating the expansion a further 4 times on the square beside P3 gives another yet smaller outside, and so on infinitely.

The convex hull around the points of complex base i+1 shape is the same as the twindragon with its unit squares (diamonds), which correspond to the complex base, reduced to points at their lower left, and division by $i-1$ to rotate $-135°$ and scale (that being the inverse of *PlusToDragon* from (259)).

The twindragon vertices are $P1(k{+}1)$ etc which go as powers b^k. It's convenient to apply the $i-1$ division by decreasing the index k. That divides by $b = i{+}1$ and is adjusted to $i-1$ by factor $b/(i-1) = -i$. Offset *coff* is a combination of move to lower left corner and change of offset in the P index reduction.

$$\begin{aligned}
\mathit{CP1}(k) &= -i\,P1(k) + \mathit{coff}(k) & \mathit{CP1}'(k) &= -i\,(b^k{-}1) - \mathit{CP1}(k)\\
\mathit{CP2}(k) &= -i\,P2(k) + \mathit{coff}(k{-}1) & \mathit{CP2}'(k) &= -i\,(b^k{-}1) - \mathit{CP2}(k)\\
\mathit{CP3}(k) &= -i\,P3(k) + \mathit{coff}(k{-}2) & \mathit{CP3}'(k) &= -i\,(b^k{-}1) - \mathit{CP3}(k)\\
\mathit{CP4}(k) &= -i\,P4(k) + \mathit{coff}(k{-}3) & \mathit{CP4}'(k) &= -i\,(b^k{-}1) - \mathit{CP4}(k)
\end{aligned}$$

$$\mathit{coff}(m) = \begin{cases} i & \text{if } k \equiv 0 \text{ to } 3 \bmod 8 \\ 0 & \text{if } k \equiv 4 \text{ to } 7 \bmod 8 \end{cases} \qquad i{\times}\text{A131078}$$

Lai[33] shows in general that the hull around a fractal complex base $p.e^{\frac{2\pi i}{n}}$ has n vertices when n even, or $2n$ vertices when n odd. Here base $b = 1{+}i = \sqrt{2}.e^{\frac{2\pi i}{8}}$ is $n{=}8$.

The area enclosed by the complex base hull is

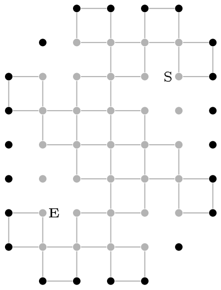
$$\begin{aligned}
\mathit{CHA}_k &= \tfrac{5}{3}2^k - [\tfrac{10}{3}, 5, \tfrac{11}{3}, 5].2^{\lfloor k/2 \rfloor} + \tfrac{5}{3} \qquad \text{area}\\
&= 0, 0, 1, 5, 15, 35, 79, 175, 375, 775, 1591, \dots
\end{aligned}$$

$\mathit{CHA}_0 = 0$ is the single point at the origin of $k{=}0$. $\mathit{CHA}_1 = 0$ is two points $0, 1$ so no width. $\mathit{CHA}_1 = 1$ is sheared $0, 1, 1{+}i, 2{+}i$.

The limit is the same as the twindragon when scaled to a unit length.

$$\frac{\mathit{CHA}_k}{2^k} \to \frac{5}{3} \qquad \text{same as } \mathit{THA} \text{ at (297)}$$

Twindragon hull sides can also be measured by how many Gaussian integers are on the sides, like theorem 41 of the dragon hull.



twindragon hull $k{=}4$
points
boundary $\mathit{THBP}_4 = 22$
inside $\mathit{THIP}_4 = 33$
total $\mathit{THP}_4 = 55$

The twindragon hull sides are all straight or 45° so the integer points goes as the side lengths. The total is

$$\begin{aligned}
\mathit{THBP}_k &= [6, 8].2^{\lfloor k/2 \rfloor} - 2\\
&= 4, 6, 10, 14, 22, 30, 46, 62, 94, \dots \qquad \text{A027383}
\end{aligned}$$

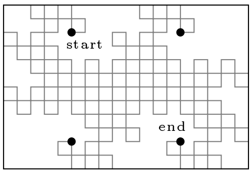
Then with Pick's theorem (220), the hull interior and total points are

$$\begin{aligned}
\mathit{THIP}_k &= \mathit{THA}_k - \tfrac{1}{2}\mathit{THBP}_k + 1 \qquad \text{hull interior points}\\
&= \tfrac{10}{3}2^k - \tfrac{1}{3}[17, 24, 19, 24]\,2^{\lfloor k/2 \rfloor} + \tfrac{7}{3}\\
&= 0, 1, 3, 13, 33, 77, 165, 365, 765, \dots
\end{aligned}$$

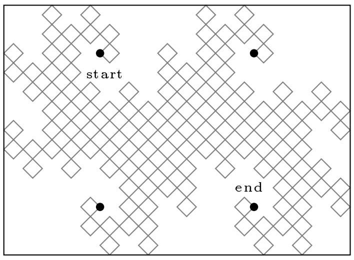
$$\begin{aligned} THP_k &= THIP_k + THBP_k && \text{hull total integer points} \\ &= \tfrac{10}{3} 2^k + \tfrac{1}{3} [1,0,-1,0]\, 2^{\lfloor k/2 \rfloor} + \tfrac{1}{3} \\ &= 4, 7, 13, 27, 55, 107, 211, 427, 859, \ldots && \text{A136408} \end{aligned}$$

11.6 Twindragon Minimum Area Rectangle

k=6
area $TMR_6 = 216$



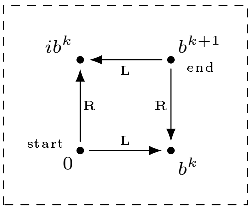
k=7
area $TMR_7 = 468$



Theorem 58. *The minimum area rectangle around the twindragon curve is aligned to a b^k side of the square of 4 dragons making up the twindragon. Its area is*

$$\begin{aligned} TMR_k &= \tfrac{1}{9}\Big(35.2^k - [34,38,38,34].2^{\lfloor k/2 \rfloor} + [8,4]\Big) \\ &= 1, 4, 8, 24, 48, 108, 216, 468, 936, 1924, 3848, 7844, \ldots \end{aligned}$$

$[34,38,38,34]$ *means the respective value as* $k \equiv 0 \,\text{to}\, 3 \bmod 4$. *Similarly* $[8,4]$ *as* $k \equiv 0 \,\text{or}\, 1 \bmod 2$.



Minimum area rectangle
aligned to b^k direction

Proof. A minimum area rectangle has at least one side aligned to a side of the convex hull. The sides of the twindragon hull have two directions, straight and 45°. It suffices to consider rectangles in those two directions, relative to b^k.

The rectangle area in the straight b^k direction is the claimed minimum

$$TMR_k = |b^k|^2 . \operatorname{Im} \frac{P2'(k{+}1) - P2(k{+}1)}{b^k} . \operatorname{Re} \frac{P4'(k{+}1) - P4(k{+}1)}{b^k}$$

The rectangle area in the diagonal b^{k+1} direction is

$$\begin{aligned} TMRdiag_k &= |b^{k+1}|^2 . \operatorname{Im} \frac{P2'(k{+}1) - P2(k{+}1)}{b^{k+1}} . \operatorname{Re} \frac{P4'(k{+}1) - P4(k{+}1)}{b^{k+1}} \\ &= \tfrac{1}{9}\Big(40.2^k - [26,26,28,28].2^{\lceil k/2 \rceil} + [4,8]\Big) \\ &= 2, 4, 12, 24, 60, 120, 260, 520, 1092, 2184, \ldots \end{aligned}$$

For $k < 4$, the sample values shown above have $TMRdiag_k \geq TMR_k$. They are equal at k=1 and k=3. For $k \geq 4$, the difference 5.2^k overcomes the negatives in the other terms. $TMRdiag_k$ is written with $2^{\lceil k/2 \rceil}$ whereas TMR_k is with $2^{\lfloor k/2 \rfloor}$ to make the coefficients more attractive. Adapt them to both $2^{\lceil k/2 \rceil}$ for the difference

$$\begin{aligned} TMRdiag_k - TMR_k &= \tfrac{1}{9}\left(5.2^k + [8,-7,10,-11].2^{\lceil k/2 \rceil} + [-4,4]\right) \\ &\geq \tfrac{1}{9}\left(5.2^k \qquad\qquad - 11.2^{\lceil k/2 \rceil} - 4\right) \\ &\geq \tfrac{1}{9}\left(4.4.2^{\lceil k/2 \rceil} + 4 - 11.2^{\lceil k/2 \rceil} - 4\right) \qquad \text{for } k \geq 4 \\ &= \tfrac{5}{9}2^{\lceil k/2 \rceil} \quad > 0 \end{aligned}$$

□

11.7 Twindragon XY Convex Hull

The XY convex hull around the twindragon follows by subtracting the side indentations as from section 7.3.

Taking the twindragon as two dragons k+1 back-to-back, the sides P2--P3 and P2′--P3′ are $XYindentA_{k-1}$. The sides P1--P4′ and P1′--P4 are 2 levels bigger.

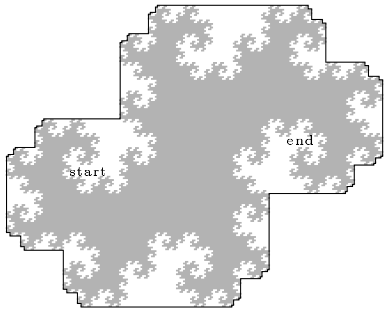
$$\begin{aligned} TXYhullA_k &= THA_k - 2XYindentA_{k+1} - 2XYindentA_{k-1} \\ &= \tfrac{190}{63}\,2^k - \tfrac{2}{9}\,[13,26,14,28]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{7}\,[-4,6,-9,10]\,2^{\lfloor k/4 \rfloor} + \tfrac{4}{9}\,[1,2] \\ &= 0, 2, 5, 14, 36, 76, 166, 340, 724, 1456, \ldots \end{aligned}$$

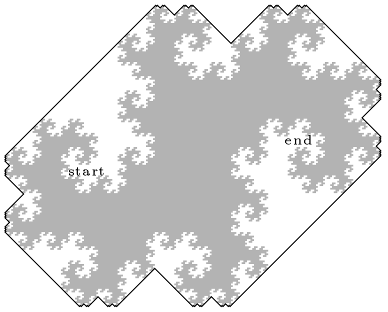
Similarly a diagonally aligned hull, with the indents 1 level smaller,

$$\begin{aligned} TDXhullA_k &= THA_k - 2XYindentA_k - 2XYindentA_{k-2} \\ &= \tfrac{200}{63}\,2^k - \tfrac{2}{9}\,[17,19,19,17]\,2^{\lfloor k/2 \rfloor} + \tfrac{1}{7}\,[5,-4,6,-9]\,2^{\lfloor k/4 \rfloor} + \tfrac{4}{9}\,[2,1] \\ &= 1, 2, 6, 17, 38, 84, 172, 374, 756, 1556, \ldots \end{aligned}$$

Limits for the twindragon scaled to endpoints a unit length are

$$\frac{TXYhullA_k}{2^{k+1}} \to \frac{95}{63} = 1.507\ldots \qquad \frac{TDXhullA_k}{2^{k+1}} \to \frac{100}{63} = 1.587\ldots \quad \text{A021067}$$





11.8 Twindragon Inertia

Theorem 59. *Consider the twindragon curve to have a unit mass in the middle of each line segment. With the x axis aligned to the endpoints, the moment of inertia tensor for rotation at the centroid (which is midway between the endpoints) is,*

$$TI(k) = \begin{pmatrix} TI_x & -TI_{xy} & 0 \\ -TI_{xy} & TI_y & 0 \\ 0 & 0 & TI_z \end{pmatrix}$$

where

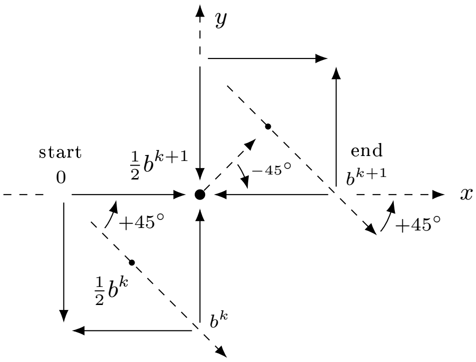
$$\begin{aligned} TI_x(k) &= \tfrac{4}{5}4^k \; - \tfrac{1}{10}[3,1,7,9].2^k \\ &= \tfrac{1}{2}, 3, 10, 44, 200, 816, 3232, 12992, \dots \\ TI_y(k) &= \tfrac{6}{5}4^k \; - \tfrac{1}{10}[7,9,3,1].2^k \\ &= \tfrac{1}{2}, 3, 18, 76, 296, 1200, 4896, 19648, \dots \\ TI_{xy}(k) &= \tfrac{2}{5}4^k \; + \tfrac{1}{5}[-2,1,2,-1].2^k \\ &= 2^k \operatorname{Im} sG_{k+1} \qquad \textit{from } (246) \\ &= 0, 2, 8, 24, 96, 416, 1664, 6528, \dots \qquad 2^k \times \mathrm{A007910} \\ TI_z(k) &= TI_x(k) + TI_y(k) \\ &= 2.4^k - 2^k \\ &= 1, 6, 28, 120, 496, 2016, 8128, 32640, \dots \qquad \mathrm{A171476} \end{aligned}$$

Proof. Twindragon k is two back-to-back copies of dragon curve $k{+}1$. The inertia I of the dragon curve from theorem 51 is at its centroid GS_{k+1}. The centroid of the twindragon is instead the middle $\frac{1}{2}b^{k+1}$. Apply the difference as an offset, and rotated using eighth root of unity ω_8 so as to be directed relative to the twindragon endpoint b^{k+1}.

$$\begin{aligned} \Delta_k &= \left(\tfrac{1}{2}b^{k+1} - GS_{k+1}\right) .\overline{\omega_8}^{k+1} \\ TI_x(k) &= 2\big(I_x(k{+}1) \;\; + 2^{k+1}(\operatorname{Im}\Delta_k)^2\,\big) \\ TI_y(k) &= 2\big(I_y(k{+}1) \;\; + 2^{k+1}(\operatorname{Re}\Delta_k)^2\,\big) \\ TI_{xy}(k) &= 2\big(I_{xy}(k{+}1) \; + 2^{k+1}(\operatorname{Im}\Delta_k)(\operatorname{Re}\Delta_k)\,\big) \end{aligned}$$

□

Second Proof of Theorem 59. A direct calculation from twindragon k as two copies of twindragon $k{-}1$ can be made too. Both copies have the x axis at +45° relative to the start to end line in the sub-part.



The x axis is at angle $+45$ relative to the two halves, and the offset at $-45°$ to their centroid can be applied with the parallel axis theorem. So with matrix of rotation R from (244).

$$TI(k) = 2R^{-1}.\,TI(k{-}1).R \ + 2.4.2^{k-1}\big(\tfrac{1}{2}\sqrt{2}^k\big)^2.R\left(\begin{smallmatrix}0&0&0\\0&1&0\\0&0&1\end{smallmatrix}\right)R^{-1} \qquad k \geq 1$$

Multiplying is mutual recurrences

$$TI_x(k) = TI_x(k{-}1) \ + TI_y(k{-}1) \ - 2\,TI_{xy}(k{-}1) \ + \tfrac{1}{2}4^k \tag{298}$$

$$TI_y(k) = TI_x(k{-}1) \ + TI_y(k{-}1) \ + 2\,TI_{xy}(k{-}1) \ + \tfrac{1}{2}4^k \tag{299}$$

$$TI_{xy}(k) = TI_x(k{-}1) \ - TI_y(k{-}1) \qquad\qquad + \tfrac{1}{2}4^k \tag{300}$$

$I_x{+}I_y = I_z$ is true of any plane figure so (298) + (299) is

$$\begin{aligned} &TI_z(k) = 2\,TI_z(k{-}1) \ + \ 4^k \qquad k \geq 2 \\ &\text{starting} \quad TI_z(0) = 4.(\tfrac{1}{2})^2 = 1 \end{aligned}$$

Difference (298) − (299) $TI_x(k) - TI_y(k) = -4\,TI_{xy}(k{-}1)$ into (300) is a second order recurrence for TI_{xy}, starting from values calculated explicitly at $k{=}0,1$

$$\begin{aligned} &TI_{xy}(k) = -4\,TI_{xy}(k{-}2) \ + \ \tfrac{1}{2}4^k \qquad k \geq 2 \\ &\text{starting} \quad TI_{xy}(0) = 0, \quad TI_{xy}(1) = 2 \end{aligned}$$

And then

$$\begin{aligned} TI_x(k) &= TI_z(k{-}1) - 2\,TI_{xy}(k{-}1) + \tfrac{1}{2}4^k \\ TI_y(k) &= TI_z(k{-}1) + 2\,TI_{xy}(k{-}1) + \tfrac{1}{2}4^k \\ &\text{with initial} \quad TI_x(0) = TI_y(0) = 4.(\tfrac{1}{2}\sqrt{2})^2 = \tfrac{1}{2} \end{aligned}$$

□

There is a factor 2^k in each TI component. It is kept so the components have the same mass basis as the dragon curve calculation in section 9. It could be eliminated by taking the whole twindragon to have a unit mass in total and distributed uniformly at the midpoint of each segment, or alternatively keeping the mass but scaling to the endpoint a unit length.

Similar to (250), the principal axes of inertia are

$$\begin{aligned}
T\alpha_{min}(k) &= \tfrac{1}{2}\arctan \frac{-2\,TI_{xy}(k)}{TI_x(k) - TI_y(k)} \qquad k \ge 2 \\
&= \tfrac{1}{2}\arctan\Big(2 + \frac{[0,-2,0,2]\,2^k}{TI_x(k) - TI_y(k)}\Big) \\
&\to \tfrac{1}{2}\arctan 2 \;\; = 31.717474^\circ \ldots && \text{radians A195693} \\
& \text{slope } 1/\phi = 0.618033\ldots && \text{A094214} \\
& \text{where golden ratio } \phi = \tfrac{1+\sqrt{5}}{2} = 1.618033\ldots && \text{A001622} \\
T\alpha_{max} &= \alpha_{min} + \tfrac{\pi}{2}
\end{aligned}$$

The minimum principal axis is exactly $\frac{1}{2}\arctan 2$ when k is even since the periodic terms in the TI parts coincide.

$$\frac{-2\,TI_{xy(k)}}{TI_x(k) - TI_y(k)} \;=\; 2 \qquad \text{when } k \text{ even} \ge 2$$

For k odd, the remaining term is powers $2^k/4^k \to 0$.

For the curve scaled to unit length and mass 2, the inertia limit is the coefficients of 4^k in $TI_x(k)$ etc.

$$\frac{TI(k)}{4^{k+1}} \to \mathit{TIf} = \begin{pmatrix} \frac{1}{5} & -\frac{1}{10} & 0 \\ -\frac{1}{10} & \frac{3}{10} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \qquad \text{inertia limit} \tag{301}$$

Mass 2 for this limit is the same density as the dragon inertia limit (253). Both are unit length. The dragon limit is mass 1 area $\frac{1}{2}$. The twindragon is mass 2 area 1. The twindragon inertia limit is two dragon limits the same as in the first proof above, with centroids Gf shifted up to the midpoint $\frac{1}{2}$.

$$\mathit{TIf} = 2\Big(\mathit{If} + \mathit{Ipoint}(\mathit{Gf} - \tfrac{1}{2})\Big)$$

The inertia limit rotated by $T\alpha_{min}$ limit to principal axes is

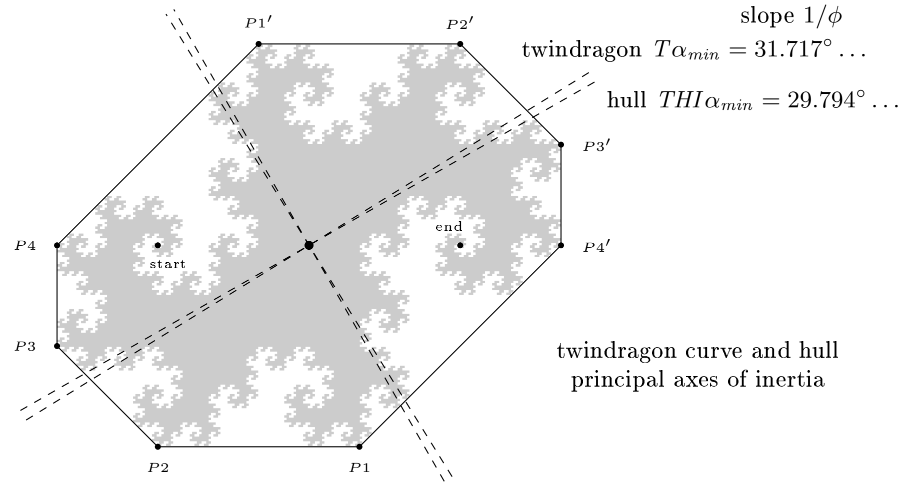
$$\begin{pmatrix} \frac{1}{4}-\frac{1}{20}\sqrt{5} & 0 & 0 \\ 0 & \frac{1}{4}+\frac{1}{20}\sqrt{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{pmatrix} = \tfrac{1}{10}\begin{pmatrix} 3-\phi & 0 & 0 \\ 0 & 2+\phi & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad \begin{matrix}\text{principal} \\ \text{axes}\end{matrix}$$

$$\begin{aligned}
TI_{\min} &= \tfrac{1}{10}(3-\phi) \;\; = 0.138196\ldots && \text{A094874} \\
TI_{\max} &= \tfrac{1}{10}(2+\phi) \;\; = 0.361803\ldots
\end{aligned}$$

The inertia of the convex hull around the twindragon (section 11.5) can be calculated from its polygon vertices and compared to the curve it surrounds. For the unit length limit and density mass = 2 × area the same as the dragon and twindragon, the hull inertia limit is

$$\mathit{THI}_x = \tfrac{95}{243} \qquad \mathit{THI}_y = \tfrac{271}{486} \qquad \mathit{THI}_{xy} = \tfrac{23}{162} \qquad \text{hull inertia}$$

$$\mathit{THI}\alpha_{min} = \tfrac{1}{2}\arctan\tfrac{46}{27} \;\; = 29.794459^\circ \ldots$$



twindragon curve and hull
principal axes of inertia

The principal axis $THI\alpha_{min}$ of the hull is slightly smaller, ie. slightly nearer the x axis, than the curve axis. Roughly speaking, there is a little more blank space filled by the hull on that side than the other (weighted by squared distance).

12 Blobs

The dragon curve comprises blobs of enclosed area with bridges in between. A blob is a run of consecutive segments all of which are sides of some enclosed unit square. Those squares touch at a corner or on a side. A bridge is segments in between which are not sides of an enclosed unit square.

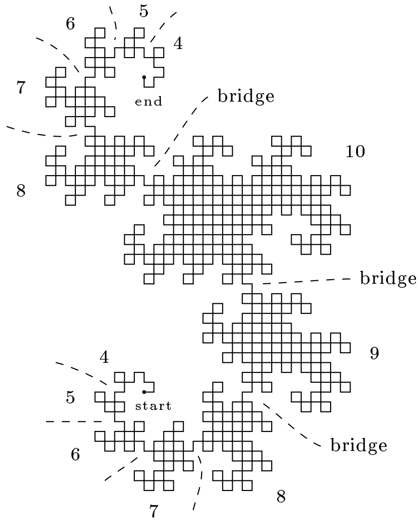


Figure 59: blobs in dragon k=10

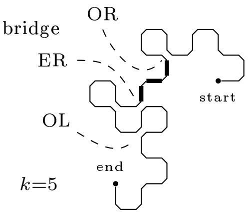
blob 4 is a unit square,
blob 10 is the middle biggest,
bridges of 3 segments between

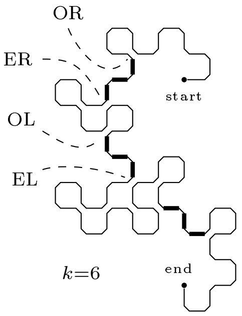
Blobs are numbered by increasing size. It's convenient to follow Daykin and Tucker [13] and number so that blob k is the biggest blob in curve k. This aligns indices for various sums and powers. The first blob (a single unit square) is then k=4.

There is a single sequence of blob sizes, each expanding to the next. The curve comprises a certain set of those sizes ($BlobList_k$). Blobs at the end of the curve are the same shape as the start but rotated −90° since they are unfolds of the start.

Theorem 60 (Ngai and Nguyen[38]). *The blobs in the dragon curve are separated by bridges of 3 segments and those bridges are maintained on each curve expansion.*

Proof. The bridges in curves k=5 and k=6 are each 3 segments.





In curve k=6, the bridges bend either right or left and the two ends are each either an odd or even point (coordinate sum x+y odd or even). The two forms expand as

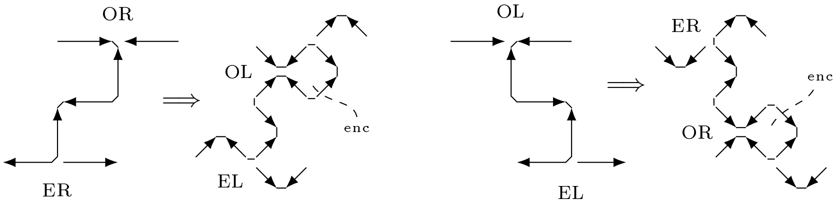
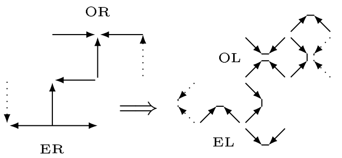


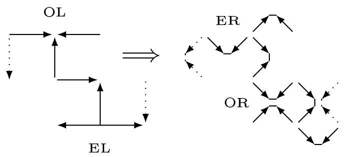
Figure 60: bridge expansions

The 3 bridge segments expand to 6 new segments but 3 of them become an enclosed unit square leaving 3 as a new bridge. The right bending ER,OR bridge expands to left bending EL,OL. In turn, EL,OL expands back to ER,OR.

This expansion means the initial bridge in k=5 is maintained in subsequent expansions. In $k \geq 6$, the 7 segments at the start of the curve expand to a new unit square blob 4 and a bridge after it. This is an OR,ER the same as in k=5 so that new bridge is maintained in subsequent expansions too. The 7 segments at the end of the curve are the same as the start of the curve, but rotated −90°. ☐

In the bridge expansions, it can be noted that it doesn't matter if there is a further segment beside the bridge (away from its direction of bending). In the following diagram, those prospective segments are shown dotted.





OL expands to ER with a segment beside, and ER expands to EL with a segment beside.

EL expands to OR without a segment beside. The segment beside would have come from completing the 3-side of the EL left bend, making it not a bridge. Similarly OR expands to OL without a segment beside. The segment beside would have come from 3-side of the OR right bend, making it not a bridge.

So there is never a segment beside OR or OL and there is always a segment beside ER and EL.

Theorem 61. *The blob sizes in dragon curve k are, in sequence from the start of the curve,*

$$\mathit{BlobList}_k = \begin{cases} 4 & \text{if } k = 4 \\ 4,\ 5 & \text{if } k = 5 \\ 4, 5, \ldots, k{-}2,\ \ k{-}1, k,\ \ k{-}2, \ldots, 5, 4 & \text{if } k \geq 6 \end{cases}$$

The $k \geq 6$ case applies for k=4 and k=5 too if terms which would be <4 are omitted.

Proof. The development of a given blob is determined by its shape and the bridge end before and after which will add new enclosed squares in the cycle above.

In curve k=4, the unit square blob has OL before and OR after. It expands successively to make a sequence of blob sizes.

In curve k=5, the new unit square blob at the start of the curve is the same as in k=4. In subsequent expansions this is the $k{-}1$ of *BlobList*. The segments at the start of k=5 are the same as in k=4 and so give a new unit square blob on each expansion. This means a sequence of blobs $4, \ldots, k{-}1, k$ at the start of the curve.

In curve k=6, there is a new unit square blob at the end of the curve with OL before (towards the end of the curve) and OR after (towards the middle of the curve). This and the non-blob segments at the very end of the curve are the same as the start of k=5 and so give a sequence of blobs $4, \ldots, k{-}2$ from the end of the curve towards the middle. ☐

From *BlobList* the number of blobs and bridges in curve level k is

$$\begin{aligned} \mathit{BlobCount}_k &= \begin{cases} 0 & k < 4 \\ 1 & k = 4 \\ 2k{-}8 & k > 4 \end{cases} \\ &= 0, 0, 0, 0,\ 1, 2, 4, 6, 8, 10, 12, \ldots \qquad \text{A004277} \end{aligned}$$

$$\begin{aligned} \mathit{BridgeCount}_k &= \begin{cases} 0 & k<4 \\ \mathit{BlobCount}_k - 1 \ = \ 2k{-}9 & k \ge 4 \end{cases} \\ &= 0,0,0,0,\ 0,1,3,5,7,\ 9,11,\ldots \end{aligned}$$

Theorem 62. *Blob k starts at point number (starting $n{=}0$ at the origin),*

$$\mathit{BlobN}_k = \tfrac{2}{5}\,2^k + \tfrac{1}{5}\,[3,6,12,9] \qquad k \ge 4 \tag{302}$$

$$= 7, 14, 28, 53, 103, 206, 412, 821, 1639, 3278, \ldots \qquad k \ge 4$$

$$= \textit{binary } 1100\ldots \textit{ repeated zero or more} \tag{303}$$

$$\textit{and final } 111, 1110, 11100, \textit{ or } 110101 \quad \textit{for } k{-}1 \textit{ bits} \tag{304}$$

$$= \textit{binary } 111,\ 1110,\ 11100,\ 110101,\ 1100111,\ 11001110, \ldots$$

Proof. In the bridge expansions of figure 60, the 3 segments which make the new enclosed unit square are at one end of the bridge. When they're at the start of blob k the expanded blob $k{+}1$ starts 3 vertices earlier than doubling segments to $2\mathit{BlobN}_k$. The bridge type at the start of a blob goes in a cycle,

$$\begin{tikzcd}
k \equiv 0 \bmod 4 & \mathrm{OL} \arrow[d] & \mathrm{OR} \arrow[l, "-3"'] & k \equiv 3 \bmod 4 \\
k \equiv 1 \bmod 4 & \mathrm{ER} \arrow[r] & \mathrm{EL} \arrow[u, "-3"'] & k \equiv 2 \bmod 4
\end{tikzcd}$$

The start of blob $k{=}4$ is an OL. It expands to ER with no extra segments. Then ER expands to EL with no extra segments. But EL to OR and OR to OL both have the new square at their ends. So

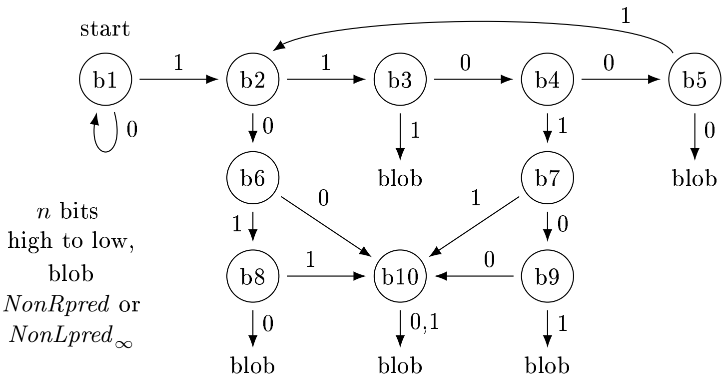
$$\begin{aligned} \mathit{BlobN}_{k+1} &= 2\mathit{BlobN}_k \ - \ [0,0,3,3]_k \qquad\qquad k \ge 4 \\ &\quad \text{starting } \mathit{BlobN}_4 = 7 \end{aligned}$$

Repeatedly expanding this recurrence gives the power form (302) and the generating function. ☐

Second Proof of Theorem 62. The non-blob segments can be characterized by the boundary predicates from section 3.4 and section 3.5. Non-blob segments of the curve continued infinitely are both left and right boundary,

$$\mathit{Rpred}(n) = 1 \ \text{ and } \ \mathit{Lpred}_\infty(n) = 1 \qquad \text{non-blob segments } n$$

The state transitions for Lpred_∞ are the same as Rpred but $0{\leftrightarrow}1$ bit-flipped and different start. Rpred and Lpred_∞ states can be followed simultaneously by the bits of n. If either reaches "non-boundary" then that segment is part of a blob. With some usual DFA state machine manipulations the result of simultaneous traversal is



b5 loops back to b2. b5 and b1 can be treated together as both going to b2, with b5 preceded by one or more repeats of 1100. Starting at b1 or b5 the bit patterns which do not go to “blob” and not taking the b5 loop are

empty	1	10	100	1101	11010	110101
		11	101	1011	11011	
			110			

Applying an initial 1100 to the four of length < 3 bits gives triplets n, $n{+}1$, $n{+}2$ which are the bridges, the triplet 4,5,6 preceding the first blob $k{=}4$.

	100	1011	11001	110010
+1	101	1100	11010	110011
+2	110	1101	11011	110100

The last segment of each triplet is the segment preceding vertex *BlobN*. So *BlobN* is that bit pattern +1, which is the binary form (303). ☐

BlobN is double-visited by its construction. Its other n is an offset

$$\begin{aligned} \mathit{other}(\mathit{BlobN}_k) &= \mathit{BlobN}_k + [4,8,16,4] \\ &= 11, 22, 44, 57, 107, 214, 428, 825, \ldots \qquad k{\ge}4 \end{aligned}$$

This follows from applying *other* to the *BlobN* bit pattern. The low bits (304) suffice for the flips *other* makes.

Or geometrically, the expansions in figure 60 show when *BlobN* moves or when it merely doubles. OR is a left turn because the preceding odd in the bridge is a right turn and odd turns alternate. OR expands as a 3e type square enclosing “enc” and giving OL its other visit at +4. OL to ER and ER to EL merely double. EL to OR is again an enclose going to other at +4. That EL has a left turn, to make the enclosure consecutive, follows from OL making ER with left turn, and ER to EL remaining left turn.

The last point in a blob is 3 bridge segments back from the start of the next,

$$\begin{aligned} \mathit{BlobNend}_k &= \mathit{BlobN}_{k+1} - 3 \qquad\qquad k \ge 4 \\ &= \tfrac{4}{5}\, 2^k - \tfrac{1}{5}\, [9,3,6,12] \\ &= 11, 25, 50, 100, 203, 409, 818, 1636, 3275, 6553, \ldots \qquad k \ge 4 \\ &= \text{binary } 1100\ldots \text{ repeated zero or more} \\ &\qquad\quad \text{and final } 1, 10, 100, \text{ or } 1011 \quad \text{for } k \text{ bits} \end{aligned}$$

$$= \text{binary } 1011,\ 11001,\ 110010,\ 1100100,\ 11001011, \ldots$$

The number of segments in blob k is difference *BlobN* to *BlobNend*.

$$\mathit{BlobSegments}_k = \begin{cases} 0 & \text{if } k < 4 \\ \mathit{BlobNend}_k - \mathit{BlobN}_k & \text{if } k \geq 4 \\ \quad = \frac{2}{5}2^k - \frac{1}{5}[12, 9, 18, 21] & \end{cases}$$

$$\begin{aligned} &= \text{binary } 1100\ldots \text{ repeated zero or more} \\ &\qquad \text{and final } 100, 1011, 10110, \text{ or } 101111 \quad \text{for } k{-}1 \text{ bits} \\ &= 0, 0, 0, 0, 4, 11, 22, 47, 100, 203, 406, \ldots \end{aligned}$$

$\mathit{BlobSegments}_{0,1,2,3}$ are taken as 0 for no such blob size. This will match the area and boundary of those sizes also taken as 0.

The total 2^k segments of curve k are the blob segments plus non-blob segments. The total non-blob segments are the 3 bridge segments between blobs and the segments at the start and end of the curve.

$$2^k = \left(\sum_{j = \mathit{BlobList}_k} \mathit{BlobSegments}_j \right) + \mathit{NonBlobSegments}_k$$

$$\mathit{NonBlobSegments}_k = \begin{cases} 1, 2, 4, 8, 12 & \text{if } k = 0 \text{ to } 4 \\ 6k - 13 & \text{if } k \geq 5 \end{cases}$$

$$= 1, 2, 4, 8, 12, 17, 23, 29, 35, 41, \ldots$$

For $k < 4$, all segments are non-blob so $\mathit{NonBlobSegments}_k = 2^k$. For $k{=}4$, there are only 5 segments at the end so $\mathit{NonBlobSegments}_4 = 7{+}5$. For $k \geq 5$, there are 7 non-blob segments at both start and end so

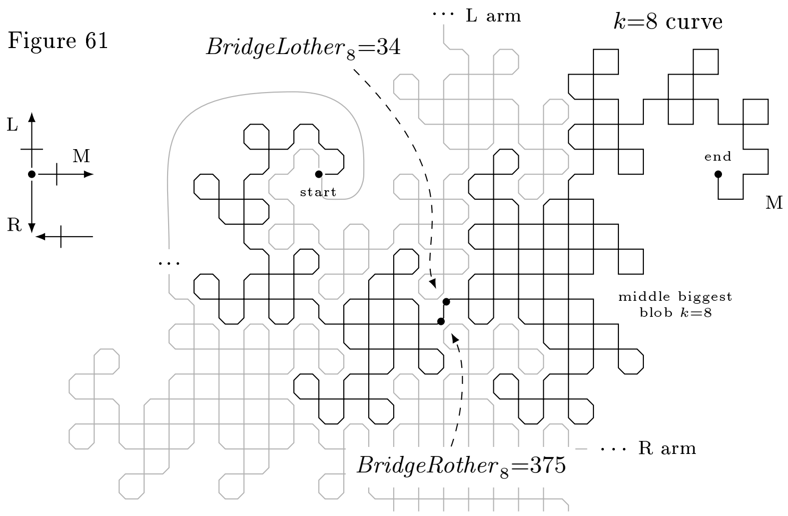
$$\mathit{NonBlobSegments}_k = 3\,\mathit{BridgeCount}_k + 14 \ = 6k - 13 \qquad k \geq 5$$

Bridge points are on the boundary so are visited a second time by curve arms on the left and right at +90° and −90°. Those start directions are 180° apart but the curve meandering takes them to just a single unit segment apart at each bridge.

The bridge before the middle biggest blob is touched on the left side by a curve of the same level. On the right side it is touched by the second half of a −90° curve of one higher level. This is an opposing right side sub-curve. Figure 61 illustrates the touches on the bridge before blob $k{=}8$ in curve $k{=}8$.

For the purpose of these bridges, the two points preceding the first blob $k{=}4$ (the single unit square) can be considered as a bridge. There is no further blob before it but the locations of the points follow the bridge pattern.

Figure 61



Theorem 63. *The n point numbers of the points of the bridge before blob k on the left and right side are*

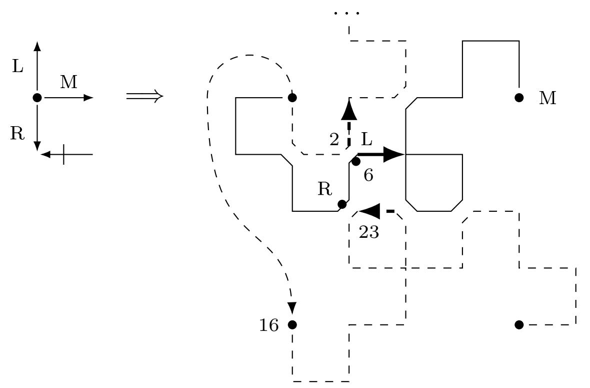
$$\begin{aligned} \mathit{BridgeLN}_k &= \mathit{BlobN}_k - [1,2] \\ &= 6, 12, 27, 51, 102, 204, 411, 819, \ldots \qquad k{\geq}4 \\ \mathit{BridgeRN}_k &= \mathit{BlobN}_k - [2,1] \\ &= 5, 13, 26, 52, 101, 205, 410, 820, \ldots \qquad k{\geq}4 \end{aligned}$$

The other n in the adjacent curve arms which are the other visits to those points are

$$\begin{aligned} \mathit{BridgeLother}_k &= \mathit{other}(\mathit{BridgeLN}_k) \quad = \tfrac{2}{15}2^k + \tfrac{1}{15}[-2,-4,7,-1] \\ &= \mathit{P3N}(k) \qquad \textit{convex hull vertex from theorem 38} \\ \mathit{BridgeRother}_k &= \mathit{other}(\mathit{BridgeRN}_k) \quad = \tfrac{22}{15}2^k + \tfrac{1}{15}[-7,1,2,4] \\ &= 2.2^k - \mathit{P1N}(k) \qquad \textit{convex hull vertex} \\ &= 23, 47, 94, 188, 375, 751, 1502, \ldots \quad k{\geq}4 \end{aligned}$$

Proof. As from theorem 60, the bend alternates left and right in successive bridges, so $\mathit{BridgeLN}_k$ and $\mathit{BridgeRN}_k$ alternately $\mathit{BlobN}_k{-}1$ and $\mathit{BlobN}_k{-}2$.

For k=4 to k=7, the n of the other visits can be calculated explicitly and are per the formulas. Level k left and right side curve arms expanded 4 times down to $k{-}4$ sub-curves is then



The bridge is within sub-curve 6 of the middle since

$$\mathit{BlobN}_k = 6.2^{k-4} + \mathit{BlobN}_{k-4} \qquad \text{for } k \ge 8$$

The left arm sub-curve at 2 is +90° to that middle sub-curve so the same arm arrangement in $k{-}4$

$$\mathit{BridgeLother}_k = \ 2.2^{k-4} + \mathit{BridgeLother}_{k-4} \qquad k \ge 8$$

For the right side, the right arm sub-curve 23 to 24 corresponds to a second $k{-}4$ sub-curve of a $-90°$ arm. The direction shown is the side of expansion. So reducing $\mathit{BridgeRother}_{k-4}$ to its offset into the second sub-curve is

$$\begin{aligned}\mathit{BridgeRother}_k &= 23.2^{k-4} + \left(\mathit{BridgeRother}_{k-4} - 2^{k-4}\right) \qquad k{\ge}8\\ &= 22.2^{k-4} + \mathit{BridgeRother}_{k-4} \qquad k{\ge}8\end{aligned}$$

These others recurrences are per the convex hull *P3N* and *P1N* shown. ☐

BridgeLother and *BridgeRother* can also be calculated by applying $\mathit{other}(n)$ of section 1.5 to the bit patterns of *BridgeLN* and *BridgeRN*.

The relation to convex hull vertices is since the blob crossings (below) in the main curve are straight lines at successive 45° angles and are all double-visited points. The bridges are directed inward as the curve curls around, so the bridge points are the furthest an adjacent curve can extend on the relevant side.

12.1 Blob Crossings

Theorem 64. *Blob $k \ge 4$ starts and ends at locations*

$$\begin{aligned}\mathit{BlobStart}_k &= \tfrac{1}{3}\left(\bar{b}\, b^k + \mathit{boff}(k)\right) \qquad (305)\\ &= -2{+}i,\ -3{-}i,\ \ -2{-}4i,\ 1{-}6i,\ 6{-}5i,\ 11{+}i,\ \ 10{+}12i, \ldots \quad k \ge 4\end{aligned}$$

$$\begin{aligned}\mathit{BlobEnd}_k &= \tfrac{1}{3}\left(2\, b^k - i\, \mathit{boff}(k{-}1)\right) \qquad (306)\\ &= -2{+}i,\ -3{-}2i,\ -1{-}5i,\ 4{-}6i,\ 10{-}i,\ 11{+}10i,\ 1{+}21i, \ldots \quad k \ge 4\end{aligned}$$

$$\mathit{boff}(m) = \begin{cases} 2{+}i,\ \ 1{+}3i,\ -2{+}4i,\ -3{+}2i, & m \equiv 0 \text{ \it to } 3 \bmod 8\\ -2{-}i,\ -1{-}3i,\ \ 2{-}4i,\ \ 3{-}2i & m \equiv 4 \text{ \it to } 7 \bmod 8\end{cases}$$

Proof. In the bridge expansions of figure 60, when a the new unit square is enclosed that end moves by 1 in the direction of the new bridge.

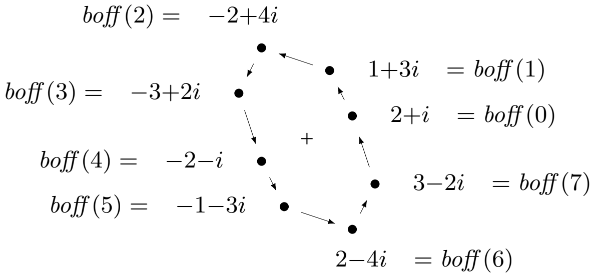
The expanded shapes are rotated +45° each time to maintain the curve initial segment direction. The result is that ER to EL and OR to OL keep the same direction and EL to OR and OL to ER rotate +90°.

$$\begin{tikzcd}
\text{OL} \arrow[d, "+90^\circ"'] & \text{OR} \arrow[l, "\text{move}"'] \\
\text{ER} \arrow[r] & \text{EL} \arrow[u, "\text{move}\ +90^\circ"']
\end{tikzcd}$$

The two moves are in the same direction, then there is two rotates +90° before the next two. For the first blob k=4, the start is OL directed East and the end is OR directed South, so the following pattern of moves in direction and they give the power forms above.

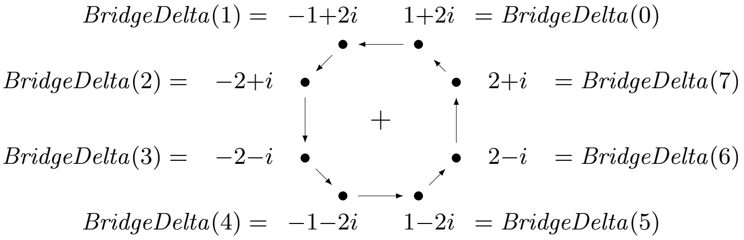
$$\begin{aligned}
\mathit{BlobStart}_{k+1} &= b\,\mathit{BlobStart}_k + [0,0,1,1,\ 0,0,-1,-1] \\
\mathit{BlobEnd}_{k+1} &= b\,\mathit{BlobEnd}_k\ \ + [i,0,0,-i,\ -i,0,0,i]
\end{aligned} \qquad \square$$

The cycle of *boff*(m) offsets can be illustrated



The step from end of blob k to start of blob k+1 is the bridge type and its direction is a period-8 pattern,

$$\begin{aligned}
\mathit{BridgeDelta}_k &= \mathit{BlobStart}_{k+1} - \mathit{BlobEnd}_k \\
&= \begin{cases} 1{+}2i,\ -1{+}2i,\ -2{+}i,\ -2{-}i & k \equiv 0 \text{ to } 3 \bmod 8 \\ -1{-}2i,\quad 1{-}2i,\quad 2{-}i,\quad 2{+}i & k \equiv 4 \text{ to } 7 \bmod 8 \end{cases}
\end{aligned}$$



The step across a blob from start to end is

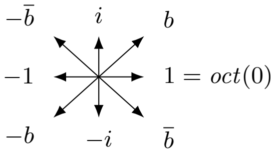
$$\mathit{BlobDelta}_k = \mathit{BlobEnd}_k - \mathit{BlobStart}_k$$

$$
\begin{aligned}
&= \mathit{BlobDeltaLen}_k \,.\, \mathit{oct}(k{+}1) \\
&= 0,\ -i,\ 1{-}i,\ 3,\ 4{+}4i,\ 9i,\ -9{+}9i,\ -19,\ -20{-}20i,\ \dots \qquad k \ge 4
\end{aligned}
$$

$$\mathit{oct}(m) = [1, b, i, -\bar{b},\ \ -1, -b, -i, \bar{b}] \qquad \text{as } m \equiv 0 \text{ to } 7 \bmod 8$$

$$
\begin{aligned}
\mathit{BlobDeltaLen}_k &= \tfrac{1}{3}\Big(2^{\lceil k/2 \rceil} - [4,5,5,7]\Big) \\
&= 0,1,1,3,4,9,9,19,20,41,41,83,84,169,169,\dots \qquad k \ge 4
\end{aligned}
$$

$\mathit{oct}(m)$ is a 8-way direction starting East and going +45° each time, as a way to express the cycle.



The half-power in *BlobDeltaLen* makes pairs $k = 2m{-}1,\ 2m$ either equal for m odd which is the $5,5$ periodic terms, or differ by $+1$ for m even which is the $7,4$ periodic terms (wrapping around) for extra $-\frac{1}{3}(4-7) = +1$.

Total deltas for all blobs and bridges are the curve endpoint b^k. The deltas at the end of the curve are an unfold and reversal so factor i. The non-blob segments at the start of the curve are $-2{+}i$, and for $k \ge 5$ the same at the end of the curve (unfolded).

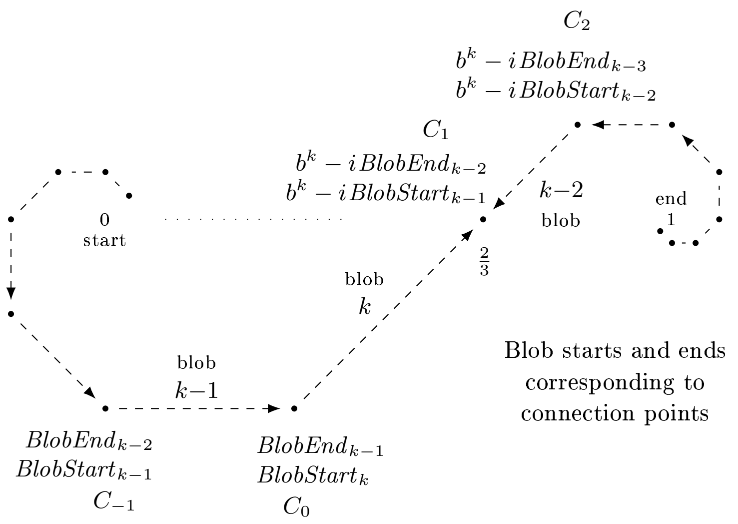
The sum here has a bridge delta included after each blob so the middle bridge between middle biggest blob k and end blob $k{-}2$ is counted twice, so subtract one copy of that.

$$
\begin{aligned}
b^k \ &= (-2{+}i) \ + \ \left(\sum_{j=4}^{k} \mathit{BlobDelta}_j + \mathit{BridgeDelta}_j \right) \qquad\qquad k \ge 5 \\
&\quad + \ i\,(-2{+}i) \ + \ i \left(\sum_{j=4}^{k-2} \mathit{BlobDelta}_j + \mathit{BridgeDelta}_j \right) - \mathit{BridgeDelta}_k
\end{aligned}
$$

Ngai and Nguyen[38] calculate blob connection points in the dragon fractal. Their C_0 is the start of the middle biggest blob, C_n is towards the end of the curve, and C_{-n} is towards the start of the curve.

$$
\begin{aligned}
C_{-n} &= \tfrac{1}{3}\, 2^{-(n-1)/2}\ \ \operatorname{cis}\big((n{+}3)\tfrac{\pi}{4}\big) && n \ge 0 \\
C_{n} &= \tfrac{1}{3}\, 2^{-(n-1)/2}\ \ \operatorname{cis}\big((n{-}3)\tfrac{\pi}{4}\big)\ \ + i && n \ge 1 \\
&\operatorname{cis}\theta = \cos\theta + i \sin\theta
\end{aligned}
$$

They draw the curve directed upwards and expanding on the left so the corresponding limits for *BlobStart* and *BlobEnd* here are conjugate for mirror image and factor i to rotate.



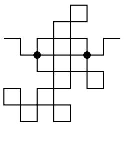
Blob starts and ends corresponding to connection points

$$i\,\mathrm{conj}\left(\frac{BlobStart_{k-n}}{b^k}\right) \;\to\; i\,\mathrm{conj}\left(\frac{BlobEnd_{k-1-n}}{b^k}\right) \;\to\; C_{-n} \qquad n{\ge}0$$

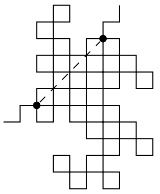
$$i\,\mathrm{conj}\left(1-i\frac{BlobStart_{k-n}}{b^k}\right) \to i\,\mathrm{conj}\left(1-i\frac{BlobEnd_{k-1-n}}{b^k}\right) \to C_n \qquad n{\ge}1$$

For C_n with $n \geq 1$, the blobs are from the end of the curve towards the middle. So at 1 and factor $-i$ to unfold and reverse. C_1 is the end of blob $k{-}2$ so index $BlobEnd_{k-1-n}$ and correspondingly $BlobStart_{k-n}$. When $n{=}1$ there is no blob $k{-}1$ in the curve end, but after the bridge after blob $k{-}2$ is where a $k{-}1$ would start.

The powers in *BlobStart* (305) and the cis in C_n show how the blob positions spiral out from (or in towards) the curve ends.



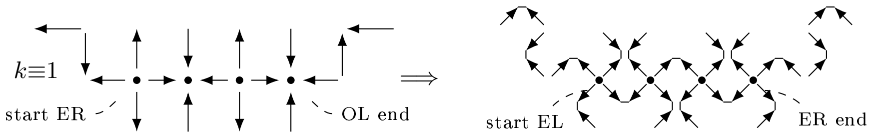
blob k=7, odd, straight crossing

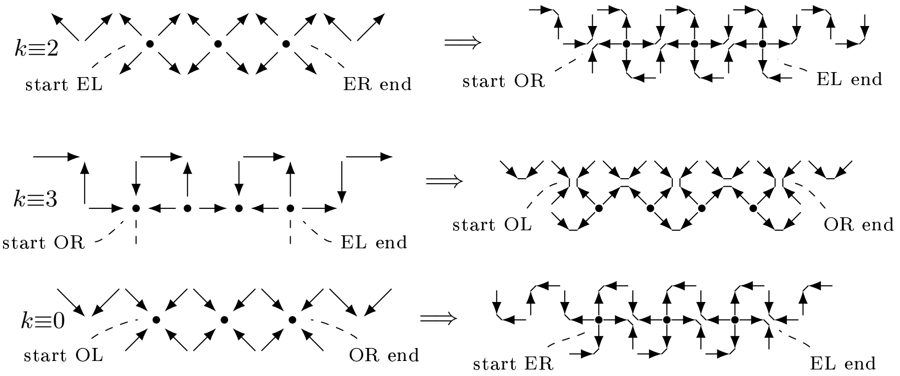


blob k=8, even, diagonal crossing

Theorem 65. *A line from blob start to end goes straight along curve segments when k odd, or* 45° *angle diagonally across enclosed unit squares when k even.*

Proof. Blob 5 is the first with a non-empty line from start to end. It is straight across with ER start and OL end. The following diagrams show the pattern of successive expansions for such a crossing.





In the first diagram, straight ER--OL expands to diamonds EL--ER.

In the second diagram, diamonds EL--ER expand to straight OR--EL.

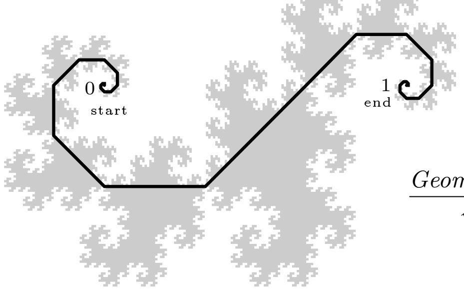
In the third diagram, straight OR--EL expands to diamonds OL--OR. For this expansion, only the unit squares above the straight line are required, even though the second diagram shows the diamonds give unit squares both above and below.

In the fourth diagram, diamonds OL--OR expand back to the first straight ER--EL again. Those diamonds expand to unit squares above and below but only the verticals of those squares are required in the first diagram. ☐

A length curve start to end can be calculated by lines going either along curve segments or through enclosed unit squares. Such a path must go along the bridge segments between blobs and then the shortest path across the blobs is the line start to end as above. So length

$$\begin{aligned}
\mathit{GeomLength}_k &= \mathit{NonBlobSegments}_k + \sum_{j=\mathit{BlobList}_k} \left|\mathit{BlobDelta}_j\right| \\
&= \begin{cases} 2^k & \text{if } k<4 \\ (1+\sqrt{2}).\sqrt{2}^k \;+ (4-\tfrac{3}{2}\sqrt{2})k \;+ \left[-8+2\sqrt{2},\ -10+\tfrac{7}{2}\sqrt{2}\right] & \text{if } k \ge 4 \end{cases} \\
&= 1, 2, 4, 8, 12, 18,\ 24+\sqrt{2},\ 34+\sqrt{2},\ 40+6\sqrt{2},\ 58+6\sqrt{2}, \dots
\end{aligned}$$

Scaled by $\sqrt{2}^k$ for curve endpoints a unit length the limit is the “silver ratio”,



$$\frac{\mathit{GeomLength}_k}{\sqrt{2}^k} \to 1+\sqrt{2}$$
$$= 2.414213\ldots \qquad \text{A014176}$$

Variations could be made going diagonally across the bridges if desired. But they are only the linear part in *GeomLength* and do not change the $\sqrt{2}^k$ term which becomes limit $1+\sqrt{2}$.

The rational and $\sqrt{2}$ parts of *GeomLength* separately are

$$GeomLength_k = GeomLengthRat_k + GeomLengthSqrt_k\sqrt{2}$$

$$GeomLengthRat = \begin{cases} 2^k & \text{if } k < 4 \\ 2^{\lceil k/2 \rceil} + 4k - [8, 10] & \text{if } k \ge 4 \end{cases}$$

$$= 1, 2, 4, 8, 12, 18, 24, 34, 40, 58, 64, 98, 104, 170, \dots$$

$$GeomLengthSqrt = \begin{cases} 0 & \text{if } k < 4 \\ 2^{\lfloor k/2 \rfloor} - 3\lfloor k/2 \rfloor + 2 & \text{if } k \ge 4 \end{cases}$$

$$= 0, 0, 0, 0,\ \ 0, 0,\ \ 1, 1,\ \ 6, 6,\ \ 19, 19,\ \ 48, 48, \dots \qquad k{\ge}6 \text{ undup A005578}$$

GeomLengthSqrt is unchanged at k odd since for $k{\ge}6$ the *BlobList* gains k and $k{-}2$ which are both odd and so rational crossing lengths.

GeomLengthRat at k even similarly does not gain any blob crossings since both k and $k{-}2$ even are irrational crossing lengths, but it does gain the two bridges +6.

12.2 Middle Nearest

Theorem 66. *In dragon curve k, the right boundary point or points nearest the curve middle $\frac{1}{2}b^k$ are on the bridge after the middle biggest blob. Their distance to the middle is*

$$Rnear_k = \begin{cases} \frac{1}{2} & \textit{if } k{=}0 \\ m_k & \textit{if } k \equiv 0, 2, 3 \bmod 4 \textit{ and } k \ge 1 \\ \sqrt{m_k^2 + \frac{1}{2}} & \textit{if } k \equiv 1 \bmod 4 \ \ \textit{(two point equal distance)} \end{cases}$$

$$= \tfrac{1}{2},\ \tfrac{1}{2}\sqrt{2},\ 0, 0, 0, 1, 1,\ \sqrt{2},\ 2,\ \sqrt{13},\ 5,\ 5\sqrt{2},\ 10,\ \sqrt{221}, \dots$$

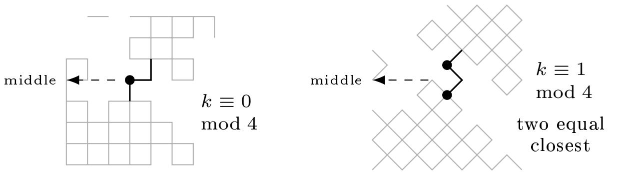
$$k{\ge}4 \text{ even A000975},\quad k{\ge}3 \text{ odd } \sqrt{}\text{A241892}$$

$$\textit{where } m_k = \tfrac{1}{6}\big(\sqrt{2}^k - [4,\ \sqrt{2},\ 2,\ 2\sqrt{2}]\big)$$

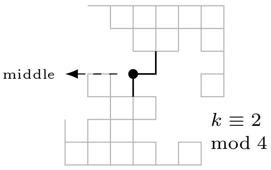
$$= -\tfrac{1}{2},\ 0, 0, 0, 0,\ \tfrac{1}{2}\sqrt{2},\ 1,\ \sqrt{2},\ 2,\ \tfrac{5}{2}\sqrt{2},\ 5,\ 5\sqrt{2},\ 10, \dots$$

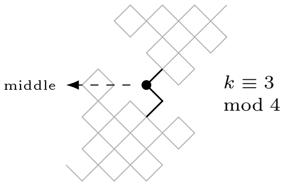
$$k{\ge}4 \text{ even A000975},\quad k{\ge}3 \text{ odd } 1/\sqrt{2}{\times}\text{A000975}$$

For $k \equiv 1 \bmod 4$, there are two equal nearest points, one above and one below the line from curve start to end. For other k, the nearest point is unique.



middle nearest cases





Proof. For $k < 4$, the points nearest the middle can be calculated explicitly.

For $k \geq 4$, consider the surrounding curves on the right side and take convex hulls around their segments and their boundary squares.

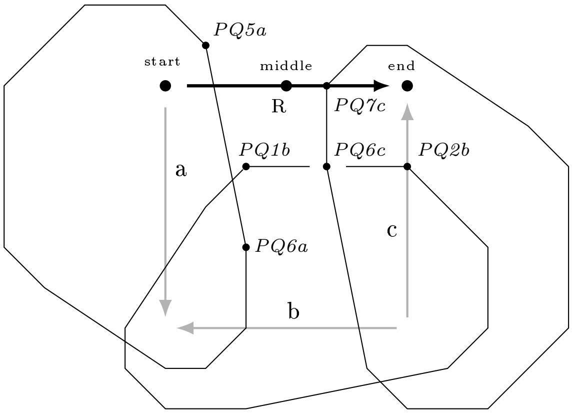


Figure 62:
convex hulls of curves
surrounding right side

The hull including boundary squares can be calculated the same as the segments hull in theorem 37, since the curve with boundary squares comprises an unfold of sub-curves $k{-}1$ and their boundary squares. For $k \geq 4$ there are 10 vertices.

The boundary squares push the vertices out by 1 unit square on each straight side. Working through the hull recurrences the result is the same location forms as segments vertices P1 etc (196), but a change of index and direction for the offset terms (p becomes pq).

$$\begin{aligned} &PQ1(k) = -\tfrac{1}{3}\left(b^{k+3} + pq(k{+}3)\right), \quad \text{etc} \\ &pq(m) = i.p(m{+}2) \end{aligned}$$

The boundary squares of the surrounding curves push into the right boundary R so that minimum extents for the boundary points are given by maximum extents of the surrounding hulls.

In figure 62, and working through the vertex formulas, the surrounding curve c hull side PQ7c--PQ6c is vertical and nearest to the middle, with PQ7c on or above the midline.

Curve a has the same left side as c but PQ6a to PQ5a slopes further away.

Curve b side PQ1b--PQ2b is horizontal and is always further away than PQ7c. In the factors of b^k, PQ1b is at $\frac{2}{3}$ distance up so $\frac{1}{3}$ from the middle whereas PQ7c is $\frac{1}{6}$. ☐

Vertex PQ7c is the single minimum point for $k \equiv 0, 2, 3 \bmod 4$. For $k \equiv 1 \bmod 4$, it is the upper of the two equal minimums.

$$PQ7(k) = \tfrac{1}{3}\left((3{+}i)b^k \quad + i\, pq(k) \right)$$

$$PQ7c(k) = (1-i)b^k + i.PQ7(k)$$
$$Rnear_k = \left|\tfrac{1}{2}b^k - PQ7c(k)\right|$$

The middle biggest blob is the lower end of the bridge. In each case, the end of that blob end is the required horizontal distance,

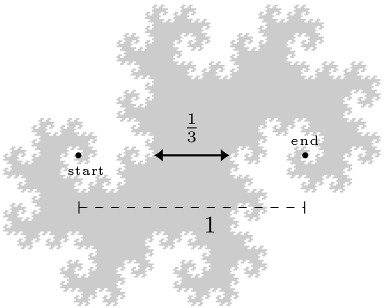
$$m_k = \operatorname{Re}\frac{BlobEnd_k - \frac{1}{2}b^k}{\omega_8^k}$$

For $k \equiv 1 \bmod 4$, the extra vertical half diagonal is $\frac{1}{2}\sqrt{2}$ so $\sqrt{m_k^2+\frac{1}{2}}$.

Scaled by $\sqrt{2}^k$ for curve endpoints a unit length, the limit for $Rnear_k$ is the coefficient $\frac{1}{6}$ in m_k.

$$\frac{Rnear_k}{\sqrt{2}^k} \to \frac{1}{6}$$

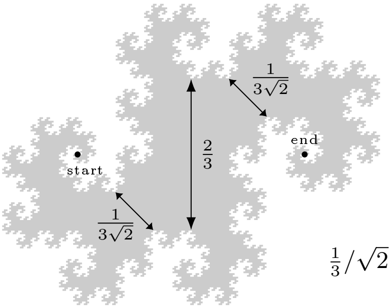
The twindragon boundary is two dragon right sides, so $2Rnear_{k+1}$ is the shortest distance across twindragon k by a line passing through its middle. Scaled by $\sqrt{2}^{k+1}$ for endpoints a unit length, this is limit $\frac{1}{3}$ from side to side across its middle.

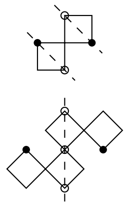


twindragon nearest opposite points

$$\frac{2Rnear_{k+1}}{\sqrt{2}^{k+1}} \to \frac{1}{3}$$

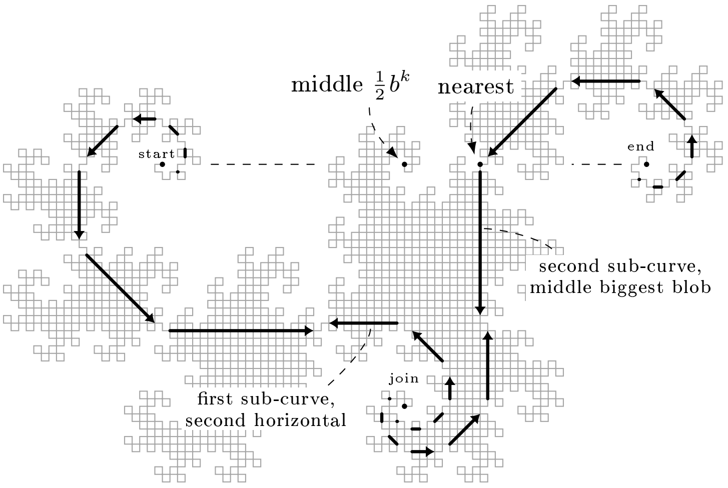
Diagonal distances across are in the first and second halves as two twindragons $1/\sqrt{2}$ smaller. Or a vertical distance is from four twindragons $\frac{1}{2}$ smaller. The whole midpoint is the end of the two vertical small twindragons so nearest point at $\frac{2}{3}$ each from the but scaled $\frac{1}{2}$ for $\frac{2}{3}$ vertically.





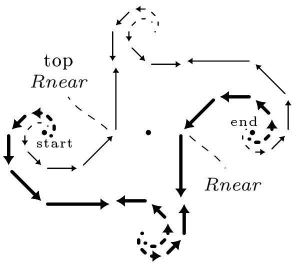
$\frac{1}{3}/\sqrt{2} = 0.235702\ldots$ A020775

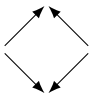
Another approach for the nearest point is to consider blob crossings within $k-1$ sub-curves.



The crossings are straight and 45° diagonal. Working through their lengths and endpoints shows that among the crossings and bridge points the *Rnear* bridge is the nearest to the middle.

The curve right boundary points are on the right of these crossings. As the curve curls around some of them are facing in towards the middle. To show those boundary points are still not closer than *Rnear*, consider the same crossings in sub-curves on the top side, completing a square in the manner of the twindragon.





Dragon curves arranged like this do not overlap so the right side of the curve is outside the crossing lines of these top curves. Their closest to the middle is the top *Rnear* and the curve right side does not touch that because there must be room for a further 2 curve arms in between for 4-arm plane filling at both start and end.

12.3 Blob Boundary

Theorem 67. *Blob k has boundary length*

$$\begin{aligned} BlobB_k = {} & 2BlobB_{k-1} - 2BlobB_{k-2} + 4BlobB_{k-3} \\ & - 3BlobB_{k-4} + 2BlobB_{k-5} - 2BlobB_{k-6} \qquad k \ge 10 \\ = {} & 0, 0, 0, 0, 4, 10, 20, 38, 68, 122, 212, \ldots \end{aligned} \tag{307}$$

$$\textit{Generating function} \quad gBlobB(x) = x^4 \, \frac{4 + 2x + 8x^2 + 2x^3 + 4x^4 + 4x^5}{(1{-}x)\,(1{+}x^2)\,(1{-}x{-}2x^3)}$$

$$= 4 + 2x + 4x^2 + 2x^3 - \frac{6}{1{-}x} + \frac{2x}{1{+}x^2} + \frac{2}{1{-}x{-}2x^3} \tag{308}$$

Proof. The total curve boundary B is all the blob boundaries plus both sides of the non-blob segments.

$$B_k = \Bigg(\sum_{j=\mathit{BlobList}_k} \mathit{BlobB}_j \Bigg) + 2\,\mathit{NonBlobSegments}_k \tag{309}$$

Taking a difference of successive *BlobList*,

$$\begin{aligned} B_k - B_{k-1} &= \mathit{BlobB}_k + \mathit{BlobB}_{k-2} \ + \ \mathit{dNonBlobSegments}_{k-1} \qquad k \ge 2 \\ \mathit{dNonBlobSegments}_k &= \mathit{NonBlobSegments}_{k+1} - \mathit{NonBlobSegments}_k \\ &= \begin{cases} 1, 2, 4, 4, 5 & \text{if } k = 0 \text{ to } 4 \\ 6 & \text{if } k \ge 5 \end{cases} \end{aligned}$$

is a recurrence for *BlobB* using B

$$\mathit{BlobB}_k = (B_k{-}B_{k-1}) - \mathit{dNonBlobSegments}_{k-1} - \mathit{BlobB}_{k-2} \qquad k \ge 2 \tag{310}$$

Usual recurrence manipulations give (307) and from that the generating function. In those manipulations, it can be convenient to start from $k{-}1 \ge 5$ for $\mathit{dNonBlobSegments}_{k-1} = 6$ constant. □

Repeatedly expanding (310) gives *BlobB* as an alternating sum of B.

$$\begin{aligned} \mathit{BlobB}_k = \quad & B_k \quad - B_{k-1} \quad - \mathit{dNonBlobSegments}_{k-1} \\ & - B_{k-2} + B_{k-3} \quad + \mathit{dNonBlobSegments}_{k-3} \\ & + B_{k-4} - B_{k-5} \quad - \mathit{dNonBlobSegments}_{k-5} \\ & - B_{k-6} + B_{k-7} \quad + \mathit{dNonBlobSegments}_{k-7} \quad \cdots \end{aligned}$$

The signs on B are pattern $+--+$. $\mathit{dNonBlobSegments} = 6$ for $k \ge 5$ so its alternating sum cancels out until the bottom-most terms. The sum can be continued down to B_0 with a suitable adjustment. (Stopping earlier or taking pairs of B etc is a different adjustment.)

$$\begin{aligned} \mathit{BlobB}_k = B_k - B_{k-1} - B_{k-2} + B_{k-3} \ \ + - - + \ \ \cdots B_0 \\ - \begin{cases} 2, 2, 2, 6 & \text{if } k = 0 \text{ to } 3 \\ [6, 4, 6, 8]_k & \text{if } k \ge 4 \end{cases} \end{aligned} \tag{311}$$

This is seen in the *gBlobB* partial fractions (308),

$$\mathit{gBlobB}(x) = \frac{1{-}x}{1{+}x^2}\, gB(x) \quad + \frac{2x}{1{+}x^2} - \frac{6}{1{-}x} \ + 4{+}2x{+}4x^2{+}2x^3$$

Factor $\frac{1-x}{1+x^2}$ is a descending alternating sum $+--+$ of B and it cancels factor $\frac{1+x^2}{1-x}$ in gB (122). The $\frac{2x}{1+x^2} - \frac{6}{1-x}$ part is periodic $-[6, 4, 6, 8]$. The fixed terms adjust the initial four of those to $-[2, 2, 2, 6]$ instead.

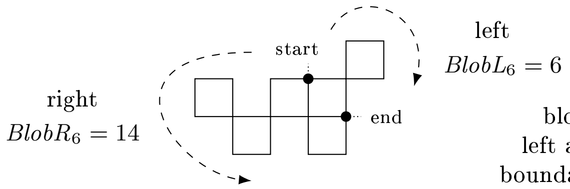
Factor $\frac{1-x}{1+x^2}$ is the usual way to take a descending sum $+--+$ in a generating function.

$$\begin{aligned} \mathit{Sum}(x) &= gB(x) - x\,gB(x) - x^2gB(x) + x^3gB(x) \ + x^4\mathit{Sum}(x) \\ \mathit{Sum}(x) &= \frac{1-x-x^2+x^3}{1-x^4}\,gB(x) \ \ = \frac{1-x}{1+x^2}\,gB(x) \end{aligned} \tag{312}$$

The cubic part of the generating function partial fractions is a multiple of gJA, giving an identity with the remaining periodic terms

$$\mathit{BlobB}_k = 4\,JA_{k+2} - [4,2,4,6] \qquad k \ge 4$$

The left side and right side boundary lengths can be taken separately. The sides are reckoned going blob start to end as the blob appears at the start of the full curve.



blob k=6
left and right
boundary lengths

Theorem 68. *Blob k has left boundary length*

$$\begin{aligned} \mathit{BlobL}_k &= \mathit{BlobL}_{k-4} \ + \ L_{k-3} & k \ge 8 \quad & (313) \\ &= \ \ \mathit{BlobL}_{k-1} \ \ + 2\mathit{BlobL}_{k-3} + \mathit{BlobL}_{k-4} & & (314) \\ &\quad - \mathit{BlobL}_{k-5} \ \ - 2\mathit{BlobL}_{k-7} & k \ge 11 \quad & \\ &= 0,0,0,0,\ \ 0,3,6,11,20,39,66,111,192,\ldots & & \\ \mathit{gBlobL}(x) &= x^4\ \frac{3x+3x^2+5x^3+3x^4+4x^5+2x^6}{(1{-}x)\,(1{+}x)\,(1{+}x^2)\,(1{-}x{-}2x^3)} & & \\ &= 2{+}x{+}2x^2{+}x^3 - \tfrac{5}{2}\frac{1}{1{-}x} - \tfrac{1}{4}\frac{1}{1{+}x} + \frac{x}{1{+}x^2} + \tfrac{1}{4}\frac{3{-}2x{+}2x^2}{1{-}x{-}2x^3} & & \end{aligned}$$

And right boundary length

$$\begin{aligned} \mathit{BlobR}_k &= \mathit{BlobR}_{k-4} + L_{k-3} + 2L_{k-4} & k \ge 8 \quad & (315) \\ &= \textit{same recurrence } (314) \textit{ as BlobL} & k \ge 11 \quad & \\ &= 0,0,0,0,\ \ 4,7,14,27,48,83,146,247,420,\ldots & & \\ \mathit{gBlobR}(x) &= x^4\ \frac{4+3x+7x^2+5x^3+3x^4+4x^5+2x^6}{(1{-}x)\,(1{+}x)\,(1{+}x^2)\,(1{-}x{-}2x^3)} & & \\ &= 2{+}x{+}2x^2{+}x^3 - \tfrac{7}{2}\frac{1}{1{-}x} + \tfrac{1}{4}\frac{1}{1{+}x} + \frac{x}{1{+}x^2} + \tfrac{1}{4}\frac{5{+}2x{-}2x^2}{1{-}x{-}2x^3} & & \end{aligned}$$

Proof. Blobs $k{-}2$, …, 4 at the end of the curve are directed from the end towards the middle so their left boundary is on the right side of the curve, and their right boundary is on the left side of the curve. The boundary on each side is then

$$L_k = \left(\sum_{j=4}^{k} \mathit{BlobL}_j \right) + \left(\sum_{j=4}^{k-2} \mathit{BlobR}_j \right) + \mathit{NonBlobSegments}_k$$

$$R_k = \left(\sum_{j=4}^{k} \mathit{BlobR}_j\right) + \left(\sum_{j=4}^{k-2} \mathit{BlobL}_j\right) + \mathit{NonBlobSegments}_k$$

Taking the differences of successive *BlobList*,

$$\begin{aligned} L_k - L_{k-1} &= \mathit{BlobL}_k + \mathit{BlobR}_{k-2} + \mathit{dNonBlobSegments}_{k-1} \qquad k \ge 2 \\ R_k - R_{k-1} &= \mathit{BlobR}_k + \mathit{BlobL}_{k-2} + \mathit{dNonBlobSegments}_{k-1} \end{aligned}$$

which are mutual recurrences for *BlobL* and *BlobR*

$$\mathit{BlobL}_k = (L_k - L_{k-1}) - \mathit{BlobR}_{k-2} - \mathit{dNonBlobSegments}_{k-1} \qquad k \ge 2 \tag{316}$$

$$\mathit{BlobR}_k = (R_k - R_{k-1}) - \mathit{BlobL}_{k-2} - \mathit{dNonBlobSegments}_{k-1} \tag{317}$$

Substitute (317) into (316) then $\mathit{dNonBlobSegments}_{k-3} = 6$ constant when $k-3 \ge 5$ so

$$\mathit{BlobL}_k \;=\; \mathit{BlobL}_{k-4} + (L_k - L_{k-1}) - (R_{k-2} - R_{k-3}) \qquad k \ge 8$$

The R difference is $R_{k-2} - R_{k-3} = L_{k-3}$ from unfolding (110). Then one application of the L recurrence (100) simplifies to (313).

Similarly for BlobR_k substitute (316) for *BlobL* into (317). This time applying the L recurrence leaves $L_{k-3} + 2L_{k-4}$ per (315).

$$\mathit{BlobR}_k = \mathit{BlobR}_{k-4} + (R_k - R_{k-1}) - (L_{k-2} - L_{k-3}) \qquad k \ge 8 \qquad \square$$

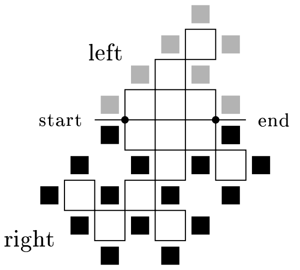
Some recurrence or generating function manipulations can express the left and right boundary in terms of the whole

$$\begin{aligned} \mathit{BlobR}_k &= \tfrac{3}{4}\mathit{BlobB}_k - \tfrac{1}{4}\mathit{BlobB}_{k+1} + \tfrac{1}{8}\mathit{BlobB}_{k+2} \;+ [1, -\tfrac{1}{4}, 0, \tfrac{1}{4}] \\ \mathit{BlobL}_k &= \tfrac{1}{4}\mathit{BlobB}_k + \tfrac{1}{4}\mathit{BlobB}_{k+1} - \tfrac{1}{8}\mathit{BlobB}_{k+2} \;- [1, -\tfrac{1}{4}, 0, \tfrac{1}{4}] \end{aligned}$$

BlobB grows as a power of r (section 2) so these give limits for the proportion of right or left boundary out of the whole blob. Notice the blob right boundary is a bigger fraction than in the whole curve right boundary (125).

$$\frac{\mathit{BlobR}_k}{\mathit{BlobB}_k} \to \tfrac{3}{4} - \tfrac{1}{4}r + \tfrac{1}{8}r^2 \;= 0.685486\ldots \tag{318}$$

$$\frac{\mathit{BlobL}_k}{\mathit{BlobB}_k} \to \tfrac{1}{4} + \tfrac{1}{4}r - \tfrac{1}{8}r^2 \;= \frac{1}{r^3 - r} \;= 0.314513\ldots$$



blob k=7 boundary squares

$\mathit{BlobLQ} = 7$ ▨

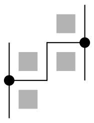
$\mathit{BlobRQ} = 15$ ■

$\mathit{BlobBQ} = 22$ total

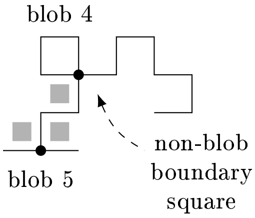
Theorem 69. *The number of boundary squares on blob k is*

$$\begin{aligned} \mathit{BlobBQ}_k &= \mathit{BlobBQ}_{k-1} - \mathit{BlobBQ}_{k-2} + 3\mathit{BlobBQ}_{k-3} + 2\mathit{BlobBQ}_{k-5} \quad k{\ge}10 \\ &= 0, 0, 0, 0, 4, 8, 13, 22, 37, 64, 109, 182, 309, \dots \\ \mathit{gBlobBQ}(x) &= -1 - 2x - x^2 - 2x^3 - x^4 + \frac{x}{1{+}x^2} + \frac{1}{1{-}x{-}2x^3} \end{aligned} \tag{319}$$

Proof. At a bridge the boundary squares of blobs do not overlap and are all the squares on the bridge. The bridge from blob 4 to 5 has 1 boundary square which is part of the initial non-blob squares (or final squares at the end of the curve).



Bridge k to $k{+}1$
for $k \ge 5$, all squares
on blob boundary



Total blob boundary squares are a sum of blobs like (309) then take differences etc.

$$\mathit{BQ}_k = \left(\sum_{j=\mathit{BlobList}_k} \mathit{BlobBQ}_j \right) + \mathit{NonBlobBQ}_k$$

$$\mathit{NonBlobBQ}_k = \begin{cases} 2, 3, 5, 9, 11, 13 & \text{if } k \le 5 \\ 14 & \text{if } k \ge 6 \end{cases} \qquad \square$$

The generating function partial fractions (319) is an identity

$$\mathit{BlobBQ}_k = \mathit{dJA}_{k+2} - [0, 1, 0, -1] \qquad k \ge 5$$

Left and right blob boundary squares also similar to left and right boundary segments are

$$\begin{aligned} \mathit{BlobLQ}_k &= 2\mathit{BlobLQ}_{k-3}{+}3\mathit{BlobLQ}_{k-4}{+}2\mathit{BlobLQ}_{k-5}{+}2\mathit{BlobLQ}_{k-6} \quad k{\ge}11 \\ &= 0, 0, 0, 0, 0, 3, 4, 7, 11, 21, 34, 57, 97, \dots \\ \mathit{gBlobLQ}(x) &= -x - x^3 - x^4 - \tfrac{3}{8}\frac{x}{1{+}x} + \tfrac{1}{2}\frac{x}{1{+}x^2} + \tfrac{1}{8}\frac{3 - 2x + 2x^2}{1{-}x{-}2x^3} \\ \mathit{BlobRQ}_k &= \text{same recurrence as } \mathit{BlobLQ}\text{, different initial values} \quad k{\ge}10 \\ &= 0, 0, 0, 0, 4, 5, 9, 15, 26, 43, 75, 125, 212, \dots \\ \mathit{gBlobRQ}(x) &= -1 - x - x^2 - x^3 + \tfrac{3}{8}\frac{x}{1{+}x} + \tfrac{1}{2}\frac{x}{1{+}x^2} + \tfrac{1}{8}\frac{5 + 2x - 2x^2}{1{-}x{-}2x^3} \end{aligned}$$

Blob boundary squares and segments are related

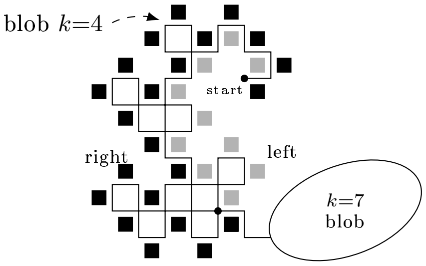
$$\mathit{BlobB}_k = 2\mathit{BlobBQ}_k - \begin{cases} 0, 0, 0, 0,\ 4 & \text{if } k \le 4 \\ 6 & \text{if } k \ge 5 \end{cases}$$

$$\mathit{BlobL}_k = 2\mathit{BlobLQ}_k - \begin{cases} 0 & \text{if } k \le 4 \\ [2, 3]_k & \text{if } k \ge 5 \end{cases}$$

$$\mathit{BlobR}_k = 2\mathit{BlobRQ}_k - \begin{cases} 0,0,0,0,2 & \text{if } k \le 4 \\ [4,3]_k & \text{if } k \ge 5 \end{cases}$$

They are each a factor of 2 so the limits (318) for proportions of blob boundary segments are the same for blob boundary squares.

Boundary squares up to and including a given blob k are sums



boundary squares pre blob k=7

$\mathit{PreBlobLQ} = 10$ ▨

$\mathit{PreBlobRQ} = 22$ ■

$\mathit{PreBlobBQ} = 32$ total

$$\begin{aligned}
\mathit{PreBlobRQ}_k &= 4 + \sum_{j=4}^{k-1} \mathit{BlobRQ}_j \qquad \text{for } k \ge 4 \\
&= \tfrac{1}{2}\Big(\mathit{BlobRQ}_{k+2} + [-1,1,0,1]\Big) && (320) \\
&= 4, 8, 13, 22, 37, 63, 106, 181, 306, 518, \dots \qquad k{\ge}4 \\
\mathit{PreBlobLQ}_k &= 3 + \sum_{j=4}^{k-1} \mathit{BlobLQ}_j \qquad \text{for } k \ge 4 \\
&= \tfrac{1}{2}\Big(\mathit{BlobLQ}_{k+2} + [0,-1,1,-1]\Big) \qquad k{\ge}5 && (321) \\
&= 3, 3, 6, 10, 17, 28, 49, 83, 140, 237, \dots \qquad k{\ge}4 \\
\mathit{PreBlobBQ}_k &= \mathit{PreBlobLQ}_k + \mathit{PreBlobRQ}_k \qquad \text{for } k \ge 4 \\
&= \tfrac{1}{2}\Big(\mathit{BlobBQ}_{k+2} + [-1,0,1,0]\Big) \qquad k{\ge}5 \\
&= \mathit{JA}_{k+2} + [0,0,1,1] \qquad k{\ge}5 && (322) \\
&= 7, 11, 19, 32, 54, 91, 155, 264, 446, 755, \dots \qquad k{\ge}4
\end{aligned}$$

For $\mathit{PreBlobLQ}_4 = 3$, blob k=4 has no left boundary squares. It is taken as the 3 squares preceding k=5 so as to have difference for all $k \ge 4$

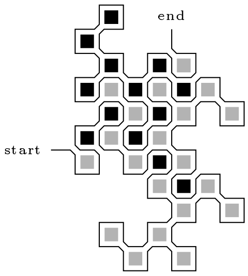
$$\mathit{BlobLQ}_k = \mathit{PreBlobLQ}_{k+1} - \mathit{PreBlobLQ}_k$$

Forms (321),(320),(322) follow from some recurrence or generating function manipulations.

The whole curve right comprises blob rights up to middle biggest k, and from the end blob lefts up to $k{-}2$ biggest end. For k=5 there are no end blobs and there is 1 fewer squares there than $\mathit{PreBlobLQ}$ would count, so -1 in that case. The whole curve left is similar, but the k=5 end is already right.

$$\begin{aligned}
\mathit{PreBlobRQ}_{k+1} + \mathit{PreBlobLQ}_{k-1} - (1 \text{ if } k{=}5) &= \mathit{RQ}_k \\
\mathit{PreBlobLQ}_{k+1} + \mathit{PreBlobRQ}_{k-1} &= \mathit{LQ}_k \\
\mathit{PreBlobBQ}_{k+1} + \mathit{PreBlobBQ}_{k-1} - (1 \text{ if } k{=}5) &= \mathit{BQ}_k
\end{aligned}$$

12.4 Blob Area



blob k=8 area

$BlobAL_8 = 20$ left, grey
$BlobAR_8 = 13$ right, black

$BlobA_8 = BlobAL_8 + BlobAR_8 = 33$

Theorem 70 (Daykin and Tucker[13]). *Blob k has area*

$$\begin{aligned} BlobA_k &= A_k - A_{k-1} - A_{k-2} + A_{k-3} \quad + - - + \quad \cdots A_0 & (323)\\ &= \quad 3BlobA_{k-1} \ - 3BlobA_{k-2} \ + 5BlobA_{k-3} & (324)\\ &\quad - 6BlobA_{k-4} \ + 2BlobA_{k-5} \ - 4BlobA_{k-6} \qquad k \geq 6 \\ &= 0, 0, 0, 0, 1, 3, 6, 14, 33, 71, 150, \ldots & \text{A003477} \end{aligned}$$

$$\textit{Generating function} \quad gBlobA(x) = \frac{x^4}{(1{-}2x)(1{+}x^2)(1{-}x{-}2x^3)} \qquad (325)$$

$$= \tfrac{1}{5}\,\frac{1}{1{-}2x} + \tfrac{1}{10}\,\frac{3{+}x}{1{+}x^2} - \tfrac{1}{2}\,\frac{1}{1{-}x{-}2x^3}$$

Proof. Per Daykin and Tucker, the total of all blob areas is the curve area

$$A_k = \sum_{j = BlobList_k} BlobA_j$$

Taking the difference of successive *BlobList*,

$$\begin{aligned} A_k - A_{k-1} &= BlobA_k + BlobA_{k-2} \\ BlobA_k &= A_k - A_{k-1} - BlobA_{k-2} & (326) \end{aligned}$$

(326) expanded repeatedly is the pattern of signs in (323). Usual manipulations on that sum give recurrence (324) and the generating function. ☐

Each enclosed unit square in a blob has all four sides traversed and satisfies the conditions of lemma 1. So blob area, boundary and segments are related

$$4\,BlobA_k + BlobB_k \ = \ 2\,BlobSegments_k$$

This allows any one to be derived from the other two if desired. *BlobB* is the slightly more difficult one due to non-blob segments in the curve total.

In (323), the repeating signs $+--+$ are the same as on B in *BlobB* at (311). Here $A_0 = A_1 = A_2 = A_3 = 0$ so terms can be taken down to A_0 or in pairs or in fours as desired and there is no extra adjustment term.

Generating function *gBlobA* is related to area *gA* by factor $\frac{1-x}{1+x^2}$ which is the descending alternating sum $+--+$ the same as for *gBlobB* at (312).

$$gBlobA(x) \ = \ \frac{1-x}{1+x^2}\, gA(x) \qquad (327)$$

Area difference $A_k - A_{k-1} = AL_k$ from (146) gives blob area as an alternating sum of every second left side area.

$$BlobA_k = AL_k \ - AL_{k-2} \ + AL_{k-4} \ - AL_{k-8} \ + \cdots \tag{328}$$

gBlobA is related to left area *gAL* in a similar way to (327), but this time factor $\frac{1-x^2}{1-x^4} = \frac{1}{1+x^2}$ for alternating sum of every second term.

$$gBlobA(x) = \frac{1}{1+x^2}\, gAL(x) \tag{329}$$

The converse, *AL* in terms of *BlobA*, is per Daykin and Tucker,

$$AL_k = BlobA_k + BlobA_{k-2}$$

This sum makes terms AL_{k-2} and below in (328) cancel, or in (329) factor $1+x^2$ moves up to the left side.

In *gBlobA* partial fractions (325), term $\frac{3+x}{1+x^2}$ is a 4-period repeating

$$3,\, 1, -3, -1, \quad 3,\, 1, -3, -1, \quad 3,\, 1, -3, -1,\, \ldots \qquad \text{A112030}$$

This and writing the cubic part as *dJA* or *JA* gives identities

$$\begin{aligned} BlobA_k &= \tfrac{1}{5}\, 2^k \ - \tfrac{1}{2}\, dJA_{k+2} \ + \tfrac{1}{10}\, [3,1,-3,-1] && k \ge 0 && (330) \\ &= \tfrac{1}{5}\, 2^k \ - \ JA_k \ - \tfrac{1}{5}\, [1,2,4,3] && k \ge 0 \end{aligned}$$

When the curve is scaled to a unit length, each enclosed square has area $1/2^k$. The limit for the middle biggest blob is the factor on its 2^k term (since *dJA* grows only a power of the cubic root r).

$$\frac{BlobA_k}{2^k} \to \frac{1}{5}$$

The blobs towards the start or end of the curve are smaller. The limit for size n levels down from the biggest is

$$\frac{BlobA_{k-n}}{2^k} \to BlobAf_n \ = \frac{1}{5} \cdot \frac{1}{2^n} \tag{331}$$

The limit for the area of the whole fractal is the total of blob areas counted down successively away from the middle. The middle is $n{=}0$ and there is just one $n{=}1$, then both start and end $n \ge 2$.

$$\begin{aligned} \frac{A_k}{2^k} \to \frac{1}{2} \ &= \ BlobAf_0 + BlobAf_1 + 2 \sum_{n=2}^{\infty} BlobAf_n && (332) \\ &= \tfrac{1}{5} \ + \tfrac{1}{2} \cdot \tfrac{1}{5} \ + 2 \cdot \tfrac{1}{5} \left(\tfrac{1}{4} + \tfrac{1}{8} + \cdots \right) \end{aligned}$$

Chang and Zhang[9] use (332) for area of the blobs in the dragon fractal as fraction of the total dragon area. Knowing blob areas decrease by factor $1/2^n$ gives the following, with solution $BlobAf_0 = \frac{2}{5} Af$.

$$Af \ = BlobAf_0 + \tfrac{1}{2} BlobAf_0 + 2 \sum_{n=2}^{\infty} \frac{1}{2^n} BlobAf_0$$

The areas enclosed on the left side and right side of a blob can be calculated separately. The sides are reckoned from blob start to end as the blob appears at the start of the full curve.

Theorem 71. *Blob k has left side area*

$$\begin{aligned} \mathit{BlobAL}_k &= AL_k - AL_{k-1} - AR_{k-2} + AR_{k-3} \ + - - + \cdots AL_0 \text{ or } AR_0 && (333)\\ &= \ 3\mathit{BlobAL}_{k-1} - 2\mathit{BlobAL}_{k-2} + 2\mathit{BlobAL}_{k-3} - 3\mathit{BlobAL}_{k-4} && (334)\\ &\quad - 3\mathit{BlobAL}_{k-5} + 2\mathit{BlobAL}_{k-6} - 2\mathit{BlobAL}_{k-7} + 4\mathit{BlobAL}_{k-8} && k \ge 8\\ &= 0,0,0,0,\ 1,2,4,9,20,41,85,176,361,736,1498,\dots \end{aligned}$$

$$\begin{aligned} \mathit{gBlobAL}(x) &= x^4\,\frac{1-x-x^3}{(1-x)\,(1+x)\,(1+x^2)\,(1-2x)\,(1-x-2x^3)}\\ &= -\tfrac{1}{8}\frac{1}{1-x} + \tfrac{1}{16}\frac{1}{1+x} + \tfrac{1}{20}\frac{3+x}{1+x^2} + \tfrac{1}{10}\frac{1}{1-2x} - \tfrac{1}{16}\frac{3-2x+2x^2}{1-x-2x^3} \end{aligned}$$

and right side area

$$\begin{aligned} \mathit{BlobAR}_k &= AR_k - AR_{k-1} - AL_{k-2} + AL_{k-3} \ + - - + \cdots AL_0 \text{ or } AR_0 && (335)\\ &= \textit{same recurrence as BlobAL, different initial values} && k \ge 8\\ &= 0,0,0,0,\ 0,1,2,5,13,30,65,142,304,639,1332,\dots \end{aligned}$$

$$\begin{aligned} \mathit{gBlobAR}(x) &= x^5\,\frac{1-x+x^2}{(1-x)\,(1+x)\,(1+x^2)\,(1-2x)\,(1-x-2x^3)}\\ &= \tfrac{1}{8}\frac{1}{1-x} - \tfrac{1}{16}\frac{1}{1+x} + \tfrac{1}{20}\frac{3+x}{1+x^2} + \tfrac{1}{10}\frac{1}{1-2x} - \tfrac{1}{16}\frac{5+2x-2x^2}{1-x-2x^3} \end{aligned}$$

Proof. The sum of blob areas is the curve area on the respective sides. The blobs at the end of the curve are an unfold so their left and right swap.

$$\begin{aligned} AL_k &= \left(\sum_{j=4}^{k} \mathit{BlobAL}_j\right) + \left(\sum_{j=4}^{k-2} \mathit{BlobAR}_j\right)\\ AR_k &= \left(\sum_{j=4}^{k} \mathit{BlobAR}_j\right) + \left(\sum_{j=4}^{k-2} \mathit{BlobAL}_j\right) \end{aligned}$$

Taking the difference of successive sums,

$$\begin{aligned} AL_k - AL_{k-1} &= \mathit{BlobAL}_k + \mathit{BlobAR}_{k-2} && k \ge 2\\ AR_k - AR_{k-1} &= \mathit{BlobAR}_k + \mathit{BlobAL}_{k-2} \end{aligned}$$

is mutual recurrences for BlobAL and BlobAR

$$\begin{aligned} \mathit{BlobAL}_k &= (AL_k - AL_{k-1}) - \mathit{BlobAR}_{k-2} && k \ge 2\\ \mathit{BlobAR}_k &= (AR_k - AR_{k-1}) - \mathit{BlobAL}_{k-2} \end{aligned}$$

Substitute each into the other for recurrences

$$\begin{aligned} \mathit{BlobAL}_k &= \mathit{BlobAL}_{k-4} + (AL_k - AL_{k-1}) - (AR_{k-2} - AR_{k-3}) && k \ge 4\\ \mathit{BlobAR}_k &= \mathit{BlobAR}_{k-4} + (AR_k - AR_{k-1}) - (AL_{k-2} - AL_{k-3}) \end{aligned}$$

which are the alternating signs and sides (333),(335). Since $AL_0 = AR_0 = \cdots = AL_3 = AR_3 = 0$ the sum can be continued down to the 0 term (either AL or AR). The recurrence and generating functions follow from those sums.

$$\begin{aligned} gBlobAL(x) &= \frac{1}{1-x^4}\Big(\, gAL(x) - x\, gAL(x) - x^2 gAR(x) + x^3 gAR(x)\,\Big) \\ gBlobAR(x) &= \frac{1}{1-x^4}\Big(\, gAR(x) - x\, gAR(x) - x^2 gAL(x) + x^3 gAL(x)\,\Big) \qquad \square \end{aligned}$$

With $AR_{k+1} - AR_k = AL_k$ from (145) the two blob sides are can be expressed with just AL if desired in a pattern of 3 terms out of each 4.

$$\begin{aligned} BlobAL_k &= AL_k - AL_{k-1} \qquad\qquad - AL_{k-3} \cdots && \text{coefficients } +1, -1, 0, -1 \\ BlobAR_k &= \qquad\quad AL_{k-1} - AL_{k-2} + AL_{k-3} \cdots && \text{coefficients } 0, +1, -1, +1 \end{aligned}$$

Or similar with AR (or likewise $A_k = AR_{k+1}$). In $BlobAL$, there is a high AR_{k+1} before the repeating part.

$$\begin{aligned} BlobAL_k &= AR_{k+1} \; - 2AR_k + AR_{k-1} - AR_{k-2} + 2AR_{k-3} \cdots \\ &\qquad\qquad \text{coefficients } -2,\ +1,\ -1,\ +2 \\ BlobAR_k &= \qquad\qquad AR_k - 2AR_{k-1} + 2AR_{k-2} - AR_{k-3} \cdots \\ &\qquad\qquad \text{coefficients } +1,\ -2,\ +2,\ -1 \end{aligned}$$

Some recurrence manipulation gives $BlobAL$ and $BlobAR$ in terms of a power of 2 similar to (330).

$$\begin{aligned} BlobAL_k &= \begin{cases} 1 & \text{if } k{=}4 \\ \frac{1}{10}2^k - \frac{1}{10}[1,-3,4,3] - \frac{1}{2}BlobLQ_k & \text{if } k \ge 5 \end{cases} \\ BlobAR_k &= \ \frac{1}{10}2^k + \frac{1}{10}[4,3,1,-3] - \frac{1}{2}BlobRQ_k \end{aligned}$$

The limits $BlobAL/2^k \to BlobAR/2^k \to \frac{1}{10}$ are the same. $BlobAL$ is always bigger, with difference

$$\begin{aligned} BlobAL - BlobAR &= \begin{cases} 1 & \text{if } k = 4 \,\text{or}\, 5 \\ BlobLQ_{k-1} - [0,0,1,0] & \text{if } k \ge 6 \end{cases} \\ &= 0,0,0,0,\ 1,1,2,4,7,11,20,34,57,97,166,\ldots \end{aligned}$$

12.4.1 Blob Area by Join

Blob area can also be calculated by considering how the join area JA and the blobs on each side of it combine to make a new middle biggest blob.

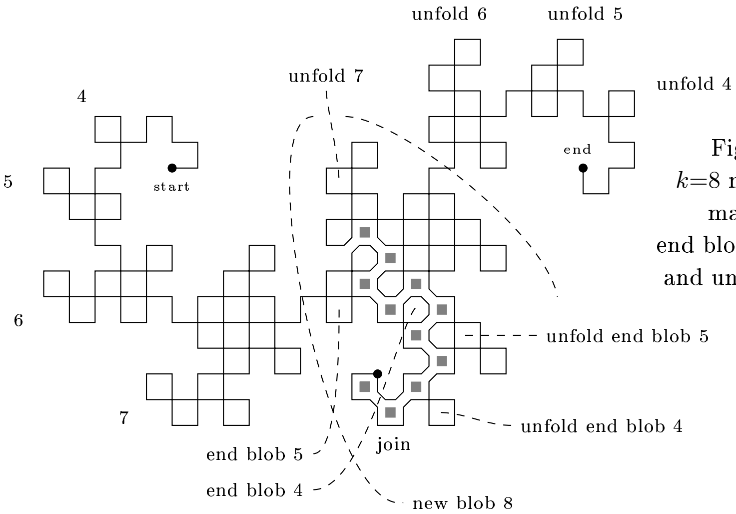


Figure 63:
k=8 middle blob
made from
end blobs, join area,
and unfolded blobs

Theorem 72. *On unfolding curve $k-1$ to make k, the new middle biggest blob k is formed by the join area connecting end blobs and unfolded copies of the end blobs and middle biggest blob so that*

$$\begin{aligned} BlobA_k &= 2\left(\sum_{j=0}^{k-3} BlobA_j\right) + BlobA_{k-1} + JA_{k-1} && k \geq 1 \\ BlobAL_k &= \left(\sum_{j=0}^{k-3} BlobA_j\right) + BlobAR_{k-1} + JA_{k-1} \\ BlobAR_k &= \left(\sum_{j=0}^{k-3} BlobA_j\right) + BlobAL_{k-1} \end{aligned}$$

Proof. The unit squares of the join area are connected at their corners since it is a contiguous part of the left boundary squares. All the blobs which touch the join area combine to a single new blob. In curve level k, this must be the new middle biggest blob k because the join area is all the touching for the unfold and there is no other way to make a new bigger blob k.

Some of the end of $BlobList_{k-1}$ and its unfold reversal make the new blob k. To form $BlobList_k$ there must be $4,\ldots,k-1$ before the join and there must be $k-2,\ldots,4$ after the join.

$$\begin{array}{l} BlobList_{k-1} \\ \text{unfold reversal} \end{array} \quad \underbrace{\begin{array}{r} 4,5,\ldots,k-3,\ k-2,k-1, \\ \end{array} \overset{\text{new blob } k}{\boxed{\begin{array}{l} k-3,\ldots,5,4 \\ 4,5,\ldots,k-3,k-1, \end{array}}} \begin{array}{l} \\ k-2,\ k-3,\ldots,5,4 \end{array}}_{BlobList_k}$$

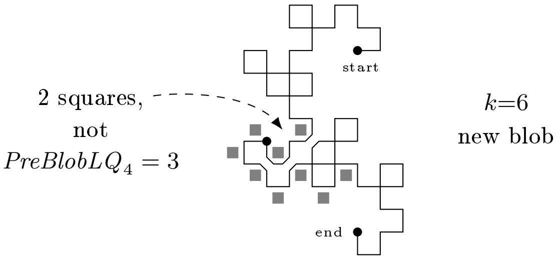
The blob sizes in the dashed box are combined for the new biggest blob and are per the sum in the theorem.

For the left and right side areas, unfolded blobs have their left and right sides swapped. Sizes 4 to $k{-}3$ are in pairs so are the BlobA_j sum. The $k{-}1$ blob from the unfold is reversed so its left goes to the new right and vice versa. ☐

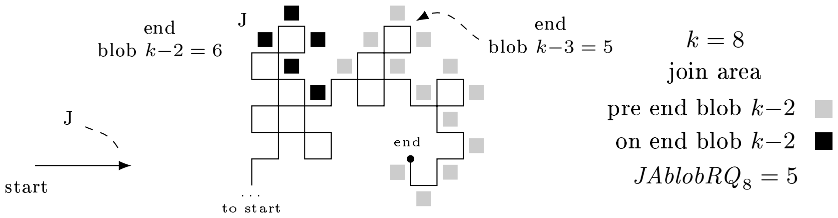
This construction gives an identity for *BlobRQ*. On the curve right side, the middle unfold towards start is left sides of blobs up to $k{-}3$. Middle towards end is right sides also of blobs up to $k{-}3$, plus left side of the $k{-}1$ which had been middle biggest in the second half. So

$$\mathit{BlobRQ}_k = \mathit{PreBlobBQ}_{k-2} + \mathit{BlobLQ}_{k-1} - (1 \text{ if } k{=}6) \qquad k \geq 6$$

For $k{=}6$, the middle towards start curve end lefts are only 2 squares, so -1 in that case.



The enclosed *JA* squares shown in figure 63 go part way around the last end blob of the curve first half.



Numerically, which blob the join extends to can be found by recurrence manipulations falling $\mathit{PreBlobRQ}_{k-2} < \mathit{JA}_k < \mathit{PreBlobRQ}_{k-1}$.

The amount of join on end blob $k{-}2$ is by subtracting the preceding blobs. Since they are end blobs, this is their right sides.

$$\begin{aligned}\mathit{JAblobRQ}_k &= \mathit{JA}_k - \mathit{PreBlobRQ}_{k-2}\\ &= 2, 3, 5, 9, 17, 28, 48, 82, 140, 237, \ldots \qquad k \geq 6\end{aligned}$$

For $k{=}6$, there is no $k{-}3$ blob to go upto, but the $\mathit{PreBlobRQ}_4 = 4$ squares preceding blob $k{-}2 = 4$ is as desired.

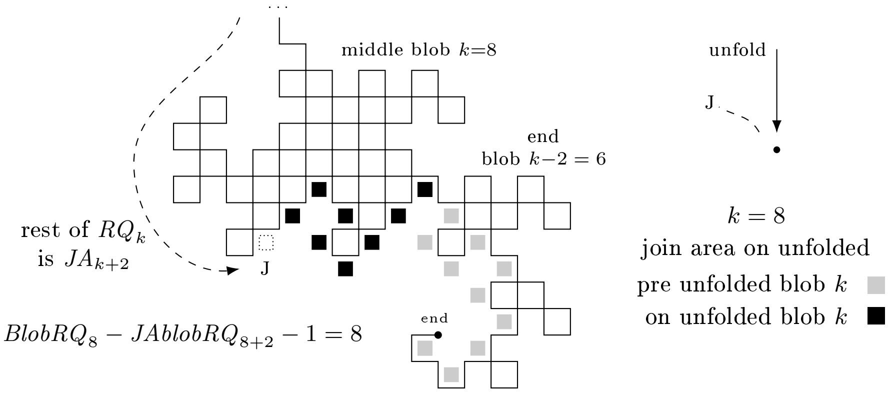
A limit proportion of *JAblobRQ* within its *BlobRQ* follows by some recurrence manipulations for an identity

$$\mathit{JAblobRQ}_k = \tfrac{1}{14}\Big(4\mathit{BlobRQ}_{k-2} - 2\mathit{BlobRQ}_{k-1} + 3\mathit{BlobRQ}_k - [14, 11, 5, 5]\Big)$$

ready to divide, for limit proportion a little under $\frac{2}{3}$,

$$\frac{\mathit{JAblobRQ}_k}{\mathit{BlobRQ}_{k-2}} \to \tfrac{1}{14}(4 - 2r + 3r^2) = 0.659581\ldots$$

The join area is also some of the unfolded curve boundary squares. This is its right side and the join extends to some of its middle biggest blob.



From theorem 29, the curve right squares comprise two join areas and middle join square (dotted J). JA_k is the black and grey squares, and the rest is JA_{k+2}.

But $JAblobRQ_{k+2}$ is JA_{k+2} on the right of blob $BlobRQ_k$, starting at its blob start. So the part of unfold blob k boundary squares which are JA_k is difference

$$BlobRQ_k - JAblobRQ_{k+2} - 1 \;= 1, 1, 3, 5, 8, 14, 26, 42, 71, 123, \dots \quad k{\geq}4$$

12.5 Blob Sub-Parts

Ngai and Nguyen [38] show that a blob in the dragon fractal comprises 3 sub-blobs and a twindragon. Their argument holds in finite iterations too.

Theorem 73 (Finite form of Ngai and Nguyen)**.** *Blob $k \geq 8$ can be subdivided into four parts: a twindragon $k-4$, a blob $k-2$, and two blobs $k-4$.*

Proof. The parts are seen in blob k=8,

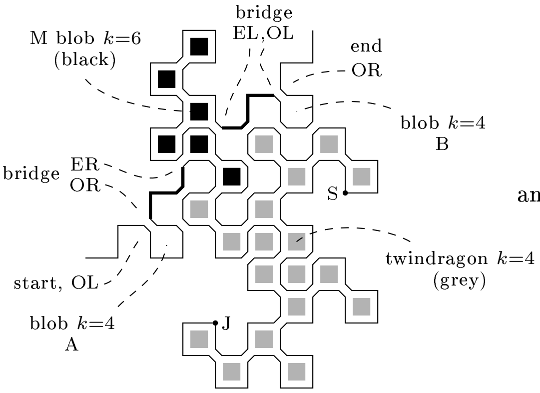


Figure 64:
Blob k=8
comprising
3 sub-blobs
and twindragon

The unit square A at the start and B at the end are blobs k=4. They both have bridge ends OL before and OR after which is how they appear at the start of the full curve (but rotated 180°), so they expand as subsequent blob levels.

Blob k=6 in the middle has ER before and EL after. This is how it appears in the end part of the full curve (and is oriented vertically the same as it appears there too, per the top of figure 59).

As a remark, M blob k=6 and A blob k=4 are as they appear as middle biggest and first end blob in curve k=6, as for example in figure 1.

The twindragon is an excursion part-way along the k=6 blob. The twindragon is a closed loop so it returns to the same point.

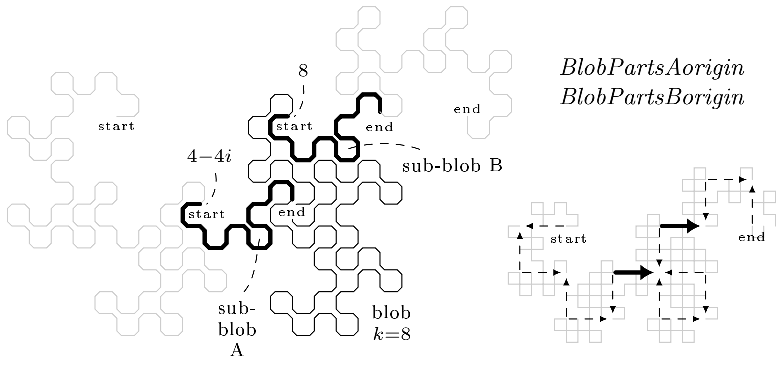
When all the segments expand in the next level, the division into these respective parts and bridges between is maintained. ☐

In figure 64, point S at $12-4i$ is the start of the twindragon in its usual orientation (first segment East). Its end marked J is the curve join where two k=7 curves unfold. On each expansion these points grow by factor b so

$$\begin{aligned} \mathit{BlobPartsTstart}_k &= (\tfrac{3}{4}-\tfrac{1}{4}i)\, b^k \\ \mathit{BlobPartsTend}_k &= (\tfrac{1}{2}-\tfrac{1}{2}i)\, b^k \end{aligned}$$

The excursion off the $k-2$ blob is at a twindragon corner. The twindragon is the enclosed square of sub-curves for the join area in figure 25.

The $k-4$ sub-blobs A, B are middle biggest blobs of $k-4$ sub-curves,



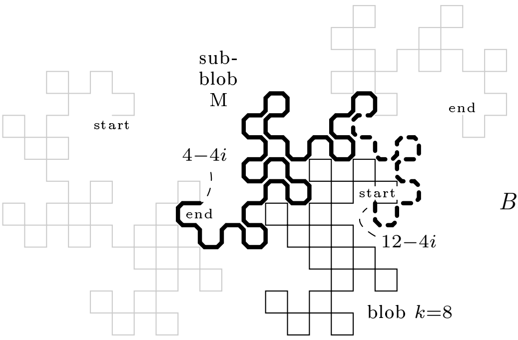
The origin locations of these sub-curves are $4-4i$ and 8 respectively in k=8 and on each expansion grow by factor b so

$$\begin{aligned} \mathit{BlobPartsAorigin}_k &= (\tfrac{1}{4}-\tfrac{1}{4}i)\, b^k \qquad (336) \\ \mathit{BlobPartsBorigin}_k &= \tfrac{1}{2}\, b^k \end{aligned}$$

These blobs are turned 180°. The start segment of each is directed West, rather than the normal curve start East. This means factor -1 to rotate when taking a location in these sub-blobs from their origins. For example, the start of blob A is the start of the whole blob,

$$\mathit{BlobPartsAorigin}_k + (-1).\mathit{BlobStart}_{k-4} \;= \mathit{BlobStart}_k$$

The $k-2$ sub-blob M also occurs as the middle biggest blob of a sub-curve, but its initial segments are not part of the curve as such. That absent part is shown dashed, then the sub-curve is the two $k-4$ sub-blob curves and further sub-curve between.



BlobPartsMorigin

The start is at $12-4i$ in $k{=}8$ and on each expansion grows by factor b so

$$BlobPartsMorigin_k = (\tfrac{3}{4}-\tfrac{1}{4}i)\,b^k \qquad (337)$$

This sub-curve is turned $-90°$ from its normal direction, as can be seen by start segment South, rather than the normal East. This means factor $-i$ to rotate when taking a location from this origin.

The two $k{-}4$ blob A, B crossings go in the same direction as blob k. The $k{-}2$ blob M is the opposite direction. The A, B unit squares in figure 64 don't make their direction particularly clear, though it follows from the bridge types. In blob $k{=}9$, the shapes $k{=}5$ show their orientations,

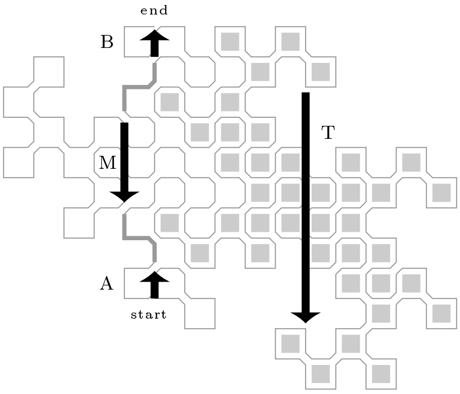


Figure 65:
blob $k{=}9$
comprising
3 sub-blobs
and twindragon

The parts give a recurrence for *BlobSegments*. Twindragon $k{-}4$ has 2^{k-2} segments and the two bridges are 6 segments so

$$BlobSegments_k = BlobSegments_{k-2} + 2\,BlobSegments_{k-4} + 2^{k-2} + 6 \qquad k{\ge}8$$

The parts also give a recurrence for blob area,

$$\begin{aligned} BlobA_k &= BlobA_{k-2} + 2\,BlobA_{k-4} + TA_{k-4} && k{\ge}8 \\ &\quad + 3LQ_{k-8} + 4RQ_{k-8} + JA_{k-8} + JA_{k-7} \end{aligned}$$

The gap between the twindragon and the blobs (as seen in $k{=}8$ figure 64) comprise 3 curve left sides and 4 curve right sides, so are those respective boundary squares. The end of the gap is an odd point with dragons meeting so join JA_{k-8}. The start of the gap is an even point with dragons pointing away so JA_{k-7} (the opposing side is a left so does not touch).

With $JA_{k-8} + JA_{k-7} = LQ_{k-8} - 1$ from theorem 29, there are 4 lefts and 4 rights so simplify to BQ.

$$BlobA_k = BlobA_{k-2} + 2BlobA_{k-4} \;\; + TA_{k-4} + 4BQ_{k-8} - 1 \qquad k{\ge}8 \tag{338}$$

Or taking out the power-of-2 part of TA,

$$BlobA_k = BlobA_{k-2} + 2BlobA_{k-4} \;\; + 2^{k-3} - dJA_{k-1} \qquad k{\ge}4 \tag{339}$$

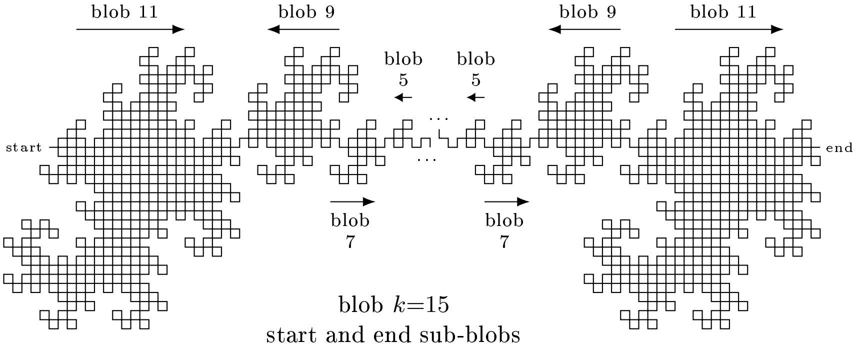
The breakdown also gives left and right side blob area. The middle $k{-}2$ blob is reversed so its left and right sides swap. The twindragon is traversed anti-clockwise so its inside is the left of the curve and is area 2^{k-4}.

The joins between the blobs and twindragon are all on the right side of the curve. The outside area of the twindragon is $2A_{k-4}$ per (256). The same simplifications as above apply to the join parts, or just subtract the left (340) from the whole (339).

$$BlobAL_k = BlobAR_{k-2} + 2BlobAL_{k-4} + 2^{k-4} \tag{340}$$

$$\begin{aligned} BlobAR_k &= BlobAL_{k-2} + 2BlobAR_{k-4} + 2A_{k-4} \\ &\qquad + 3LQ_{k-8} + 4RQ_{k-8} + JA_{k-8} + JA_{k-7} \\ &= BlobAL_{k-2} + 2BlobAR_{k-4} + 2^{k-4} - dJA_{k-1} \end{aligned}$$

The breakdown has sub-blobs $k{-}4$ at start and end, and a middle $k{-}2$. That middle then has $k{-}2{-}4 = k{-}6$ at start and end. Repeating this is sub-blobs of sizes $k{-}4, k{-}6, k{-}8, \ldots$, with bridges between, through to the middle of the blob. The bridges alternate which side they bend so that the sub-blob crossings are either on or 1 away from the whole blob crossing. For example,



blob k=15
start and end sub-blobs
leading to middle

At the middle, the curve goes down for more curve below, then through the middle gap up to curve above.

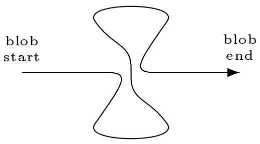
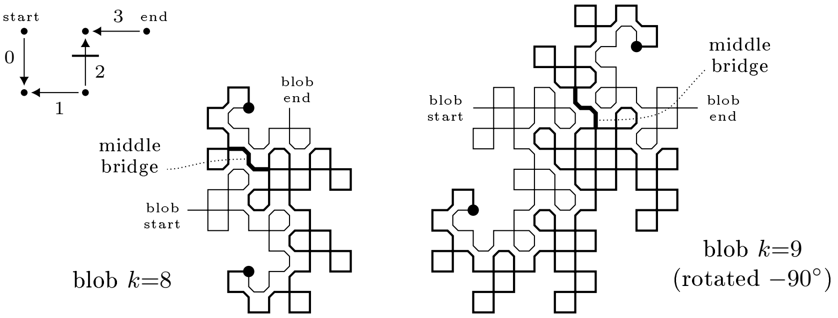


Figure 66

The bridge locations and blob crossing line locations show this middle of the blob crossing line is the bridge after the biggest blob of sub-curve 2 of level $k-2$.



The limit for the dragon fractal is

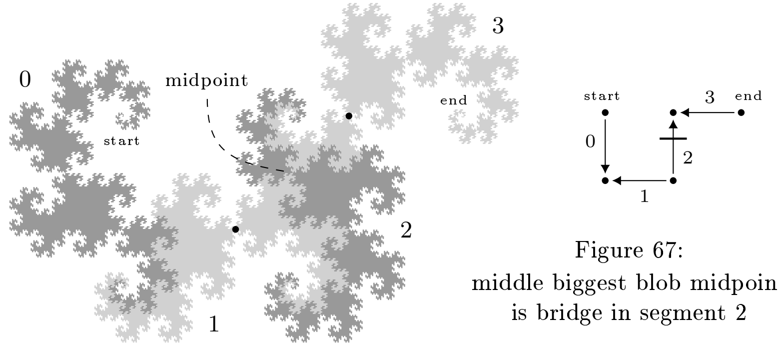


Figure 67:
middle biggest blob midpoint
is bridge in segment 2

Gosper's illustration `a260482.png` in OEIS A260482[23] is a high resolution image with the curve coloured according to how it falls between visits to this midpoint. There are 3 visits (15ths fractions) and they divide the curve into 4 parts.

In finite iterations, let the point numbers n for the two points in the middle bridge be *MidMNs* for the start side and *MidMNe* for the end side. Let *MidSN* and *MidEN* be their other visits respectively, being when the curve reaches there from curve start, or leaves to go to curve end, respectively.

These points are *BridgeLN* etc from theorem 63 in sub-curve 2. *BridgeLN* etc are before biggest blob so level +1 for after, and then -2 for sub-curve here. Left side *BridgeLN* is the blob start side, and *BridgeRN* is the blob end side.

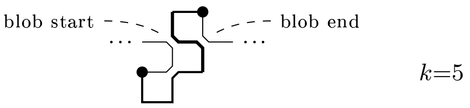
$$\begin{aligned}
\mathit{MidMNs}_k &= 2^{k-1} - \mathit{BridgeLN}_{k-1} && k \geq 5\\
&= 2^{k-1} + \mathit{BlobN}_{k-1} - [2,1]\\
&= \tfrac{7}{10}2^k + \tfrac{1}{5}[-1,-2,-4,7]\\
&= 22, 44, 91, 179, 358, 716, 1435, \ldots\\
\mathit{MidSN}_k &= \mathit{other}\big(\mathit{MidMNs}_k\big)\\
&= 2^{k-1} - \mathit{BridgeLother}_{k-1}\\
&= \tfrac{13}{30}2^k + \tfrac{1}{15}[1,2,4,-7]\\
&= 14, 28, 55, 111, 222, 444, 887, \ldots
\end{aligned}$$

$$\mathit{MidMNe}_k = 2^{k-1} + \mathit{BridgeRN}_{k-1}$$

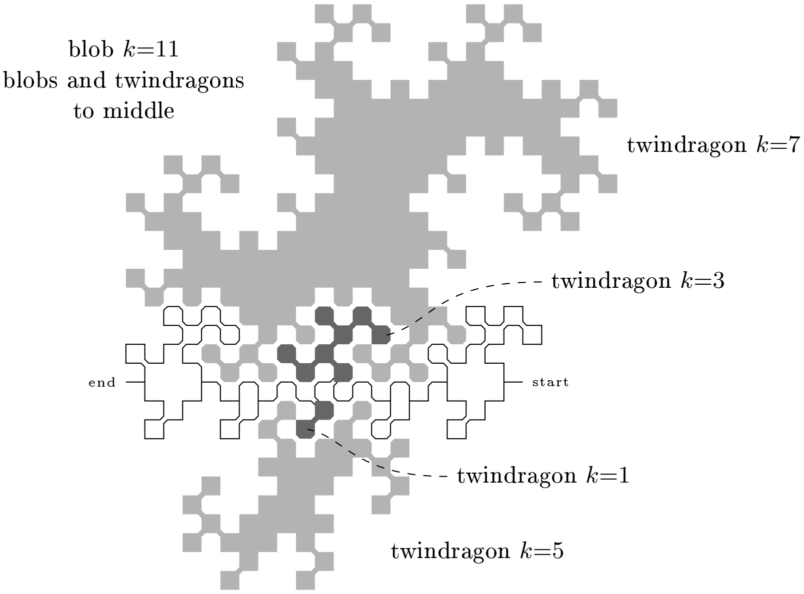
$$
\begin{aligned}
&= 2^{k-1} + \mathit{BlobN}_{k-1} - [1,2] \\
&= \tfrac{7}{10}\,2^k + \tfrac{1}{5}\,[4,-7,1,2] \\
&= 21, 45, 90, 180, 357, 717, 1434, \ldots \\
\mathit{MidEN}_k &= \mathit{other}\big(\mathit{MidMNe}_k\big) \\
&= 3.2^{k-1} - \mathit{BridgeRother}_{k-1} \\
&= \tfrac{23}{30}\,2^k + \tfrac{1}{15}\,[-4,7,-1,-2] \\
&= 25, 49, 98, 196, 393, 785, 1570, \ldots
\end{aligned}
$$

In successive levels, the bridge after middle biggest alternates in bending left or right, so *MidMNs* and *MidMNe* alternate in which is the bigger. The curve always goes to the right first and then up and through as shown in figure 66, simply since that is where sub-curve 2 goes.

k=5 is the first blob with a middle bridge, ie. with a non-zero crossing distance. The formulas above use a "bridge" imagined preceding $k-1=4$. The 3 segments there are not a bridge to another blob as such, but do follow the general pattern and suit here. The "2" sub-part is only $k-2=3$ and has no blobs or bridges at all as such, but there too the 3 segments of the general case still suit.

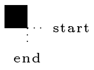


Successive middle blobs put a twindragon on alternate sides, starting from $k-4$ on the right. So twindragons $k-4, k-8, \ldots$ on the right and $k-6, k-10, \ldots$ on the left.

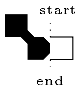


These twindragons stop at the sub-blob bridges, and they are on or +1 from the whole blob crossing line, hence their bases making a straight line across. There are bites into them by the bigger blobs, and in all cases they mesh perfectly.

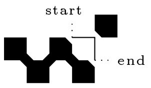
The breakdown can be applied repeatedly to all the sub-blobs to reduce entirely to a set of twindragons. Blobs $k = 4, 5$ are taken to be twindragons $k{-}4 = 0, 1$ and no sub-blobs. Blobs $k = 6, 7$ are taken as twindragons $k{-}4 = 2, 3$ and middle blob $k{-}2 = 4, 5$ which in turn contain twindragons $0, 1$.



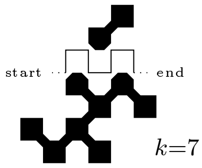
$k{=}4$



$k{=}5$



$k{=}6$



$k{=}7$

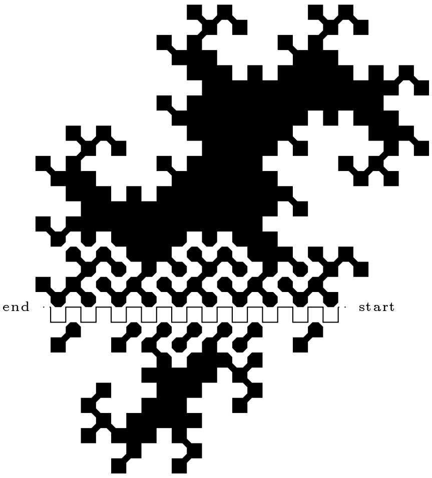


Figure 68:
blob $k{=}11$
twindragons
area
$\mathit{BlobTA}_{11} = 258$

The area due to these twindragons is a recurrence in TA.

$$\mathit{BlobTA}_k = \begin{cases} 0 & \text{if } k < 4 \\ \mathit{BlobTA}_{k-2} + 2\mathit{BlobTA}_{k-4} + \mathit{TA}_{k-4} & \text{if } k \ge 4 \end{cases} \tag{341}$$

$$= \mathit{BlobA}_k - 2RQ_{k-4} + [\tfrac{2}{3}, \tfrac{5}{6}].2^{\lfloor k/2 \rfloor} - \tfrac{1}{3}[2, 1, 1, 2] \qquad k{\ge}4 \tag{342}$$

$$= 0, 0, 0, 0, 1, 2, 5, 10, 25, 54, 121, 258, 561, 1182, \ldots$$

This is less than BlobA area by the amount of the gaps between the twindragons and at start and end. Recurrence or generating function manipulations give (342) showing total gap growing as $2RQ$. The half power $2^{\lfloor k/2 \rfloor}$ grows only as $\sqrt{2}$ which is less than RQ growing as r (98).

$$\mathit{BlobA}_k{-}\mathit{BlobTA}_k = 2RQ_{k-4} - [\tfrac{2}{3}, \tfrac{5}{6}].2^{\lfloor k/2 \rfloor} + \tfrac{1}{3}[2, 1, 1, 2] \qquad k \ge 4$$
$$= 0, 0, 0, 0, 0, 1, 1, 4, 8, 17, 29, 60, 104, 193, \ldots$$

The generating function for BlobTA is

$$g\mathit{BlobTA}(x) = \frac{x^4\,(1 - 2x + x^2 - 2x^3 + 4x^4)}{(1{-}x)(1{-}2x)(1{+}x^2)(1{-}2x^2)(1{-}x - 2x^3)}$$

This is $g\mathit{TA}$ with a factor for the BlobTA terms of (341),

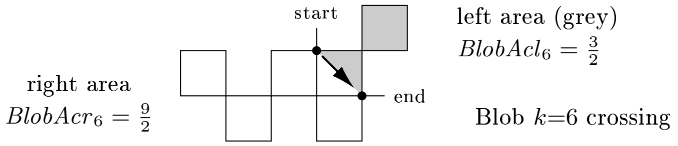
$$gBlobTA(x) = \frac{x^4}{1 - x^2 - 2x^4}\, gTA(x) \;\; = \frac{x^4}{(1{+}x^2)\,(1{-}2x^2)}\, gTA(x)$$

The total number of twindragons in this recursion is the same recurrence but counting 1 for each twindragon.

$$\begin{aligned} BlobTnum_k &= \begin{cases} 0 & \text{if } k < 4 \\ BlobTnum_{k-2} + 2BlobTnum_{k-4} + 1 & \text{if } k \geq 4 \end{cases} \\ &= \left\lfloor \tfrac{1}{3}.2^{\lfloor k/2 \rfloor} \right\rfloor \\ &= 0,0,0,0,1,1,2,2,5,5,10,10,21,21,\ldots && \scriptstyle k\geq 1 \text{ 2-rep A000975} \\ &= \text{binary } 101010\ldots && \scriptstyle k\geq 1 \text{ 2-rep A056830} \end{aligned}$$

12.6 Area Left and Right of Crossings

The blob crossings of section 12.1 divide each blob into a left and right half.



Theorem 74. *The enclosed area of blob k on the left and right of its crossing line are*

$$BlobAcl_k = \begin{cases} 0 & \textit{if } k \leq 4, \quad \textit{and otherwise} \\ \frac{2}{45}2^k + \frac{1}{9}[1{,}1{,}2{,}2].2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{90}[-59{,}17{,}-101{,}-67] - \frac{1}{2}BlobLQ_k \end{cases} \tag{343}$$

$$= 0,0,0,0,\; 0,1,\tfrac{3}{2},5,7,16,\tfrac{69}{2},76,140,295,\tfrac{1231}{2},1275,\ldots$$

$$BlobAcr_k = \begin{cases} 0 & \textit{if } k \leq 3, \quad \textit{and otherwise} \\ \frac{7}{45}2^k + \frac{1}{9}[1,1,2,2].2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{90}[86,37,74,13] - \frac{1}{2}BlobRQ_k \end{cases} \tag{344}$$

$$= 0,0,0,0,\; 1,2,\tfrac{9}{2},9,26,55,\tfrac{231}{2},242,525,1080,\tfrac{4429}{2},4523,\ldots$$

$$BlobA_k = BlobAcl_k + BlobAcr_k \qquad \textit{whole blob area} \tag{345}$$

Proof. Using the blob parts of section 12.5, for k odd the blob crossing line goes straight along segments. The two sub-blobs $k{-}4$ are on the crossing line so their left sides are the left of k. Sub-blob $k{-}2$ is 180° so its right side is on the left of k, and it is offset by one segment of the bridges. This offset is 1 unit square for each *BlobDeltaLen* plus 1 on the bridges at each end. (Eg. per figure 65.)

For k even likewise, but the offset for sub-blob $k{-}2$ is two half-squares for each *BlobDeltaLen* plus $\frac{3}{2}$ for the ends.

$$BlobAcl_k = 2BlobAcl_{k-4} + BlobAcr_{k-2} + BlobDeltaLen_{k-2} + [\tfrac{3}{2}, 2] \tag{346}$$

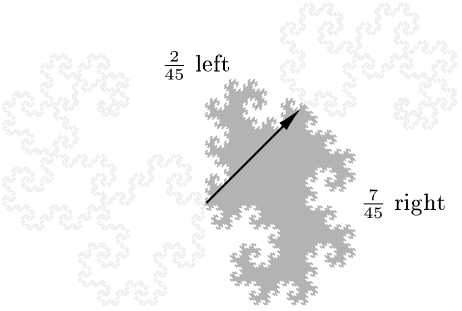
A similar recurrence applies for right side *BlobAcr*, but with the twindragon on the right side too, and area between it and the blobs as from (338). Going by difference from whole area *BlobA* per (345) is also possible.

$$\begin{aligned} \mathit{BlobAcr}_k &= 2\mathit{BlobAcr}_{k-4} + \mathit{BlobAcl}_{k-2} - \mathit{BlobDeltaLen}_{k-2} - [\tfrac{3}{2}, 2] \\ &\quad + \mathit{TA}_{k-4} + 4\mathit{BQ}_{k-8} - 1 \qquad\qquad \text{for } k \ge 8 \end{aligned} \tag{347}$$

Recurrence or generating function manipulations give (343),(344). *BlobLQ* and *BlobRQ* are convenient ways to express the cubics which arise. ☐

When k odd the crossing line is along segments so *BlobAcl* and *BlobAcr* are whole unit squares. When k even the line divides squares diagonally, giving half integers when *BlobDeltaLen* is odd, so *BlobAcl* and *BlobAcr* half integers for $k \equiv 2 \bmod 4$.

When scaled to curve start to end a unit length, limits for the middle biggest blob are coefficients of the 2^k terms



blob area limits
left and right
of crossing line

$$\frac{\mathit{BlobAcl}_k}{2^k} \to \frac{2}{45} \qquad \frac{\mathit{BlobAcr}_k}{2^k} \to \frac{7}{45}$$

Recurrences (346),(347) become for the limits

$$\text{left } \tfrac{2}{45} = 2.\tfrac{1}{16}.\tfrac{2}{45} + \tfrac{1}{4}.\tfrac{7}{45} \qquad \text{right } \tfrac{7}{45} = 2.\tfrac{1}{16}.\tfrac{7}{45} + \tfrac{1}{4}.\tfrac{2}{45} + \tfrac{1}{8}$$

The sub-blob areas scale by $\frac{1}{4}$ for blob $k-2$ and $\frac{1}{16}$ for blobs $k-4$. The twindragon is unit area at unit length and here is length $\frac{1}{4}\sqrt{2}$ so area $\frac{1}{8}$. These relations are also a way to find the proportions of left and right area without working through the exact calculation since they must satisfy these scale equations (and total $\mathit{BlobAf}_0 = \frac{1}{5}$).

Theorem 75. *The area left and right of the crossing lines of the whole curve are*

$$\begin{aligned} \mathit{Acl}_k &= \begin{cases} 0 & \text{if } k \le 3 \\ \frac{1}{6}2^k + \frac{1}{3}[1,1,1,2].2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{6}[6,4,5,7] - \frac{1}{2}\mathit{LQ}_k & \text{if } k \ge 4 \end{cases} \\ &= 0,0,0,0,\ 0,1,\tfrac{7}{2},\tfrac{21}{2},22,47,\tfrac{215}{2},\tfrac{477}{2},494,1031,\ldots \\ \mathit{Acr}_k &= \begin{cases} 0 & \text{if } k \le 3 \\ \frac{1}{3}2^k - \frac{1}{3}[1,1,1,2].2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{6}[9,7,8,10] - \frac{1}{2}\mathit{RQ}_k & \text{if } k \ge 4 \end{cases} \\ &= 0,0,0,0,\ 1,3,\tfrac{15}{2},\tfrac{35}{2},45,105,\tfrac{455}{2},\tfrac{971}{2},1045,2201,\ldots \\ A_k &= \mathit{Acl}_k + \mathit{Acr}_k \qquad \text{total area} \end{aligned}$$

Proof. The left and right side areas are sums of the blobs at start and end of the curve. The blobs at the end of the curve are directed towards the middle so their left and right sides swap.

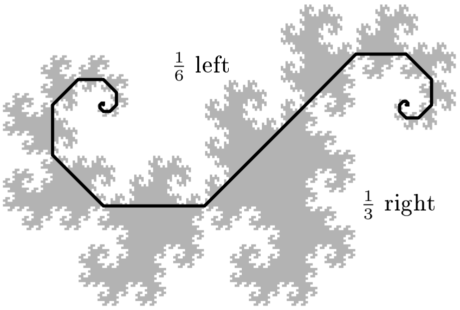
$$\begin{aligned} Acl_k &= \sum_{j=0}^{k} BlobAcl_j + \sum_{j=0}^{k-2} BlobAcr_j \\ Acr_k &= \sum_{j=0}^{k} BlobAcr_j + \sum_{j=0}^{k-2} BlobAcl_j \end{aligned}$$

The sums up to $k{-}2$ have pairs of left and right sides balancing to be $\frac{1}{2}A_k$ half total curve area. The middle biggest blob k and preceding $k{-}1$ are not balanced that way but an offset can be applied to push from half area to left or right area.

$$\begin{aligned} Acl_k &= \tfrac{1}{2}A_k - \tfrac{1}{2}\big(BlobAcdiff_k + BlobAcdiff_{k-1}\big) \\ Acr_k &= \tfrac{1}{2}A_k + \tfrac{1}{2}\big(BlobAcdiff_k + BlobAcdiff_{k-1}\big) \\ BlobAcdiff_k &= BlobAcr_k - BlobAcl_k \\ &= \begin{cases} 0 & \text{if } k \le 3, \\ 1 & \text{if } k = 4 \text{ or } 5, \quad \text{and otherwise} \\ \frac{1}{9}2^k - \frac{1}{9}[2,4,4,8].2^{\lfloor k/2 \rfloor} + \frac{1}{9}[10,2,22,8] - BlobLQ_{k-1} \end{cases} \\ &= 0,0,0,0,\ 1,1,3,4,19,39,81,166,385,785,\ldots \end{aligned}$$

☐

$BlobAcdiff_k$ is an integer since the half squares of a diagonal crossing go the same to each side. Its limit $\frac{1}{9}$ is $BlobAcr - BlobAcl$ limits $\frac{7}{45} - \frac{2}{45} = \frac{1}{9}$.



Area limits
left and right
of crossing line

$$\frac{Acl_k}{2^k} \to \frac{1}{6}$$
$$\frac{Acr_k}{2^k} \to \frac{1}{3}$$

When a new middle biggest blob is formed at the join when the curve unfolds per theorem 72, some area is on the outside of the component curves.

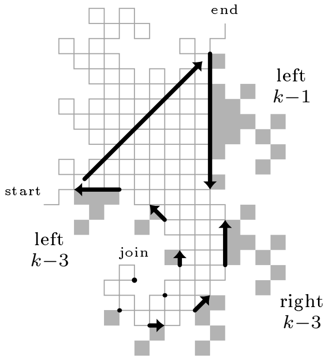


Figure 69:
Blob k=10
crossing area outside
of unfolding sub-curves
$AunfOut_{10} = 40$

The sides from join to the blob start are the blobs at the end of the first sub-curve. They are blob left sides 4 to $k{-}3$.

The sides from join to the blob end are the blobs at the end of the unfolded second sub-curve. They are blob right sides 4 to $k{-}3$, plus a left side $k{-}1$ (the middle biggest blob of the unfolded copy).

Together all these are a whole curve left Acl_{k-1} except for missing $k{-}2$ blob left,

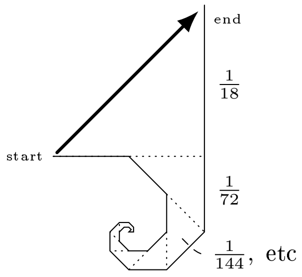
$$\begin{aligned} AunfOut_k &= Acl_{k-1} - BlobAcl_{k-2} \qquad\qquad \text{outside} \\ &= \begin{cases} 0 \quad \text{if } k = 0 \,\text{or}\, 2, \quad \text{and otherwise} \\ \frac{13}{180}2^k + \frac{1}{9}[2,1,1,2].2^{\lfloor k/2 \rfloor} - \frac{1}{90}[4,23,1,92] - \frac{1}{2}BlobRQ_k \end{cases} \\ &= 0,0,0,0,\ 0,0,1,\tfrac{5}{2},9,17,40,\tfrac{183}{2},204,418,\ldots \end{aligned}$$

The area inside these joins, and on the right side of the new blob crossing, is then by difference

$$\begin{aligned} AunfIn_k &= BlobAcr_k - AunfIn_k \qquad\qquad \text{inside} \\ &= \tfrac{1}{12}2^k - \tfrac{1}{3}[1,1,1,2].2^{\lfloor k/2 \rfloor} + \tfrac{1}{6}[6,4,5,7] \\ &= 0,0,0,0,\ 1,2,\tfrac{7}{2},\tfrac{13}{2},17,38,\tfrac{151}{2},\tfrac{301}{2},321,662,\ldots \end{aligned}$$

Taking the inside as everything not outside means some of the JA join area between the two sub-curves is included, though only what is on the right of the crossing line, not the whole JA. This is most noticeable at the sub-curve ends (marked "join" in figure 69).

A geometric interpretation of inside limit $AunfIn_k/2^k \to \frac{1}{12}$ is to take the inside area as triangles.

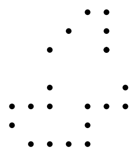


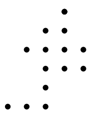
$AunfIn$ limit triangles
$\frac{1}{18} + \frac{1}{72}(1{+}\frac{1}{2}{+}\cdots) = \frac{1}{12}$

The big triangle with hypotenuse start to end and side as the $k{-}1$ crossing is area $\frac{1}{18}$. Further triangles are made by noting that the crossings are at successive 45° angles and so a crossing in the first sub-curve extends back to the end of a crossing in the unfolded second sub-curve, giving a spiral of triangles successively halving in area. The extensions back are shown dotted.

12.7 Blob Points

Blob $k{=}7$
single-visited
points
$BlobS_7 = 20$





Blob $k{=}7$
double-visited
points
$BlobD_7 = 14$
(same as area)

Theorem 76. *The number of single-visited points in blob k is*

$$\begin{aligned} BlobS_k &= \begin{cases} 0 & \text{if } k < 4 \\ \tfrac{1}{2} BlobB_k + 1 & \text{if } k \geq 4 \end{cases} \\ &= \text{same recurrence as } BlobB \ (307) \qquad k \geq 9 \\ &= 0,0,0,0,\ 3,6,11,20,35,62,107,180,307,526,\ldots \end{aligned}$$

$$\text{Generating function} \quad gBlobS(x) = x^4 \, \frac{3\ + 5x^2 - 2x^3 + 2x^4}{(1{-}x{-}2x^3)\,(1{+}x^2)\,(1{-}x)}$$

$$= 1 + x^2 - \frac{2}{1{-}x} + \frac{x}{1{+}x^2} + \frac{1}{1{-}x{-}2x^3}$$

The number of double-visited points is

$$BlobD_k = BlobA_k$$

Proof. Each blob traverses all four sides of each enclosed unit square since it is a run of connected unit squares of the full curve. A blob is a continuous run of segments so lemma 2 applies for $S = \frac{1}{2}B + 1$ and $D = A$. □

The recurrence for *BlobS* is the same as *BlobB* but for *BlobS* it can start at k=9 rather than k=10 for *BlobB*.

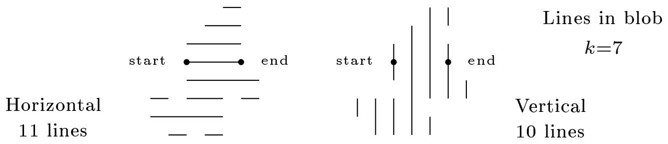
The number of distinct visited points in blob k is

$$\begin{aligned} BlobP_k &= BlobS_k + BlobD_k \\ &= 0,0,0,0,\ 4,9,17,34,68,133,257,498,972,\ldots \end{aligned}$$

$$gBlobP(x) = x^4 \, \frac{4 - 7x + 5x^2 - 12x^3 + 6x^4 - 4x^5}{(1{-}x)\,(1{+}x^2)\,(1{-}2x)\,(1{-}x{-}2x^3)}$$

$$= 1 + x^2 \ - \frac{2}{1{-}x} + \tfrac{1}{10}\frac{3{+}11x}{1{+}x^2} + \tfrac{1}{5}\frac{1}{1{-}2x} + \tfrac{1}{2}\frac{1}{1{-}x{-}2x^3}$$

Blob points can be used to count connected horizontal and vertical lines in a blob the same as for the full curve in section 5.4.



Theorem 77. *The number of horizontal and vertical lines in blob k are*

$$\begin{aligned} BlobHorizontals_k &= \tfrac{1}{2}\left(BlobS_k + [1,0,1,2]\right) \ = JA_k \qquad && k \geq 4 \\ &= 0,0,0,0,\ 2,3,6,11,18,31,54,91,\ldots && \text{A003479} \end{aligned}$$

$$gBlobHorizontals(x) = -x^3 + \tfrac{1}{2}\left(\frac{-1}{1{-}x} + \frac{1}{1{-}x{-}2x^3}\right)$$

$$\begin{aligned} BlobVerticals_k &= \tfrac{1}{2}\left(BlobS_k + [1,2,1,0]\right) \qquad && k \geq 4 \\ &= 0,0,0,0,\ 2,4,6,10,18,32,54,90,\ldots \end{aligned}$$

$$gBlobVerticals(x) = -x + \frac{x}{1{+}x^2} + \tfrac{1}{2}\left(\frac{-1}{1{-}x} + \frac{1}{1{-}x{-}2x^3}\right)$$

$$\begin{aligned} BlobLines_k &= BlobVerticals_k + BlobHorizontals_k \\ &= 0,0,0,0,\ 4,7,12,21,36,63,108,181,\ldots \end{aligned}$$

$$gBlobLines(x) = -x - x^3\ - \frac{1}{1{-}x} + \frac{x}{1{+}x^2} + \frac{1}{1{-}x{-}2x^3}$$

Proof. Every single-visited point is one end of a vertical and one end of a horizontal. The start and end of the blob are double-visited points. Each is an end of either a horizontal or vertical. Blob 4 has a horizontal start and vertical end and the subsequent bridge expansions give a period-4 pattern

start H, V, V, H
end V, V, H, H

These together are period-4 count of horizontal ends $[1,0,1,2]$ and vertical ends $[1,2,1,0]$. There are two ends for each line, so $\frac{1}{2}(BlobS + periodic)$ in each direction.

The generating functions follow from *gBlobS* and periodic parts

$$\begin{aligned} x^4\left(\frac{1}{1-x} - \frac{x}{1+x^2}\right) &= [1,0,1,2] \qquad k \ge 4 \\ x^4\left(\frac{1}{1-x} + \frac{x}{1+x^2}\right) &= [1,2,1,0] \qquad k \ge 4 \end{aligned}$$

In *gBlobHorizontals*, the periodic part of *BlobS* has cancelled out leaving $BlobHorizontals_k = JA_k$. Term $-x^3$ in *gBlobHorizontals* adjusts to make $BlobHorizontals_3 = 0$ whereas the rest is *gJA* which has $JA_3 = 1$. ☐

From the periodic terms, $BlobVerticals_k = BlobHorizontals_k$ when k even or an alternating difference $+1$ or -1 when k odd.

Summing lines in all blobs is the total in the full curve. The middle segment of each bridge is 1 further line over the lines in the blobs. At the start of the curve there are 6 further lines, and for $k \ge 5$ the same unfolded at the end of the curve.

$$Lines_k = \left(\sum_{j=BlobList_k} BlobLines_j\right) + BridgeCount_k + 12 \qquad k \ge 5$$

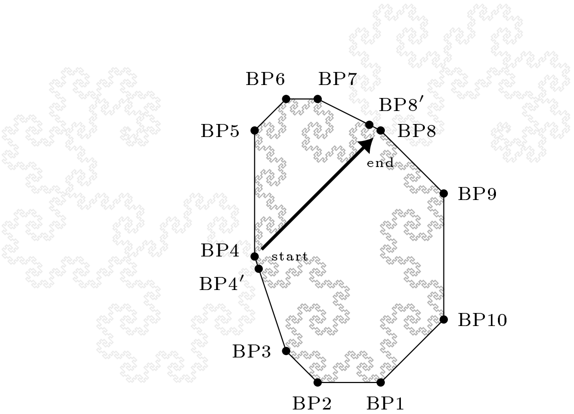
Similarly for horizontals and verticals in the full curve separately, but the blobs at the end of the curve are rotated −90° which means horizontals and verticals swap. The line in the bridge after blob k goes in a pattern H,H,V,V (per *BlobStart* of theorem 64). Including in both sums over blobs means the middle bridge after blob k is counted twice, so adjust accordingly. The lines at the start and end of the curve for $k{\ge}5$ are 3 horizontals and 3 verticals so total 6 in each direction.

$$\begin{aligned}
\mathit{Horizontals}_k &= \left(\sum_{j=4}^{k} \mathit{BlobHorizontals}_j + [1,1,0,0]_j\right) && k \ge 5 \\
&\quad + \left(\sum_{j=4}^{k-2} \mathit{BlobVerticals}_j + [0,0,1,1]_j\right) - [1,1,0,0]_k \ + 6 \\
\mathit{Verticals}_k &= \left(\sum_{j=4}^{k} \mathit{BlobVerticals}_j + [0,0,1,1]_j\right) && k \ge 5 \\
&\quad + \left(\sum_{j=4}^{k-2} \mathit{BlobHorizontals}_j + [1,1,0,0]_j\right) - [0,0,1,1]_k \ + 6
\end{aligned}$$

These sums could be used to derive blob lines from the full curve lines by successive differences as for the blob boundaries. But going from *BlobS* is a little easier and *BlobS* from *BlobB* has effectively been through those differences already.

12.8 Blob Convex Hull

Ngai and Nguyen[38] establish a 10 vertex convex hull around the blob fractal. This is the same in finite iterations except there are extra vertices at fixed distances from blob start and end for some k mod 4. Vertex numbering here is per Ngai and Nguyen. The extras are $BP4'$ and $BP8'$. (For reference, their e_6 shown $x{=}-\frac{1}{12}$ is a misprint which should be $+\frac{1}{12}$, vertically under e_7 as in their figure 5.)



Theorem 78 (Finite form of Ngai and Nguyen). *The vertices of the convex hull around blob k are, using some of P and p(m) from the whole curve hull from theorem 37,*

$$\begin{aligned}
&BP1(k) = P1(k) && \textit{same as whole curve} \\
&BP2(k) = (\tfrac{1}{2}-\tfrac{2}{3}i)\,b^k + \tfrac{1}{3}p(k{-}2) \\
&BP3(k) = (\tfrac{5}{12}-\tfrac{7}{12}i)\,b^k + \tfrac{1}{3}p(k{-}3) \\
&BP4'(k) = BP4(k) + [0,0,4{+}i,2{+}3i,\ 0,0,-4{-}i,-2{-}3i]
\end{aligned}$$

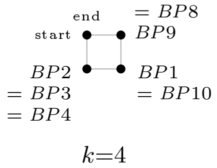
$$\begin{aligned}
\mathit{BP4}(k) &= \mathit{BlobStart}_k\ + [-i, -1, -1{-}i, -i,\ i, 1, 1{+}i, i] \\
&= (\tfrac{1}{3}{-}\tfrac{1}{3}i)\, b^k + \tfrac{1}{3}\,[2{-}2i, -2{+}3i, -5{+}i, -3{-}i,\ -2{+}2i, 2{-}3i, 5{-}i, 3{+}i] \\
\mathit{BP5}(k) &= \tfrac{1}{3}\, b^k\ \ + \tfrac{1}{3} i.p(k{+}2) \\
\mathit{BP6}(k) &= (\tfrac{5}{12}{+}\tfrac{1}{12}i)\, b^k + \tfrac{1}{3} i.p(k{+}1) \\
\mathit{BP7}(k) &= (\tfrac{1}{2}{+}\tfrac{1}{12}i)\, b^k\ \ + \tfrac{1}{3} i.p(k) \\
\mathit{BP8}'(k) &= \mathit{BP8}(k)\ + [-3{+}2i, 0, 0, 1{-}4i,\ 3{-}2i, 0, 0, -1{+}4i] \\
\mathit{BP8}(k) &= \mathit{BlobEnd}_k\ + [1, -1, i, -1{+}i,\ -1, 1, -i, 1{-}i] \\
&= \tfrac{2}{3}\, b^k + \tfrac{1}{3}[1{-}3i, -2{-}2i, 3{+}2i, 1{+}5i,\ -1{+}3i, 2{+}2i, -3{-}2i, -1{-}5i] \\
\mathit{BP9}(k) &= (\tfrac{5}{6}{-}\tfrac{1}{6}i)\, b^k\ + \tfrac{1}{3} p(k{+}9) \\
\mathit{BP10}(k) &= \mathit{P10}(k) \qquad\quad \textit{same as whole curve}
\end{aligned}$$

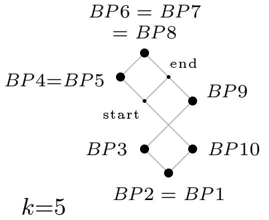
Vertex $\mathit{BP4}'$ *is present when* $k \equiv 2 \,\mathrm{or}\, 3 \bmod 4$, *and set equal to* $\mathit{BP4}$ *otherwise. Vertex* $\mathit{BP8}'$ *is present when* $k \equiv 0 \,\mathrm{or}\, 3 \bmod 4$, *and set equal to* $\mathit{BP8}$ *otherwise.*

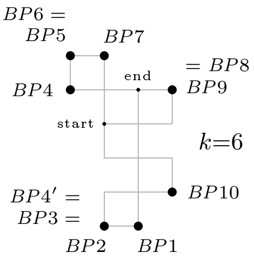
For $k < 9$, *the above points are the hull vertices but with some duplications, and some exceptions at* $k{=}4$.

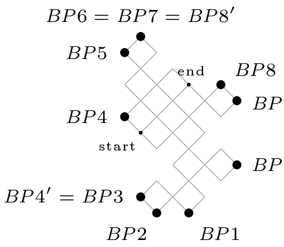
k	*vertices*	*duplication*	*exclude*
4	*4*	$BP2{=}BP3{=}BP4$ *and* $BP1{=}BP10$ *and* $BP8{=}BP9$ *and missing* $-2{+}i$	$BP5, BP6,$ $BP7, BP8'$
5	*6*	$BP1{=}BP2$ *and* $BP4{=}BP5$ *and* $BP6{=}BP7{=}BP8$	
6	*7*	$BP2{=}BP3{=}BP4'$ *and* $BP5{=}BP6$ *and* $BP8{=}BP9$	
7	*9*	$BP3{=}BP4'$ *and* $BP6{=}BP7{=}BP8'$	
8	*10*	$BP7{=}BP8'$	

Proof. Hulls around blobs $k{=}4$ to $k{=}8$ can be formed explicitly. The following diagrams are drawn oriented for curve endpoints horizontal so as to keep the relative locations of the hull vertices consistent.

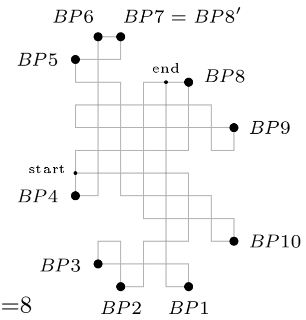








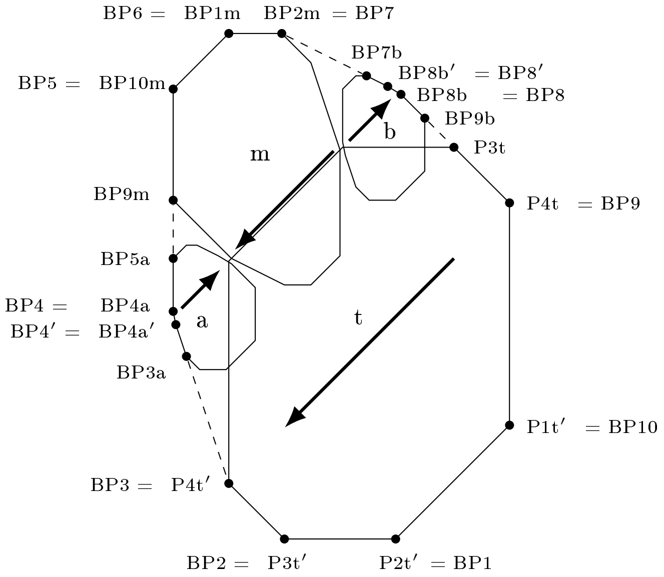
k=7



k=8

In k=6 and k=7, vertex $BP4'$ exists but the offsets make it coincide with $BP3$. In k=7 and k=8, vertex $BP8'$ exists but the offsets make it coincide with $BP7$.

Ngai and Nguyen use their blob sub-parts (section 12.5 here) to determine the hull around the fractal. The same can be done for finite iterations



For $k \geq 9$, the sub-blob vertices are positioned relative to their respective origins (336),(337), and using the rotation factors described there. The twin-dragon vertices similarly.

$$\begin{aligned}
\mathit{BP1a}(k) &= \mathit{BlobPartsAorigin} + (-1).\mathit{BP1}(k{-}4) && \text{etc}\\
\mathit{BP1b}(k) &= \mathit{BlobPartsBorigin} + (-1).\mathit{BP1}(k{-}4) && \text{each}\\
\mathit{BP1m}(k) &= \mathit{BlobPartsMorigin} + (-i).\mathit{BP1}(k{-}2) && \mathit{BPn}\\
\mathit{P1t}(k) &= \mathit{BlobPartsTstart}_k + P1(k{+}1-4)\\
\mathit{P1t}'(k) &= \mathit{BlobPartsTend}_k - P1(k{+}1-4)
\end{aligned}$$

The slopes of the sub-blob sides show those hulls do not overlap. Working through the formulas then shows left side verticals BP4a--BP5a and BP9m--BP10m are co-linear, forming BP4--BP5. Similarly BP8b--BP9b and P3t--P4t on the right 45° diagonal are co-linear forming BP8--BP9.

BP3a--BP4a′ extended back passes through P4t′ to form BP3--BP4′. Similarly BP7b--BP8b′- extended back passes through BP2m at top right forming BP7--BP8′. So vertices

$$\begin{array}{lll}
\mathit{BP1}(k) = \mathit{P2t}'(k) & \mathit{BP4}(k) = \mathit{BP4a}(k) & \mathit{BP8}'(k) = \mathit{BP8b}'(k) \\
\mathit{BP2}(k) = \mathit{P3t}'(k) & \mathit{BP5}(k) = \mathit{BP10m}(k) & \mathit{BP8}(k) = \mathit{BP8b}(k) \\
\mathit{BP3}(k) = \mathit{P4t}'(k) & \mathit{BP6}(k) = \mathit{BP1m}(k) & \mathit{BP9}(k) = \mathit{P4t}(k) \\
\mathit{BP4}'(k) = \mathit{BP4a}'(k) & \mathit{BP7}(k) = \mathit{BP2m}(k) & \mathit{BP10}(k) = \mathit{P1t}'(k)
\end{array}$$

These are not recurrences as such. BP1,2,3,9,10 are the twindragon vertices. BP5,6,7 go to sub-blob $k{-}2$ then are twindragon vertices there. BP4,4′,8,8′ inherit from their respective $k{-}4$ sub-blobs down to initial $k = 5$ to 8 base cases. ☐

At blob start and end a hanging square sticks out so that vertices BP4 and BP8 are 1 or 2 segments away. BP4′ and BP8′ are at fixed distances from BP4 and BP8 where the blob juts out at 5 segments away. In both cases, for limits $/b^k$ these differences $\to 0$ so that there are just 10 vertices around the fractal (and among them blob start and end) per Ngai and Nguyen.

The area of the blob hull is calculated by triangles like HA from section 7.

$$\begin{aligned}
\mathit{BlobHA}_k &= \begin{cases} 0 & \text{if } k \le 3 \\ \frac{29}{96} 2^k - \frac{1}{12}[11,16,11,14] 2^{\left\lfloor \frac{k}{2} \right\rfloor} + \frac{1}{6}[-1,1,-3,-5] & \text{if } k \ge 4 \end{cases} \\
&= 0,0,0,0,\ 1, \tfrac{9}{2}, \tfrac{23}{2}, \tfrac{57}{2}, \tfrac{125}{2}, \tfrac{267}{2}, \tfrac{559}{2}, \dots \\
\frac{\mathit{BlobHA}_k}{2^k} &\to \mathit{BlobHAf} = \frac{29}{96} \ = 0.30208333\ldots
\end{aligned}$$

The boundary length of the hull is calculated from its sides. For $k \ge 8$, side BP3--BP4′ is slope 1:3 so length a multiple of $\sqrt{5}$ or $\sqrt{10}$. Side length BP7--BP8′ is the same as the preceding $k{-}1$ side BP3--BP4′,

$$\big|\mathit{BP7}(k) - \mathit{BP8}'(k)\big| = \big|\mathit{BP3}(k{-}1) - \mathit{BP4}'(k{-}1)\big|$$

Sides BP4′--BP4 and BP8′--BP8, when they exist, are fixed lengths of slope 4:1 or 3:2 which are $\sqrt{17}$ and $\sqrt{13}$ respectively. The remaining 8 sides are straight or 45° diagonal.

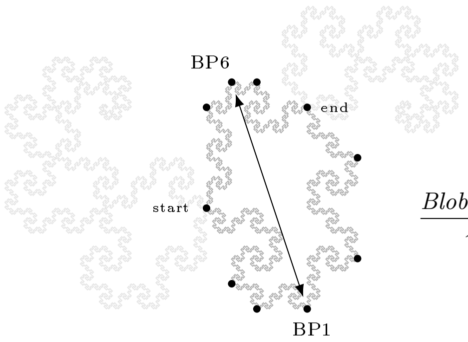
$$\begin{aligned}
\mathit{BlobHB}_k = 4 \quad & \text{if } k{=}4, \text{ or otherwise} \\
& \left(\tfrac{11}{12} + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{12}\sqrt{5} + \tfrac{1}{12}\sqrt{10} \right) .\sqrt{2}^k \qquad (348) \\
& + \tfrac{1}{3}[1,3,-4,-6] \quad - \tfrac{1}{3}[6,8,9,7].\sqrt{2} \\
& + \tfrac{1}{3}[-4,1,1,-4].\sqrt{5} \ - \tfrac{1}{3}[1,1,2,2].\sqrt{10} \\
& + [1,0,0,1].\sqrt{13} \ + [0,0,1,1].\sqrt{17} \\
= 4,\ 5{+}\sqrt{2}{+}\sqrt{5},\ & 6{+}\sqrt{2}{+}\sqrt{5}{+}\sqrt{17},\ 6{+}5\sqrt{5}{+}\sqrt{13}{+}\sqrt{17},\ \dots \quad k{\ge}4
\end{aligned}$$

Scaled by $/\sqrt{2}^k$ to endpoints a unit length, the limit is the coefficient at (348),

$$\frac{BlobHB_k}{\sqrt{2}^k} \to BlobHBf = \tfrac{11}{12} + \tfrac{1}{2}\sqrt{2} + \tfrac{1}{12}\sqrt{5} + \tfrac{1}{12}\sqrt{10} \ = 2.073635\ldots$$

The two points of the blob furthest apart must be vertices of its convex hull. They are BP1 and BP6 except for k=4 where across the unit square. This can be seen by taking vertices pairwise in the manner of theorem 39. The distance apart is

$$\begin{aligned} BlobHdiam_k &= \sqrt[2]{\begin{cases} 2 & \text{if } k=4 \\ \tfrac{5}{8}2^k - [\tfrac{3}{2},2].2^{\left\lfloor \frac{k}{2} \right\rfloor} + 1 & \text{if } k \ge 5 \end{cases}} \\ &= \sqrt[2]{\ } \ 2, 13, 29, 65, 137, 289, 593, 1217, 2465, \ldots \qquad k \ge 4 \end{aligned}$$



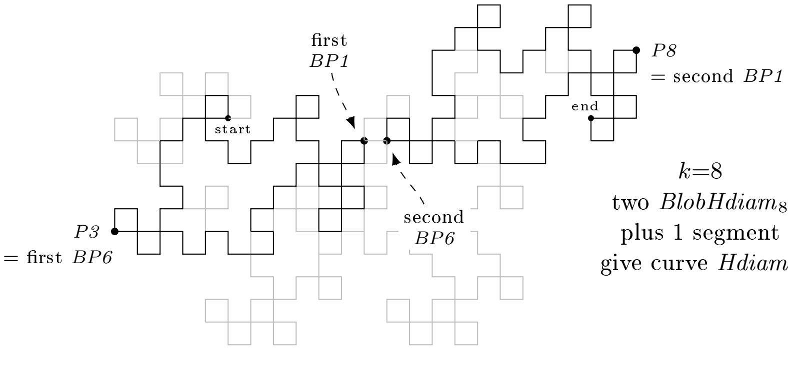
$$\frac{BlobHdiam_k}{\sqrt{2}^k} \to \sqrt{\frac{5}{8}}$$
$$= 0.790569\ldots$$
$$= \tfrac{1}{2}\,Hdiamf$$

$BlobHdiam_9 = 17$ is an integer but all other $BlobHdiam$ are irrational. That can be seen like theorem 39 again. With the same rotation as there, the x_k distance for BP1--BP6 is always a power of 2,

$$x_k = \tfrac{1}{4}2^{\lceil k/2 \rceil}$$

So $BlobHdiam_k^2$ can have at most 3 bits. But its negative half-power puts ≥ 4 bits in the high half for $k \ge 15$. For $k < 15$, it can be verified explicitly only $BlobHdiam_9$ is rational. The legs and hypotenuse of $BlobHdiam_9$ are primitive Pythagorean triple $8, 15, 17$.

$BlobHdiam_k$ is close to $\frac{1}{2}Hdiam_k$, ie. half the whole curve diameter (213). Equality holds for the limits $BlobHdiamf = \frac{1}{2}Hdiamf$. In finite iterations, the halves can be illustrated geometrically by taking two copies of the middle biggest blob, both turned +90°, and placed along the curve diameter.



k=8
two $BlobHdiam_8$
plus 1 segment
give curve $Hdiam_8$

The first blob (lower left) has its *BP6* at the whole curve diameter endpoint *P3*. At this position, the $k-2$ blob of the whole curve coincides with blob subpart m. The blob twindragon part has its start at curve start, but turned +90° so it follows the left of the curve around like a +90° arm.

The second blob (upper right) has its *BP1* at the whole curve diameter endpoint *P8*. At this position, the end of the whole curve coincides with the end of the twindragon part of the blob.

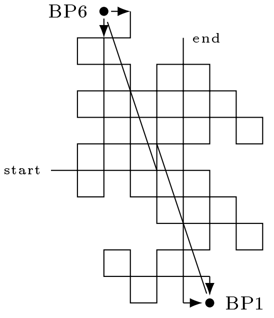
The two blob diameters arranged this way have a unit segment between *BP1* of the first and the *BP6* of the second.

$$\Big(P8(k) - P3(k)\Big) = 2i\Big(BP1(k) - BP6(k)\Big) + i^{\lceil k/2 \rceil} \qquad k \geq 5$$

The i power is the direction of the segment. For k=8 shown above, the extra is $i^{\lceil k/2 \rceil} = 1$ horizontal. If the curve is rotated $\overline{\omega}_8^k$ to endpoints horizontal then the extra is 1 horizontal when k even or 1 segment +45° when k odd.

Theorem 79. *The blob diameter line BP1–BP6 is everywhere within an enclosed unit square or a boundary square.*

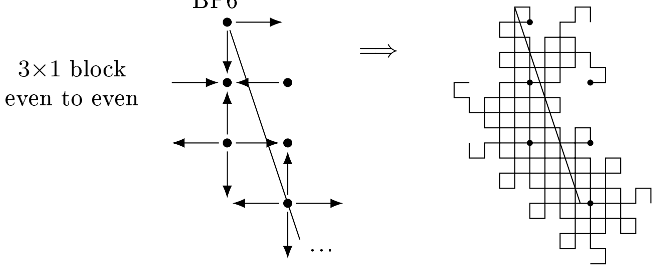
Proof. The diameter across blob k=8 is



blob k=8

The lines shown are slopes are 3:1. The actual diameter line is between them, so at most $\frac{1}{3}$ across anywhere. The claim will be that the 3:1 lines, through to the middle, are within $\frac{1}{3}$ of an enclosed or boundary square.

BP6 is an even point (x+y even) and the 3×1 block immediately below BP6 includes the following segments, as does the next 3×1 below it. These segments expand 4 times to

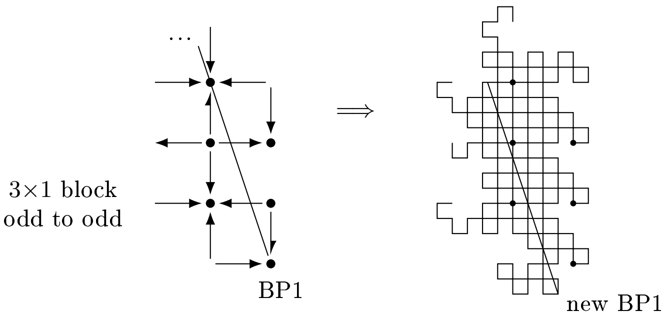


In the expansion it can be noted each new 3×1 along the path has the original set of segments and possibly more, so the 3:1 line passes though enclosed or boundary squares.

The new BP6 vertex is located above-left of the original point. This is per the following identity, noting that expanding 4 times is a 180° rotation of the start segment, so offset $1-i$ in the identity is negated to $-1+i$ in the diagram.

$$\mathit{BP6}(4j{+}8+4) = b^4.\mathit{BP6}(4j{+}8)\ + (1{-}i).(-1)^j$$

Similarly at BP1 which is an odd point and its set of segments expand

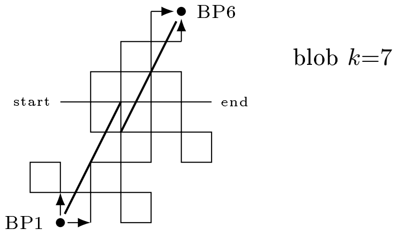


The new BP1 is at offset $-1-2i$ per identity

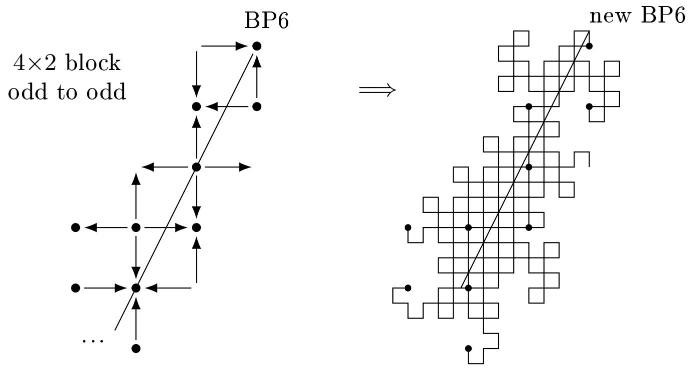
$$\mathit{BP1}(4j{+}8+4) = b^4.\mathit{BP1}(4j{+}8) + (1{+}2i).(-1)^j$$

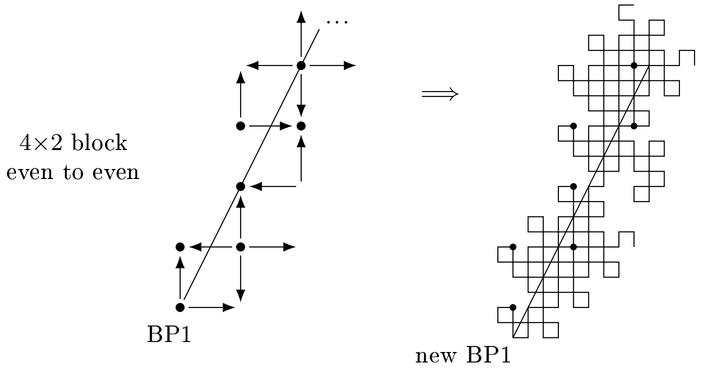
Applying successive 4-fold expansions this way starting from $k{=}8$ shows the theorem for blobs $k=4j{+}8$. Blobs $k=4j{+}6$ have the same 3:1 slope and blocks, but swapped so BP1 is the even case and BP6 is the odd case.

The diameter line in blob $k{=}7$ is



It's convenient to consider the crossing lines in 4×2 blocks so that they go even point to even point from BP1, and odd point to odd point from BP6. Segments at the respective blocks and the way they expand are then





The offsets to the new BP1 and BP6 points are per identities

$$\mathit{BP6}(4j{+}7+4) = b^4.\mathit{BP6}(4j{+}7) - i.(-1)^j$$
$$\mathit{BP1}(4j{+}7+4) = b^4.\mathit{BP1}(4j{+}7) + 2i.(-1)^j$$

Applying successive 4-fold expansions this way starting from $k{=}7$ shows the theorem for blobs $k = 4j{+}7$. Blobs $k = 4j{+}9$ have the same 3:1 slope and blocks, but swapped so BP1 is the even case and BP6 is the odd case.

Diameter lines in blobs $k \le 6$ can be verified explicitly. ☐

12.9 Blob Centroid and Inertia

Theorem 80. *Consider the segments of blob k to have mass uniformly distributed along their length. Their centroid is located at*

$$\mathit{BlobGS}_k = \frac{\mathit{BlobGStotal}_k}{\mathit{BlobSegments}_k} \qquad k \ge 4$$
$$= -\tfrac{5}{2}+\tfrac{3}{2}i,\ -\tfrac{39}{11}-\tfrac{25}{22}i,\ -\tfrac{105}{44}-\tfrac{205}{44}i,\ \tfrac{211}{94}-\tfrac{326}{47}i,\ \dots \qquad k \ge 4$$

where BlobGStotal is the sum of the midpoints of the segments (254)

$$\mathit{BlobGStotal}_k = \sum_{j=\mathit{BlobN}_k}^{\mathit{BlobN}_{k+1}-4} \mathit{midpoint}(j)$$

$$\begin{aligned}
&= \frac{8-4i}{35}\,(2b)^k \\
&\quad + \tfrac{1}{30}\Big[-39+7i, -23+9i, -51+23i, -67+21i\;\Big].b^k && (349) \\
&\quad + \tfrac{1}{42}\begin{bmatrix} 24+16i, & -26-11i, & 5-39i, & -31-68i, \\ -24-16i, & 26+11i, & -5+39i, & 31+68i \end{bmatrix} \\
&= -10+6i,\ -39-\tfrac{25}{2}i,\ -\tfrac{105}{2}-\tfrac{205}{2}i,\ \tfrac{211}{2}-326i, \dots \qquad k \ge 4
\end{aligned}$$

Proof. Let $GStotal_k$ be the total midpoints of the whole curve, as from centroid theorem 46.

$$GStotal_k = \sum_{j=0}^{2^k-1} midpoint(j) \quad = 2^k\, GS_k$$

This total of the whole curve comprises the blob segment midpoints, all the bridge segment midpoints, and the segments at the start and end of the curve.

$$\begin{aligned}
GStotal_k &= \sum_{j=4}^{k} \Big(BlobGStotal_j + 3\,BridgeMiddle_j \Big) \qquad k \ge 6 && (350) \\
&\quad - i \sum_{j=4}^{k-2} \Big(BlobGStotal_j + 3\,BridgeMiddle_j \Big) \\
&\quad - 3\,BridgeMiddle_k \qquad \text{unduplicate bridge } k \\
&\quad + b^k\, BlobN_{k-1} \qquad \text{shift segments after middle} && (351) \\
&\quad + (1{+}(-i)).\big(-1{+}\tfrac{15}{2}i\big) \qquad \text{start and end 7 segments}
\end{aligned}$$

Blobs at the end of the curve are rotated $-i$ by the unfolding back along the curve. Each point z reckoned from the end of the curve is to be at $b^k + (-i)\,z$. Term (351) is all those b^k.

$BridgeMiddle_j$ is the midpoint of the bridge after blob j, which is the centroid of its 3 segments.

$$\begin{aligned}
BridgeMiddle_k &= \tfrac{1}{2}(BlobEnd_k + BlobStart_{k+1}) \\
&= \tfrac{2}{3}b^k \ + \tfrac{1}{6}\,[-1, -1{+}2i, i, 2{+}i,\ 1, 1{-}2i, -i, -2{-}i] && (352) \\
&= -\tfrac{5}{2},\ -\tfrac{5}{2}-3i,\ -\tfrac{11}{2}i,\ 5-\tfrac{11}{2}i,\ \tfrac{21}{2}, \dots
\end{aligned}$$

The sums have *BridgeMiddle* after each blob which means the bridge after blob k is counted twice, so subtract to unduplicate.

Difference $GStotal_k - GStotal_{k-1}$ using (350) leaves two *BlobGStotal* so a 2-delayed recurrence similar to the other blob forms,

$$\begin{aligned}
BlobGStotal_k &= i\, BlobGStotal_{k-2} \\
&\qquad + GStotal_k - GStotal_{k-1} + (\text{bridge} + b^k \text{ terms})
\end{aligned}$$

Factor i gives an 8-period pattern of factors on descending *GStotal* terms

$$1, -1,\ \ i, -i,\ \ -1, 1,\ \ -i, i$$

It's convenient to write (350) in generating functions and solve for *gBlob*

GStotal rather than sum all the various b^k etc terms and their periodic factors. Partial fractions for the result gives various $\frac{1}{1 \pm bx}$ which are in (349) as b^k with periodic direction. $\square$

For the curve scaled by b^k to endpoints a unit length, the limit for *BlobGS* is the ratio of *BlobGStotal* term $(2b)^k$ and *BlobSegments* term 2^k.

$$\frac{\mathit{BlobGS}_k}{b^k} \to \mathit{BlobGSf} = \frac{4}{7} - \frac{2}{7}i \tag{353}$$

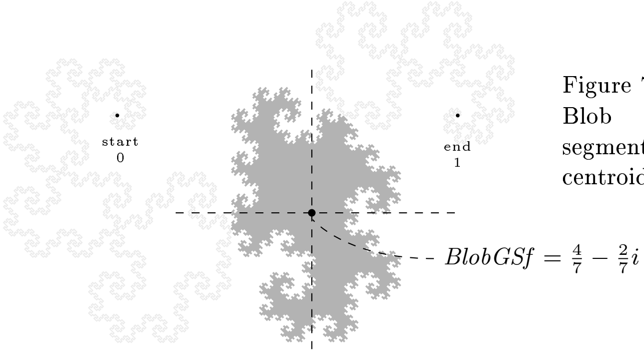
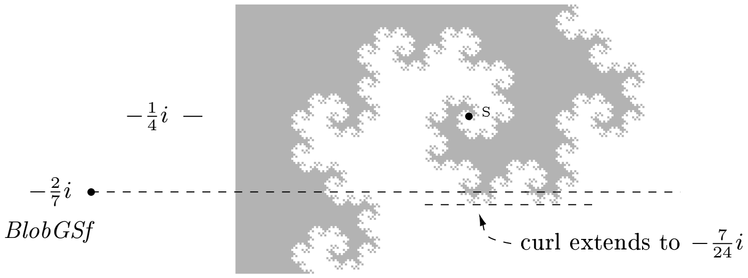


Figure 70: Blob segments centroid

The horizontal through *BlobGSf* at $-\frac{2}{7}i$ is close to the curling boundary to its right, but the curl goes lower. In general, the extents of such boundary curls follow from the convex hull (section 7) around relevant sub-curves. The hull extent limits are multiples of $\frac{1}{3}/2^k$ at k levels smaller, so never a multiple of $\frac{1}{7}$. In this case, the curl is the start S of the twindragon part of the blob. That start is at $\mathit{BlobPartsTstart}_k/b^k \to \frac{3}{4} - \frac{1}{4}i$. The sub-curve to its right in figure 64 is 8 levels smaller and hull extent $\frac{2}{3}$ downwards (the segment right side).

$$-\tfrac{1}{4} - \tfrac{2}{3}.\sqrt{2}^{-8} = -\tfrac{7}{24} \; < -\tfrac{2}{7} = \operatorname{Im} \mathit{BlobGSf} \qquad -\tfrac{7}{24} + \tfrac{1}{168} = -\tfrac{2}{7} \tag{354}$$



The centroid of the convex hull surrounding the blob (section 12.8) can be calculated from the triangles making up the hull (like *HG* at (230)).

$$\begin{aligned}
\mathit{BlobHG}_k &= \frac{\mathit{BlobHGtotal}_k}{\mathit{BlobHA}_k} \qquad\qquad k \ge 4 \qquad\qquad \text{convex hull centroid} \\
&= -\tfrac{5}{2}+\tfrac{3}{2}i,\ \ -\tfrac{32}{9}-\tfrac{29}{27}i,\ \ -\tfrac{55}{23}-\tfrac{316}{69}i,\ \ \tfrac{128}{57}-7i,\ \ \tfrac{3488}{375}-\tfrac{599}{125}i,\ \ \ldots
\end{aligned}$$

$$\mathit{BlobHGtotal}_k = \sum (\text{triangle centroid})\,.\,(\text{triangle area})$$

$$= \begin{cases} -\tfrac{5}{2}+\tfrac{3}{2}i \quad \text{if } k{=}4, \text{ and otherwise for } k \ge 5, \\ b^k \left(\begin{array}{l} \quad \big(\tfrac{599}{3456} - \tfrac{29}{324}i\big)\, 2^k \\ +\big[\tfrac{-47}{96}+\tfrac{29}{108}i,\ \tfrac{-37}{48}+\tfrac{10}{27}i,\ \tfrac{-157}{288}+\tfrac{233}{864}i,\ \tfrac{-191}{288}+\tfrac{313}{864}i\big].2^{\left\lfloor \frac{k}{2} \right\rfloor} \\ +\big[\tfrac{-17}{72}-\tfrac{11}{108}i,\ \tfrac{11}{72}+\tfrac{1}{216}i,\ \tfrac{-7}{72}+\tfrac{13}{108}i,\ \tfrac{-29}{72}+\tfrac{7}{216}i\big] \end{array} \right) \\ +\tfrac{1}{162} \begin{bmatrix} 84{+}i, & 4{-}19i, & -17{-}66i, & -64{-}85i, \\ -84{-}i, & -4{+}19i, & 17{+}66i, & 64{+}85i \end{bmatrix} \end{cases}$$

$$= \tfrac{-5+3i}{2},\ \tfrac{-96-29i}{6},\ \tfrac{-165-316i}{6},\ \tfrac{384-1197i}{6},\ \tfrac{3488-1797i}{6},\ \ldots$$

With the curve scaled to endpoint b^k a unit length, the hull centroid limit is the coefficients of the high power terms in *BlobHGtotal* and *BlobHA*

$$\frac{BlobHG_k}{b^k} \to BlobHGf = \frac{\frac{599}{3456}-\frac{29}{324}i}{29/96} = \frac{599}{1044} - \frac{8}{27}i \tag{355}$$

$$= 0.573754\ldots - 0.296296\ldots i$$

This is very close to the segments centroid limit of figure 70. *BlobHGf* is a little to the right and below *BlobGSf*.

$$BlobHGf - BlobGSf = \tfrac{17}{7308} - \tfrac{2}{189}i = 0.002326\ldots - 0.010582\ldots i \qquad \text{Im} = \text{A021949}$$

12.9.1 Blob Inertia

Theorem 81. *Consider blob k to have unit masses in the middle of each line segment. The blob moment of inertia tensor is, with axes through the blob centroid and x axis parallel to the endpoints of the whole curve,*

$$BlobI(k) = \begin{pmatrix} BlobI_x & -BlobI_{xy} & 0 \\ -BlobI_{xy} & BlobI_y & 0 \\ 0 & 0 & BlobI_z \end{pmatrix}$$

where

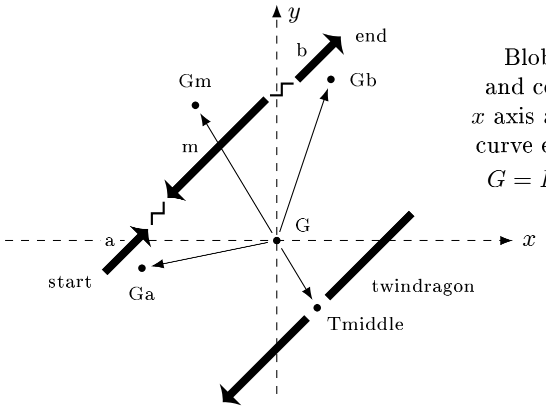
$$\begin{aligned} BlobI_x(k) = {} & \tfrac{53}{4165}\, 4^k \ - \tfrac{1}{8820}\, [1875, 1009, 1227, 2093]\, 2^k \\ & + \tfrac{1}{441}\, [159, 146, 180, 503]\, 2^{\lfloor k/2 \rfloor} \\ & - \tfrac{1}{299880}\, [239861, 81122, 289859, 565558] \\ & + \Big(\tfrac{1}{882}\, [304, 135, -55, 2079]\, 2^{\lfloor k/2 \rfloor} \\ & \quad - \tfrac{1}{1764}\, [2422, 477, 1114, 9531] \Big) / BlobSegments_k \end{aligned} \tag{356}$$

$$= \tfrac{1}{2},\ \tfrac{115}{11},\ \tfrac{4001}{88},\ \tfrac{8725}{47},\ \tfrac{39227}{50}, \ldots \qquad k{\ge}4$$

$$\begin{aligned} BlobI_y(k) = {} & \tfrac{23}{4165}\, 4^k \ - \tfrac{1}{8820}\, [815, 1229, 1807, 1393]\, 2^k \\ & + \tfrac{1}{441}\, [146, 220, 314, 367]\, 2^{\lfloor k/2 \rfloor} \\ & - \tfrac{1}{299880}\, [176501, 143202, 347779, 253918] \\ & + \Big(\tfrac{1}{882}\, [-72, 407, 585, 259]\, 2^{\lfloor k/2 \rfloor} \\ & \quad - \tfrac{1}{1764}\, [630, 1229, 3546, 1199] \Big) / BlobSegments_k \end{aligned}$$

$$\begin{aligned}
&= \tfrac{1}{2},\ \tfrac{31}{11},\ \tfrac{1249}{88},\ \tfrac{3577}{47},\ \tfrac{17147}{50},\dots \qquad k{\ge}4 \\
\mathit{BlobI}_{xy}(k) &= -\tfrac{8}{4165}\,4^k \ \ -\tfrac{1}{4410}\,[253,181,257,329]\,2^k \\
&+ \tfrac{1}{882}\,[337,158,358,326]\,2^{\lfloor k/2\rfloor} \\
&+ \tfrac{1}{59976}\,[-20635,2290,-30467,13622] \\
&+ \Big(\tfrac{1}{294}\,[68,-83,-59,-245]\,2^{\lfloor k/2\rfloor} \\
&\qquad + \tfrac{1}{588}\,[-14,241,430,1125]\Big)\,/\mathit{BlobSegments}_k \\
&= 0,\ -\tfrac{57}{22},\ -\tfrac{783}{88},\ -\tfrac{1783}{47},\ -\tfrac{3369}{25},\dots \quad k{\ge}4 \\
\mathit{BlobI}_z(k) &= \mathit{BlobI}_x(k) + \mathit{BlobI}_y(k) \\
&= \tfrac{76}{4165}\,4^k \ \ -\tfrac{1}{4410}\,[1345,1119,1517,1743]\,2^k \\
&+ \tfrac{1}{441}\,[305,366,494,870]\,2^{\lfloor k/2\rfloor} \\
&- \tfrac{1}{149940}\,[208181,112162,318819,409738] \\
&+ \Big(\tfrac{1}{441}\,[116,271,265,1169]\,2^{\lfloor k/2\rfloor} \\
&\qquad - \tfrac{1}{882}\,[1526,853,2330,5365]\Big)\,/\mathit{BlobSegments}_k \\
&= 1,\ \tfrac{146}{11},\ \tfrac{2625}{44},\ \tfrac{12302}{47},\ \tfrac{28187}{25},\dots \qquad k{\ge}4
\end{aligned}$$

Proof. Inertia can be calculated by taking the blob in the parts of Ngai and Nguyen per section 12.5. The inertia of the blob is the inertia of its parts and the bridges between the parts, all shifted to new centroid $G = \mathit{BlobGS}$ by the parallel axis theorem.

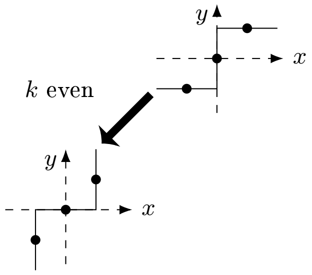


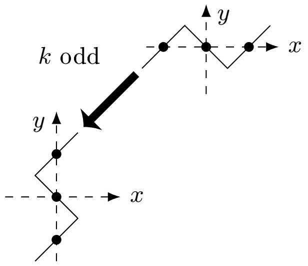
Blob parts
and centroids,
x axis aligned to
curve endpoints
$G = \mathit{BlobGS}_k$

The two $k{-}4$ sub-blobs labelled a, b are the same direction as blob k so their $\mathit{BlobI}(k{-}4)$ is used without rotation. $k{-}2$ sub-blob m is turned 180° which does not change inertia so it too is without rotation.

The twindragon endpoints are at +45° (and back 180° which is no change), so the axes in *TI* (theorem 59) must be rotated −45°.

The bridges rotated to curve endpoints horizontal have two cases,





For k even, the midpoints of the two bridges are the same configuration each. The bridge centroid is the middle point so inertia tensors, with x axis aligned to the curve endpoints, are

$$\mathit{BridgeIa}_k = \begin{cases} \mathit{Ipoint}(\tfrac{1}{2}{+}\tfrac{1}{2}i) + \mathit{Ipoint}(-\tfrac{1}{2}{+}{-}\tfrac{1}{2}i) & \text{if } k \text{ even} \\ \mathit{Ipoint}(\tfrac{1}{2}\sqrt{2}i) + \mathit{Ipoint}(-\tfrac{1}{2}\sqrt{2}i) & \text{if } k \text{ odd} \end{cases}$$

$$= \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k \text{ even,} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k \text{ odd,}$$

$$\mathit{BridgeIb}_k = \begin{cases} \mathit{BridgeIa}_k & \text{if } k \text{ even} \\ \mathit{Ipoint}(\tfrac{1}{2}\sqrt{2}) + \mathit{Ipoint}(-\tfrac{1}{2}\sqrt{2}) & \text{if } k \text{ odd} \end{cases}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ if } k \text{ odd,}$$

Offsets from the blob k centroid to the centroids of the parts follow from the blob sub-part origins (336),(337).

$$\begin{aligned} \mathit{GtoGa}_k &= (\mathit{BlobPartsAorigin}_k + (-1).\mathit{BlobGS}_{k-4}) - \mathit{BlobGS}_k \\ \mathit{GtoGb}_k &= (\mathit{BlobPartsBorigin}_k + (-1).\mathit{BlobGS}_{k-4}) - \mathit{BlobGS}_k \\ \mathit{GtoGm}_k &= (\mathit{BlobPartsMorigin}_k + (-i).\mathit{BlobGS}_{k-2}) - \mathit{BlobGS}_k \end{aligned}$$

The middles of the two bridges similarly, with (352) and reckoning the first bridge as after sub-blob A and the second as before sub-blob B.

$$\begin{aligned} \mathit{GtoBa}_k &= \big(\mathit{BlobPartsAorigin}_k + (-1).\mathit{BridgeMiddle}_{k-4}\big) - \mathit{BlobGS}_k \\ \mathit{GtoBb}_k &= \big(\mathit{BlobPartsBorigin}_k + (-1).\mathit{BridgeMiddle}_{k-5}\big) - \mathit{BlobGS}_k \end{aligned}$$

For the twindragon, its middle which is its centroid is between start and end,

$$\begin{aligned} \mathit{BlobPartsTmiddle}_k &= \tfrac{1}{2}(\mathit{BlobPartsTstart}_k + \mathit{BlobPartsTend}_k) \ = (\tfrac{5}{8}{-}\tfrac{3}{8}i)\, b^k \\ \mathit{GtoTmid}_k &= \mathit{BlobPartsTmiddle}_k - \mathit{BlobGS}_k \end{aligned}$$

Each of the G offsets above are in a curve ending at b^k, so factors $\overline{\omega_8}^k$ to rotate for curve end on the x axis. R is the +45° matrix of rotation from (244), here used to turn the twindragon axes −45°.

$$\begin{aligned} \mathit{BlobI}(k) = \mathit{BlobI}(k{-}2) \quad &+ \mathit{BlobSegments}(k{-}2).\mathit{Ipoint}(\mathit{GtoGm}_k.\overline{\omega_8}^k) \qquad (357) \\ + \mathit{BlobI}(k{-}4) \quad &+ \mathit{BlobSegments}(k{-}4).\mathit{Ipoint}(\mathit{GtoGa}_k.\overline{\omega_8}^k) \\ + \mathit{BlobI}(k{-}4) \quad &+ \mathit{BlobSegments}(k{-}4).\mathit{Ipoint}(\mathit{GtoGb}_k.\overline{\omega_8}^k) \end{aligned}$$

$$\begin{aligned}&+ R.\mathit{TI}(k{-}4).R^{-1} \;+ 4.2^{k-4}.\mathit{Ipoint}(\mathit{GtoTmid}_k.\overline{\omega_8}^k)\\ &+ \mathit{BridgeIa}(k) \quad + 3.\mathit{Ipoint}(\mathit{GtoBa}_k.\overline{\omega_8}^k)\\ &+ \mathit{BridgeIb}(k) \quad + 3.\mathit{Ipoint}(\mathit{GtoBb}_k.\overline{\omega_8}^k)\end{aligned}$$

Recurrence manipulations give (356) etc. □

The various *GtoGm* etc terms are differences of *BlobGS*. Those offsets are fractions with denominators $\mathit{BlobSegments}_k$ for G, $\mathit{BlobSegments}_{k-2}$ for *Gm*, and $\mathit{BlobSegments}_{k-4}$ for *Ga*, *Gb*. The total is fractions over $\mathit{BlobSegments}_k$ only. This is not quite obvious in the terms, but the total is offsets from G to half-integer segment midpoints, so $\mathit{BlobSegments}_k$ only.

$$\mathit{BlobI}(k) = \sum_{n=\mathit{BlobN}_k}^{\mathit{BlobN}_{k+1}-4} \mathit{Ipoint}\Big((\mathit{midpoint}(n) - \mathit{BlobGS}_k).\overline{\omega_8}^k\Big)$$

Multiplying through by $\mathit{BlobSegments}_k$ is integers, halves and quarters,

$$\begin{aligned}\mathit{BlobI}_x.\mathit{BlobSegments}_k &= 2,\ 115,\ \tfrac{4001}{4},\ 8725,\ 78454,\ 666301,\ \tfrac{21457681}{4}, \dots \quad k{\ge}4\\ \mathit{BlobI}_y.\mathit{BlobSegments}_k &= 2,\ 31,\ \tfrac{1249}{4},\ 3577,\ 34294,\ 280913,\ \tfrac{9098193}{4}, \dots\\ \mathit{BlobI}_{xy}.\mathit{BlobSegments}_k &= 0, -\tfrac{57}{2}, -\tfrac{783}{4}, -1783, -13476, -\tfrac{211789}{2}, -\tfrac{3347519}{4}, \dots\end{aligned}$$

$\mathit{BlobI}_{xy} = \sum xy$ is negative, and is then negated to positive in the *BlobI* matrix. Coordinate products xy are positive in the 1st and 3rd quadrants and negative in the 2nd and 4th. Roughly speaking, BlobI_{xy} negative means there is more in the 2nd and 4th than in 1st and 3rd, as reckoned by the products.

When the curve is scaled so the whole curve is unit length and unit mass, the limit for inertia of the middle biggest blob is

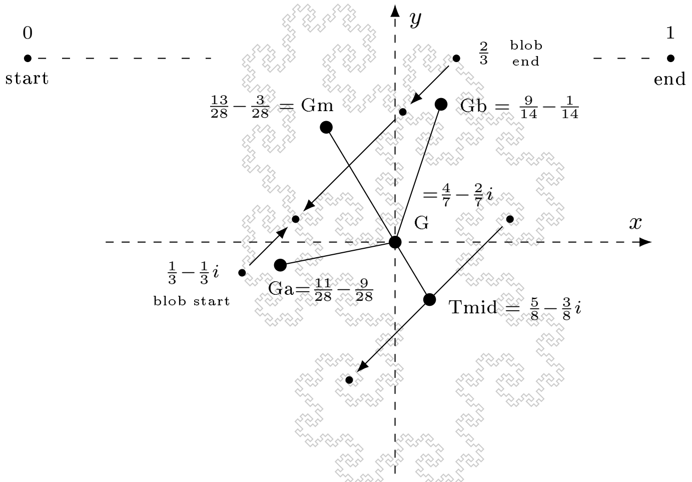
$$\frac{\mathit{BlobI}(k)}{4^k} \to \mathit{BlobIf} = \frac{1}{4165}\begin{pmatrix}53 & 8 & 0\\ 8 & 23 & 0\\ 0 & 0 & 76\end{pmatrix} \qquad \text{blob inertia limit}$$

Each entry is the coefficient of the 4^k term at (356) etc (and negate xy for the matrix). With this scaling the blob has area $\mathit{BlobAf}_0{=}\frac{1}{5}$ per (331), and mass $\frac{2}{5}$ per *BlobSegments*.

For the limit, the sum of parts at (357) can ignore the bridges, giving the following identity for *BlobIf* by parts. Each level shrinks x,y by $1/\sqrt{2}$ and mass reduced in proportion to area. So blob $k{-}2$ is reduction $\frac{1}{16}$ and blob $k{-}4$ reduction $\frac{1}{256}$. The twindragon inertia limit *TIf* from (301) is for unit length and here it is $1/\sqrt{2}^3$ (and axes at −45°) so reduction $\frac{1}{64}$.

$$\begin{aligned}\mathit{BlobIf} = {}& \tfrac{1}{16}\mathit{BlobIf} \;+ 2\tfrac{1}{256}\mathit{BlobIf} \;+ \tfrac{1}{64}R.\mathit{TIf}.R^{-1}\\ &+ \tfrac{1}{16}.\tfrac{2}{5}\,\mathit{Ipoint}(\mathit{GtoGa}) \;+ \tfrac{1}{16}.\tfrac{2}{5}\,\mathit{Ipoint}(\mathit{GtoGb})\\ &+ \tfrac{1}{4}.\tfrac{2}{5}\,\mathit{Ipoint}(\mathit{GtoGm}) \;+ \tfrac{1}{4}\,\mathit{Ipoint}(\mathit{GtoTmid})\end{aligned} \tag{358}$$

$$\begin{aligned}\mathit{GtoGa} &= -\tfrac{5}{28}-\tfrac{1}{28}i & \mathit{GtoGm} &= -\tfrac{3}{28}+\tfrac{5}{28}i\\ \mathit{GtoGb} &= \;\;\tfrac{1}{14}+\tfrac{3}{14}i & \mathit{GtoTmid} &= \tfrac{3}{56}-\tfrac{5}{56}i\end{aligned}$$



The sub-blobs are unrotated so *BlobIf* in (358) is a fraction of rotated *TIf* plus *Ipoint* offsets.

$$\mathit{BlobIf} = \tfrac{128}{119}\Big(\tfrac{1}{64}R.\mathit{TIf}.R^{-1} \; + \mathit{Ipoint}\ldots\Big)$$

All the twindragons in all the sub-blobs are at the same 45° orientation (eg. figure 68) so this is also effectively a total sum over twindragons of descending sizes and their locations.

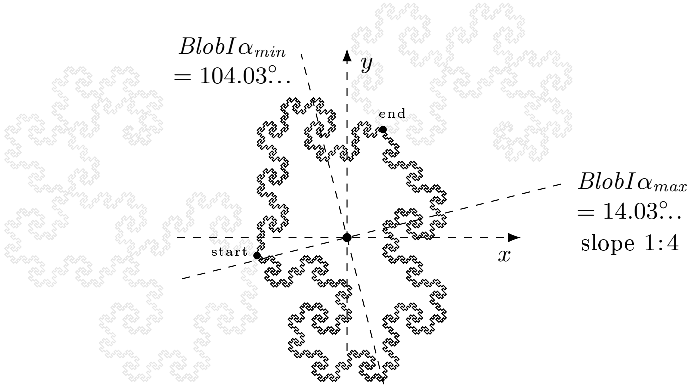
The inertia of the whole curve is the sum of the blob inertias, each shifted to the curve centroid. For finite iterations, this is complicated by the bridges and segments at curve start and end. In the inertia limits, the sum is an identity

$$\begin{aligned} \mathit{If} = \sum_{j=0}^{\infty} \frac{1}{2^j} R^{-j}.\mathit{BlobIf}.R^{j} \qquad &+ \tfrac{2}{5}\frac{1}{2^j}\mathit{Ipoint}\Big(\mathit{BlobGSf}/b^j - \mathit{GSf}\Big) \qquad (359) \\ + \sum_{j=2}^{\infty} \frac{1}{2^j} R^{-j-2}.\mathit{BlobIf}.R^{j+2} \; &+ \tfrac{2}{5}\frac{1}{2^j}\mathit{Ipoint}\Big(1{-}i\mathit{BlobGSf}/b^j - \mathit{GSf}\Big) \end{aligned}$$

The first sum (359) is from middle biggest blob to start of the curve. The second sum is the blob after the middle (which is 2 levels smaller) to curve end. Each level smaller has axes rotated +45°, and the curve end has further 90° for the unfolding. Each level shrinks coordinates by $1/\sqrt{2}$ in each of x,y so area factor $1/2^j$. Mass likewise reduced by that factor for the terms shifting blob centroids to the curve centroid.

The principal axes of inertia for the middle biggest blob limit follow in a similar way to (250).

$$\begin{aligned} \mathit{BlobI}\alpha_{max} &= \tfrac{1}{2}\arctan\tfrac{-2.-8}{53-23} = \arctan\tfrac{1}{4} = 14.036243°\ldots \\ &\hspace{10em} \text{radians A195727} \quad (360) \\ \mathit{BlobI}\alpha_{min} &= \mathit{BlobI}\alpha_{max} + 90° = 104.036243°\ldots \end{aligned}$$



$BlobI\alpha_{max}$ axis passes close to the twindragon part start S, but the axis is just above that point. S is at $\frac{3}{4}{-}\frac{1}{4}$ but slope 1:4 up from G passes the vertical there at

$$-\tfrac{2}{7} + (\tfrac{3}{4} - \tfrac{4}{7}).\tfrac{1}{4} = -\tfrac{27}{112} \;\; > -\tfrac{1}{4} \qquad \text{axis above by } -\tfrac{27}{112} - (-\tfrac{1}{4}) = \tfrac{1}{112}$$

The axis also passes close to the top of the curl there, but the curl goes higher. For example directly above S the hull extent is higher by calculating similar to (354) but k=12 levels down so the sub-curve right side faces upwards (with curve endpoint horizontal),

$$-\tfrac{1}{4} + \tfrac{2}{3}.\sqrt{2}^{-12} = -\tfrac{23}{96} \;\; > -\tfrac{27}{112} \qquad \text{axis below by } -\tfrac{23}{96} - (-\tfrac{27}{112}) = \tfrac{1}{672}$$

The blob inertia limit when rotated by arctan $\frac{1}{4}$ to its principal axes, with x the maximum, is

$$\tfrac{1}{4165}\begin{pmatrix} 55 & 0 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 76 \end{pmatrix} \qquad \text{principal axes}$$

The inertia of the blob convex hull (section 12.8) can be compared to the inertia of the blob it surrounds. The hull inertia is calculated from its polygon. For the limit with curve unit length, density $mass = 2.area$ the same as the curve, and axes through the hull centroid $BlobHGf$,

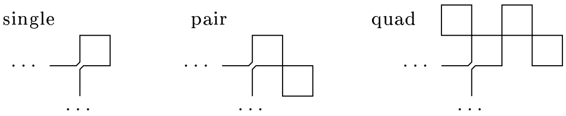
$$BlobHI_x = \tfrac{24281}{1119744} \qquad BlobHI_y = \tfrac{37471}{3608064} \qquad BlobHI_{xy} = -\tfrac{2213}{746496}$$

$$BlobHI\alpha_{max} = \tfrac{1}{2}\arctan\tfrac{192531}{366910} \;\; = 13.843813° \ldots$$

This is a little shallower slope than the segments principal axis $BlobI\alpha_{max}$ (360), and the hull centroid is a little below.

13 Hanging Squares

On the boundary there are some hanging squares attached to the rest of the curve only at a corner. Some are single unit squares, some are pairs of unit squares, and some are quads of 4 unit squares.

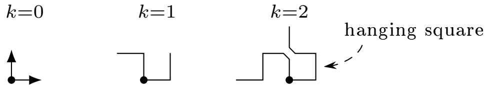


Some hanging singles occur as the end of a pair, and some occur as the two singles in a quad. For the purpose of the following *HQ1*, all the singles are counted no matter what further form they might be part of.

Theorem 82. *The number of hanging single squares on the dragon curve k is*

$$\begin{aligned} HQ1L_k &= \begin{cases} 0 & \text{if } k < 3 \\ 2\,JA_{k-3} & \text{if } k \geq 3 \end{cases} && \textit{left boundary} \\ &= 0, 0, 0, 0, 0, 0, 2, 4, 6, 12, 22, 36, 62, 108, \ldots \\ HQ1R_k &= \begin{cases} 0 & \text{if } k < 1 \\ JA_{k-1} & \text{if } k \geq 1 \end{cases} && \textit{right boundary} \\ &= 0, 0, 0, 0, 1, 2, 3, 6, 11, 18, 31, 54, 91, 154, \ldots \\ HQ1_k &= HQ1L_k + HQ1R_k && \textit{whole boundary} \\ &= \begin{cases} 0 & \text{if } k < 3 \\ JA_k - 1 & \text{if } k \geq 3 \end{cases} \\ &= 0, 0, 0, 0, 1, 2, 5, 10, 17, 30, 53, 90, 153, 262, \ldots \end{aligned}$$

Proof. When the starts of two dragon curves are at right angles they may form a hanging square comprising some segments from the first curve and some from the second.

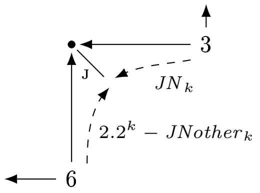


k=2 forms a hanging square and the curve start is the same for $k > 2$.

From the left boundary breakdown of figure 14, the number of hanging squares in level k+4 is the preceding k+3, two k+1, and possible new hanging squares at the two right angle meetings where those k+1 are inserted.

$$HQ1L_{k+4} = HQ1L_{k+3} + 2HQ1L_{k+1} + \begin{cases} 0 & \text{if } k < 2 \\ 2 & \text{if } k \geq 2 \end{cases} \tag{361}$$

The 2 new hanging squares are at points 6 and 7 in figure 14. The sub-curves pointing inward from there touch as an inward join but they do not touch the new hanging squares since per theorem 27 the first side touches at vertex number JN_k and this is to be ≥ 1. The second side touches at *JNother* from the start which is $2.2^k - JNother_k$ and is to be ≥ 3. This is so for $k \geq 2$.



(361) is the same recurrence as JA growth (162) but +2 instead of +1. Initial values are $HQ1L_0 = \cdots = HQ1L_5 = 0$ so the +2 becomes factor 2 on all $HQ1L$, giving $2\,JA$ with offset.

For the right boundary, the unfolding of section 3.2 is two dragon ends meeting and they create a new hanging square for $k \ge 3$, per join area $JA_3 = 1$.

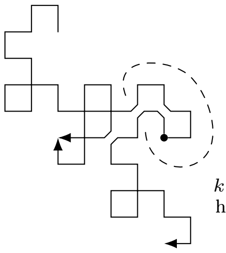
$$HQ1R_{k+1} = HQ1R_k + HQ1L_k + \begin{cases} 0 & \text{if } k < 3 \\ 1 & \text{if } k \ge 3 \end{cases} \qquad \square$$

Hanging squares on a blob can be calculated from the total for the curve in the same way as blob area section 12.4. The hanging squares of each blob sum to the hanging squares of the whole curve in the manner of the blob area calculation (since there are no hanging squares on the bridges).

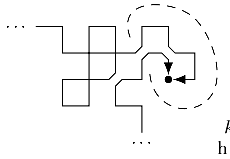
Blob k=4 is a single unit square and is reckoned as a hanging square in the sense of hanging from its start/end point, so $BlobHQ1_4$=1 here and $BlobHQA$ =1 below. This makes the whole curve the sum of the blobs.

$$\begin{aligned} BlobHQ1_k &= \text{same recurrence as } BlobA \text{ (324)} \qquad k \ge 8 \qquad (362) \\ &= 0,0,0,0,\ 1,1,2,4,5,9,18,28,45,81,138,228,\ldots \\ BlobHQ1L_k &= \text{same recurrence as } BlobAL \text{ (334)} \qquad k \ge 8 \\ &= 0,0,0,0,\ 0,0,1,1,1,3,6,8,14,26,43,71,\ldots \\ BlobHQ1R_k &= \text{same recurrence as } BlobAL \text{ (334)} \qquad k \ge 10 \\ &= 0,0,0,0,\ 1,1,1,3,4,6,12,20,31,55,95,157,\ldots \end{aligned}$$

For hanging pairs, a similar calculation is made. For the left side, the outward sub-curves form a pair for $k \ge 4$ which is $JN_k \ge 2$ and $2.2^k - JNother_k \ge 6$. For the right side, a pair is formed for k=5.



k=4 outward
hanging pair

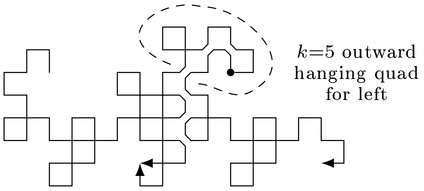


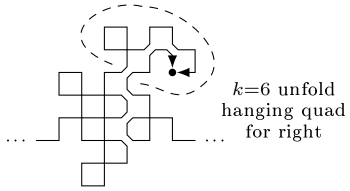
k=5 unfold
hanging pair

So pairs occur 2 levels later on both left and right, giving the same as the singles but delayed.

$$\begin{aligned} HQ2L_k &= HQ1L_{k-2} && \text{left boundary pairs} \\ HQ2R_k &= HQ1R_{k-2} && \text{right boundary pairs} \\ HQ2_k &= HQ1_{k-2} && \text{whole boundary pairs} \end{aligned}$$

Hanging quads similarly. For the left side, the outward sub-curves form a pair for $k \geq 5$ which is $JN_k \geq 4$ and $2.2^k - JNother_k \geq 12$. For the right side, a pair is formed for k=6. This is then 3 levels later than the hanging single squares.





$$\begin{aligned} HQ4L_k &= HQ1L_{k-3} && \text{left boundary pairs} \\ HQ4R_k &= HQ1R_{k-3} && \text{right boundary pairs} \\ HQ4_k &= HQ1_{k-3} && \text{whole boundary pairs} \end{aligned}$$

As noted above, a pair includes a hanging single square and a quad includes a hanging pair and a further hanging square. Each pair has 1 unit square not accounted for by the singles. Each quad has 1 unit square not accounted for by the pair and the single in it. So total hanging area

$$\begin{aligned} HQA_k &= HQ1_k + HQ2_k + HQ4_k \\ &= 0, 0, 0, 0, 1, 2, 6, 13, 24, 45, 80, 137, 236, 405, \ldots \end{aligned}$$

There are 4 boundary segments for each such total hanging square. The proportion of boundary on hanging squares out of total boundary is found by some recurrence or generating function manipulation to express HQA in terms of B. Then since B grows as a power r^k,

$$\begin{aligned} \frac{4\,HQA_k}{B_k} &= \frac{\frac{5}{2}B_k + \frac{1}{2}B_{k+1} - B_{k+2} - 14}{B_k} && k \geq 6 \\ &\to \tfrac{5}{2} + \tfrac{1}{2}r - r^2 \ = \ 0.472680\ldots && \text{as } k{\to}\infty \\ &= \tfrac{1}{r} - \tfrac{8}{r^8} && (363) \end{aligned}$$

The limit is the same for the left or right side of the curve taken separately.

In form (363), $1/r$ is the proportion of right boundary segments out of total boundary (125). The hanging square sides here is smaller by $8/r^8 = 0.117073\ldots$.

The limit is the same for a blob since the same coefficients occur on $BlobB$ and like B it grows as a power r^k.

$$4BlobHQA_k = \tfrac{5}{2}BlobB_k + \tfrac{1}{2}BlobB_{k+1} - BlobB_{k+2} - [13, 7, 11, 17] \qquad k \geq 5$$

13.1 Points on Boundary

To count points on the boundary of a blob, there is a point at the start of each boundary segment $BlobB$ but attachment points of a hanging square are then counted twice. Unduplicating those, and not reckoning the blob k=4 unit square as a hanging square the way $BlobHQA$ does, gives

$$\begin{aligned} \mathit{BlobPB}_k &= \mathit{BlobB}_k - \begin{cases} 0 & \text{if } k = 4 \\ \mathit{BlobHQA}_k & \text{if } k \neq 5 \end{cases} \qquad \text{blob boundary points} \\ &= 0, 0, 0, 0,\ 4, 9, 17, 32, 60, 107, 185, 316, \ldots \end{aligned}$$

and enclosed points by difference from total *BlobP*

$$\begin{aligned} \mathit{BlobPE}_k &= \mathit{BlobP}_k - \mathit{BlobPB}_k \qquad \text{blob enclosed points} \\ &= 0, 0, 0, 0,\ 0, 0, 0, 2, 8, 26, 72, 182, \ldots \end{aligned}$$

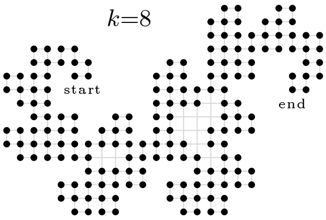
The boundary points of the whole curve are the sum of the blobs, plus non-blob points which are at curve start and end and 2 on each bridge. Or alternatively, boundary segments B_k less hanging squares and less the points duplicated by segments on both boundaries which are counted twice by B.

$$\begin{aligned} \mathit{NonBlobPoints}_k &= \begin{cases} 2^k + 1 & k = 0 \text{ to } 3 \\ 4k - 4 & k \geq 4 \end{cases} \\ &= 2, 3, 5, 9, 12, 16, 20, 24, 28, 32, \ldots \\ \mathit{PB}_k &= \mathit{NonBlobPoints}_k + \sum_{j=\mathit{BlobList}_k} \mathit{BlobPB}_j \qquad \text{boundary points} \\ &= B_k - \mathit{HQA}_k - 8k + 24 \qquad k \geq 6 \\ &= 2, 3, 5, 9, 16, 29, 54, 99, 180, 323, 572, 999, 1728, \ldots \end{aligned}$$

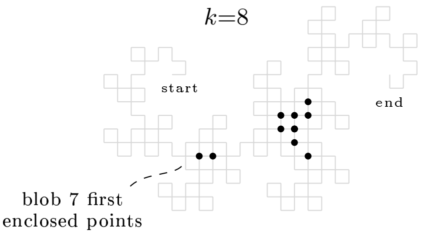
$$\begin{aligned} \mathit{gPB}(x) = {} & \tfrac{69}{2} \frac{1}{1-x} - 8 \frac{1}{(1-x)^2} + \tfrac{1}{4} \frac{13 + 9x + 14x^2}{1-x-2x^3} \\ & - \tfrac{111}{4} - 21x - \tfrac{29}{2}x^2 - 9x^3 - 5x^4 - 2x^5 \end{aligned}$$

and enclosed points are then, by difference from total points P,

$$\begin{aligned} \mathit{PE}_k &= P_k - \mathit{PB}_k \qquad \text{enclosed points} \\ &= 0, 0, 0, 0, 0, 0, 0, 2, 10, 38, 118, 326, 830, \ldots \end{aligned}$$



Boundary points
$PB_8 = 180$



Enclosed points
$PE_8 = 10$

The first enclosed points are in blob k=7. For $k < 7$, the boundary points are simply all points P_k from (175).

The enclosed points grow as 2^k (like enclosed area A_k) so eventually exceed the boundary points in either the whole curve or a blob.

$$\mathit{PE}_k > \mathit{PB}_k \qquad \text{iff } k \geq 15$$

$$\mathit{BlobPE}_k > \mathit{BlobPB}_k \qquad \text{iff } k \geq 13$$

Boundary points are either single or double-visited. All single-visited points are on the boundary and enclosed points are always double-visited,

$$\begin{aligned}S_k &= \text{single-visited on boundary}\\ \mathit{DB}_k &= \text{double-visited on boundary}\\ \mathit{PE}_k &= \text{double-visited enclosed}\end{aligned}$$

So double-visited boundary points by difference either of two ways

$$\begin{aligned}\mathit{DB}_k &= \mathit{PB}_k - S_k && \text{doubles on boundary}\\ &= D_k - \mathit{PE}_k\\ &= 0,0,0,0,1,4,11,26,57,114,217,398,709,\ldots\\ \mathit{BlobDB}_k &= \mathit{BlobPB}_k - \mathit{BlobS}_k\\ &= \mathit{BlobD}_k - \mathit{BlobPE}_k\\ &= 0,0,0,0,\ 1,3,6,12,25,45,78,136,233,\ldots\end{aligned}$$

The proportion of single-visited or double-visited points among the boundary points can be found by some recurrence or generating function manipulation to write S_k in terms of PB_k.

$$S_k = \tfrac{1}{29}\big(16\mathit{PB}_k + 24\mathit{PB}_{k+1} - 14\mathit{PB}_{k+2}\ + 208k - 721\big) \qquad \text{for } k \geq 6$$

Then since PB_k grows as power r^k,

$$\begin{aligned}\frac{S_k}{\mathit{PB}_k} &\to \tfrac{16}{29} + \tfrac{24}{29}r - \tfrac{14}{29}r^2 &&= 0.567002\ldots \qquad \text{as } k \to \infty\\ \frac{\mathit{DB}_k}{\mathit{PB}_k} &\to \tfrac{13}{29} - \tfrac{24}{29}r + \tfrac{14}{29}r^2 &&= 0.432997\ldots\end{aligned}$$

Each hanging square attachment point is a double-visited boundary point. There are HQA of them. The remaining non-hanging double-visited boundary points are then

$$\begin{aligned}\mathit{DBnH}_k &= \mathit{DB}_k - \mathit{HQA}_k && \text{non-hanging boundary doubles}\\ &= 0,0,0,0,0,2,5,13,33,69,137,261,\ldots\end{aligned}$$

The proportions within DB are found by writing in terms of DB,

$$\begin{aligned}\mathit{HQA}_k &= \tfrac{1}{19}\big(-27\mathit{DB}_k + 16\mathit{DB}_{k+1} + 2\mathit{DB}_{k+2}\ - 72k + 313\big) && k \geq 6\\ \mathit{DBnH}_k &= \tfrac{1}{19}\big(\ 46\mathit{DB}_k - 16\mathit{DB}_{k+1} - 2\mathit{DB}_{k+2}\ + 72k - 313\big) && k \geq 6\end{aligned}$$

$$\begin{aligned}\frac{\mathit{HQA}_k}{\mathit{DB}_k} &\to \tfrac{-27}{19} + \tfrac{16}{19}r + \tfrac{2}{19}r^2 &&= 0.309483\ldots \qquad \text{as } k \to \infty\\ \frac{\mathit{DBnH}_k}{\mathit{DB}_k} &\to \tfrac{46}{19} - \tfrac{16}{19}r - \tfrac{2}{19}r^2 &&= 0.690516\ldots\end{aligned}$$

Or the proportions out of the whole PB boundary points are

$$\mathit{HQA}_k = \tfrac{1}{29}\big(\ 3\mathit{PB}_k + 48\mathit{PB}_{k+1} - 28\mathit{PB}_{k+2}\ + 184k - 804\big) \qquad k \geq 6$$

$$DBnH_k = \tfrac{1}{29}\big(10PB_k - 72PB_{k+1} + 42PB_{k+2} \; - 392k + 1525\big) \qquad k \ge 6$$

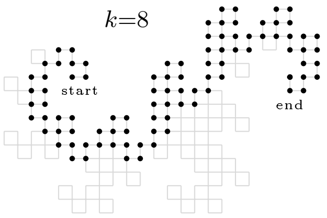
$$\frac{HQA_k}{PB_k} \;\to\; \tfrac{3}{29} + \tfrac{48}{29}r - \tfrac{28}{29}r^2 \;\; = 0.134005\ldots \qquad \text{as } k \to \infty$$

$$\frac{DBnH_k}{PB_k} \;\to\; \tfrac{10}{29} - \tfrac{72}{29}r + \tfrac{42}{29}r^2 \;\; = 0.298991\ldots$$

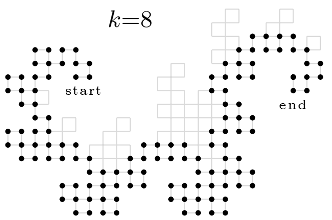
All of these limits are the same in a blob.

Left and right side boundary points can be counted separately. Similar to above, points are left or right segments L or R and unduplicate the left or right side hanging square attachment points.

$$\begin{aligned} PL_k &= L_k + 1 - HQAL_k && \text{left boundary points} \\ &= 2, 3, 5, 9, 13, 21, 35, 57, 93, 155, 261, \ldots \\ PR_k &= R_k + 1 - HQAR_k && \text{right boundary points} \\ &= 2, 3, 5, 9, 16, 27, 45, 76, 129, 218, \ldots \end{aligned}$$



Left boundary points
$PL_8 = 93$



Right boundary points
$PR_8 = 129$

The curve unfolding means boundary points PB in level k become right boundary points PR in k+1. Points on both the left and right boundaries in PB are separated by the unfold, except those at the curve end which continue to touch. For $k \ge 6$, they are 4 points of a hanging quad.

$$Pboth_k = \begin{cases} 2, 3, 5, 9, 13, 19 & k = 0 \,\text{to}\, 5 \\ 8k - 22 & k \ge 6 \end{cases} \qquad \begin{array}{c}\text{points on both}\\ \text{left and right boundary}\end{array}$$

$$PR_{k+1} \;=\; PB_k + Pboth_k - \begin{cases} 1, 1, 1, 2, 2, 3 & \text{if } k = 0 \,\text{to}\, 5 \\ 4 & \text{if } k \ge 6 \end{cases} \tag{364}$$

Relation (113) between left and right boundary squares has an equivalent for points,

$$2\,PR_k = PL_{k+2} - \begin{cases} 1 & \text{if } k{=}0 \\ 3 & \text{if } k \ge 1 \end{cases}$$

Left plus right boundary points exceed total boundary points by $Pboth$ but it is only linear so the limits of left and right over total boundary approach 1.

$$PL_k + PR_k \;=\; PB_k + Pboth_k$$

$$\frac{PL_k}{PB_k} + \frac{PR_k}{PB_k} \to 1 \qquad \text{as } k \to \infty$$

The proportion of left or right boundary points out of total points are the same as left or right boundary segments out of total segments at end of section 3.3. The *PR* limit is from (364). The *PL* limit follows from writing $PL_k = PB_k - PB_{k-1} + 12$ for $k \geq 7$.

$$\frac{PR_k}{PB_k} \to \frac{1}{r} = 0.589754\ldots \qquad \frac{PL_k}{PB_k} \to 1 - \frac{1}{r} = 0.410245\ldots$$

13.1.1 Twindragon Boundary Points

Twindragon boundary points can be found similarly as boundary segments less right-side hanging squares *HQAR*. At curve start and end there are up to 6 boundary points of the two $k{+}1$ component dragons which touch and are to be unduplicated for the total.

$$\begin{aligned} TPB_k &= TB_k - 2\,HQAR_{k+1} - \begin{cases} 0,1,3,5 & \text{if } k = 0 \,\text{to}\, 3 \\ 6 & \text{if } k \geq 4 \end{cases} && \text{boundary} \\ &= 4, 7, 13, 25, 46, 82, 144, 250, 428, 730, \ldots \\ TPE_k &= TP_k - TPB_k && \text{enclosed} \\ &= 0, 0,\ \ 0,\ \ 0,\ \ 1,\ \ 7,\ \ 27,\ \ 79, 207, 503, \ldots \end{aligned}$$

The first enclosed point of the twindragon occurs in $k{=}4$ as $TPE_4 = 1$.

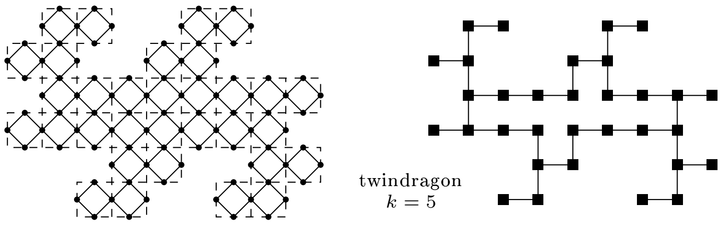
The boundary points *TPB* are either single or double-visited. All single-visited points (theorem 53) are on the boundary so the double-visited boundary points by difference two ways

$$\begin{aligned} TDB_k &= TD_k - TPE_k \quad = TPB_k - TS_k \qquad \text{double-visited boundary} \\ &= 0, 1, 3, 7, 16, 32, 58, 104, 182, 312, \ldots \end{aligned}$$

Theorem 83. *For $k \geq 4$, the number of double-visited boundary points on the twindragon is equal to the number of complex base 8-side boundary squares from* (270)*.*

$$TDB_k = EightB_k - (1 \text{ if } k < 4)$$

Proof. The unit squares of complex base $i{+}1$ correspond to diamonds of the twindragon and where the sides of unit squares touch is a double-visited twindragon point. The double-visited boundary points are those touching sides where one end of the side is on the boundary.



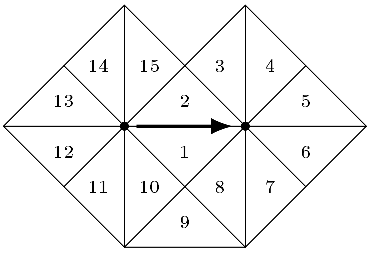
Construct a graph where vertices are the unit squares with at least one corner on the boundary. Such a square shares a side with 1 or more neighbours. These sides are where the twindragon diamonds connect. Graph edges are across these touching sides. So the vertices are 8-side boundary unit squares and edges are twindragon double-visited boundary points.

A touching side can have both ends on the boundary (two corners of the square). These are the hanging squares (diamonds) of the twindragon. They form little trees off a cycle around the rest of the complex base shape. There is a single cycle since only hanging squares have a single corner attachment to the rest of the curve.

For $k \geq 4$, there is a cycle and such a graph has $\textit{vertices} = \textit{edges}$ so $\textit{TDB} = \textit{EightB}$. For $k < 4$, there is only a tree which is $\textit{vertices} = \textit{edges}+1$ so $\textit{TDB} = \textit{EightB} - 1$. These are cases of Euler's formula $\textit{vertices}+\textit{regions} = \textit{edges}+2$ for a connected planar graph. Here the cycle divides the plane into $\textit{regions} = 2$, an inside and an outside. ☐

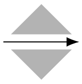
14 Triangles in Regions

Consider the following regions around the ends of the dragon curve. Number the regions the same as Duvall and Keesling use for the Lévy C curve.



TR
region numbers
around curve
start to end

Treat each segment of the dragon curve as having a triangle on each side. These triangles fall variously into the regions. The number in each region can be counted as a density measure.



Theorem 84. *The number of triangles in each region above for dragon curve k is*

$$TR(k,1) = \begin{cases} 1,2,3,4 & k = 0 \,\text{to}\, 3 \\ \frac{53}{100}2^k + k\,[\frac{-3}{10},\frac{2}{5},\frac{3}{10},\frac{-2}{5}] \;+[\frac{43}{25},\frac{1}{25},\frac{-43}{25},\frac{-1}{25}] & k \geq 4 \end{cases} \tag{365}$$

$$TR(k,2) = \begin{cases} 1,0,1,3 & k = 0 \,\text{to}\, 3 \\ \frac{11}{50}2^k - k\,[\frac{-3}{10},\frac{2}{5},\frac{3}{10},\frac{-2}{5}] \;-[\frac{43}{25},\frac{1}{25},\frac{-43}{25},\frac{-1}{25}] & k \geq 4 \end{cases}$$

$$TR(k,8) = \begin{cases} 0,1,0,1 & k = 0 \,\text{to}\, 3 \\ \frac{61}{300}2^k + k\,[\frac{1}{20},\frac{7}{20},\frac{-1}{20},\frac{-7}{20}] \;+[\frac{41}{75},\frac{-377}{300},\frac{-107}{150},\frac{427}{300}] & k \geq 4 \end{cases}$$

$$TR(k,9) = \begin{cases} 0 & k = 0 \,\text{to}\, 3 \\ \frac{1}{20}2^k + [\frac{-4}{5},\frac{-3}{5},\frac{4}{5},\frac{3}{5}] & k \geq 4 \end{cases}$$

$$TR(k,10) = \begin{cases} 0,1,3,3 & k = 0 \text{ to } 3 \\ \tfrac{113}{300}2^k + k\,[\tfrac{-7}{20},\tfrac{1}{20},\tfrac{7}{20},\tfrac{-1}{20}] + [\tfrac{28}{75},\tfrac{209}{300},\tfrac{-31}{150},\tfrac{-259}{300}] & k \geq 4 \end{cases}$$

$$TR(k,11) = \begin{cases} 0,0,1,3 & k = 0 \text{ to } 3 \\ \tfrac{16}{75}2^k + k\,[\tfrac{-1}{20},\tfrac{-7}{20},\tfrac{1}{20},\tfrac{7}{20}] + [\tfrac{-16}{75},\tfrac{-23}{300},\tfrac{7}{150},\tfrac{73}{300}] & k \geq 4 \end{cases}$$

$$\begin{aligned} TR(k,3) &= TR(k{-}1,11) & TR(k,6) &= TR(k{-}4,11) \\ TR(k,4) &= TR(k{-}2,11) & TR(k,7) &= TR(k{-}5,11) \\ TR(k,5) &= TR(k{-}3,11) \\ TR(k,12) &= TR(k{-}1,11) & TR(k,14) &= TR(k{-}3,11) \\ TR(k,13) &= TR(k{-}2,11) & TR(k,15) &= TR(k{-}4,11) \end{aligned}$$

$$\begin{aligned}
TR(k,1) &= 1,2,3,4,9,19,34,65,135,275,544,1081,\ldots \\
TR(k,2) &= 1,0,1,3,3,5,14,31,57,109,224,455,\ldots \\
TR(k,3) &= 0,0,0,1,3,3,5,14,30,54,106,219,441,\ldots \\
TR(k,4) &= 0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,5) &= 0,0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,6) &= 0,0,0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,7) &= 0,0,0,0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,8) &= 0,1,0,1,4,7,12,25,53,106,207,414,\ldots \\
TR(k,9) &= 0,0,0,0,0,1,4,7,12,25,52,103,204,\ldots \\
TR(k,10) &= 0,1,3,3,5,13,26,47,94,194,389,770,1539,\ldots \\
TR(k,11) &= 0,0,1,3,3,5,14,30,54,106,219,441,873,\ldots \\
TR(k,12) &= 0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,13) &= 0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,14) &= 0,0,0,0,0,1,3,3,5,14,30,54,106,\ldots \\
TR(k,15) &= 0,0,0,0,0,0,1,3,3,5,14,30,54,106,\ldots
\end{aligned}$$

Generating functions

$$gTR1(x) = \tfrac{53}{100}\frac{1}{1-2x} + \tfrac{1}{25}\frac{58-9x}{1+x^2} - \tfrac{1}{5}\frac{3-4x}{(1+x^2)^2} \quad - \tfrac{5}{4} + \tfrac{1}{2}x + 2x^2 + x^3$$

$$gTR2(x) = \tfrac{11}{50}\frac{1}{1-2x} - \tfrac{1}{25}\frac{58-9x}{1+x^2} + \tfrac{1}{5}\frac{3-4x}{(1+x^2)^2} \quad + \tfrac{5}{2} - x^2$$

$$gTR8(x) = \tfrac{61}{300}\frac{1}{1-2x} + \tfrac{1}{100}\frac{53-169x}{1+x^2} + \tfrac{1}{10}\frac{1+7x}{(1+x^2)^2} - \tfrac{1}{12}\frac{1}{1+x} \quad -\tfrac{3}{4}+\tfrac{3}{2}x-x^3$$

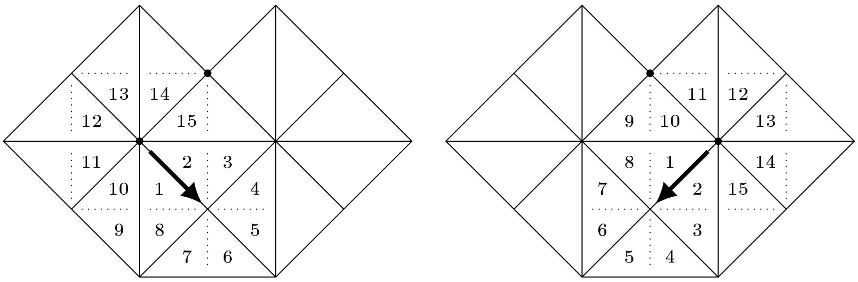
$$gTR9(x) = \tfrac{1}{20}\frac{1}{1-2x} \quad - \tfrac{1}{5}\frac{4+3x}{1+x^2} \quad + (\tfrac{3}{4} + \tfrac{1}{2}x - x^2 - x^3)$$

$$gTR10(x) = \tfrac{113}{300}\frac{1}{1-2x}+\tfrac{1}{12}\frac{1}{1+x}+\tfrac{1}{100}\frac{99+73x}{1+x^2}+\tfrac{1}{10}\frac{-7+x}{(1+x^2)^2} \quad -(\tfrac{3}{4}+\tfrac{1}{2}x-x^2-x^3)$$

$$gTR11(x) = \tfrac{16}{75}\frac{1}{1-2x} - \tfrac{1}{12}\frac{1}{1+x} + \tfrac{1}{100}\frac{-3+19x}{1+x^2} - \tfrac{1}{10}\frac{1+7x}{(1+x^2)^2}$$

$$\begin{aligned} gTR3(x) &= x.gTR11(x) & gTR6(x) &= x^4.gTR11(x) \\ gTR4(x) &= x^2.gTR11(x) & gTR7(x) &= x^5.gTR11(x) \\ gTR5(x) &= x^3.gTR11(x) \\ gTR12(x) &= x.gTR11(x) & gTR14(x) &= x^3.gTR11(x) \\ gTR13(x) &= x^2.gTR11(x) & gTR15(x) &= x^4.gTR11(x) \end{aligned}$$

Proof. k=0 is a single line segment with a triangle on each side which are regions 1 and 2 so $TR(0,1) = TR(0,2) = 1$. Thereafter a curve level k consists of two level k−1 curves directed as



The total triangles in each region are the triangles from the two k−1 sub-curves which fall in it. For example in the bottom middle region which is 9 there are triangles from k−1 regions 7 and 6 of the left sub-curve and regions 5 and 4 of the right sub-curve. So a set of 15 mutual recurrences

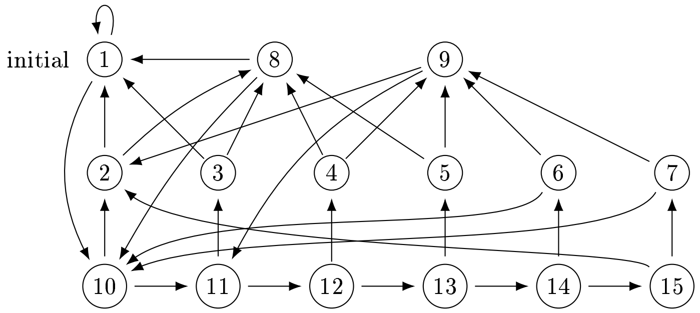
$$\begin{aligned} TR(k,1) &= TR(k{-}1,\ \ 1) + TR(k{-}1,2) + TR(k{-}1,3) + TR(k{-}1,8) \\ TR(k,2) &= TR(k{-}1,\ \ 9) + TR(k{-}1,10) + TR(k{-}1,15) \\ TR(k,3) &= TR(k{-}1,11) \\ TR(k,4) &= TR(k{-}1,12) \\ TR(k,5) &= TR(k{-}1,13) \\ TR(k,6) &= TR(k{-}1,14) \\ TR(k,7) &= TR(k{-}1,15) \\ TR(k,8) &= TR(k{-}1,2) + TR(k{-}1,3) + TR(k{-}1,4) + TR(k{-}1,5) \\ TR(k,9) &= TR(k{-}1,4) + TR(k{-}1,5) + TR(k{-}1,6) + TR(k{-}1,7) \\ TR(k,10) &= TR(k{-}1,1) + TR(k{-}1,6) + TR(k{-}1,7) + TR(k{-}1,8) \\ TR(k,11) &= TR(k{-}1,9) + TR(k{-}1,10) \\ TR(k,12) &= TR(k{-}1,11) \\ TR(k,13) &= TR(k{-}1,12) \\ TR(k,14) &= TR(k{-}1,13) \\ TR(k,15) &= TR(k{-}1,14) \end{aligned}$$

Repeated substitution or a little linear algebra gives the power forms (365) for each. The generating functions follow from the power forms. ☐

There are 2 triangles per segment so total of all regions is

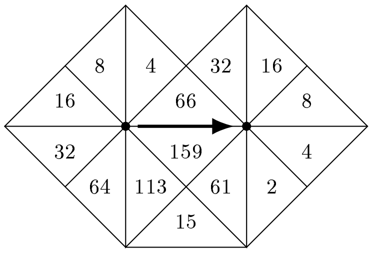
$$2.2^k = \sum_{j=1}^{15} TR(k,j)$$

The way the recurrences form the next level from copies of the preceding can be illustrated,



Notice 11 propagates to 3, 12, 13, 14, 15, and from them in turn to 4, 5, 6, 7. Each of those receive nothing else so give region 11 delayed. But 11 is not a delayed 10 because it also receives 9.

The proportions as $k{\to}\infty$ are the factors of 2^k in the power formulas or $1/(1{-}2x)$ terms in the generating functions. With common denominator 300 they are

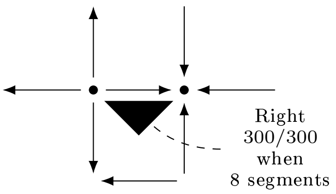


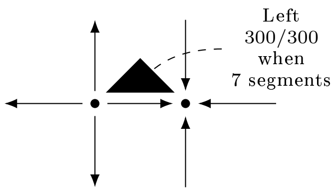
$$\frac{1}{300} \text{ density as } k \to \infty$$

$$\left(\text{total } \frac{600}{300}\right)$$

Full weight 300/300 occurs on the right side of a segment when there are a total 8 segments all contributing their fractions. This occurs first in $k{=}5$ at $n = 21$.

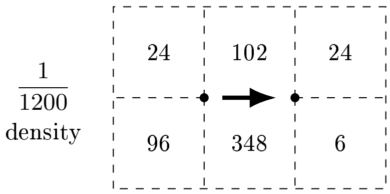
Full weight 300/300 occurs on the left side of a segment when there are 7 surrounding segments all contributing their fractions. This occurs first in $k{=}5$ at $n = 14$ and $n = 21$.

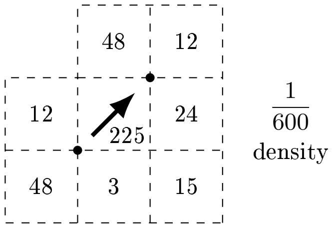




One use for these densities could be in computer graphics to approximate the fractal by some grey-scale colours at the limit of desired resolution.

If a line segment is the side of a square pixel then that line contributes to 6 surrounding pixels. If a line segment is a diagonal across a pixel then it contributes to 8 surrounding pixels. When curve start to end is horizontal, these straight and diagonal cases are k even or odd respectively.

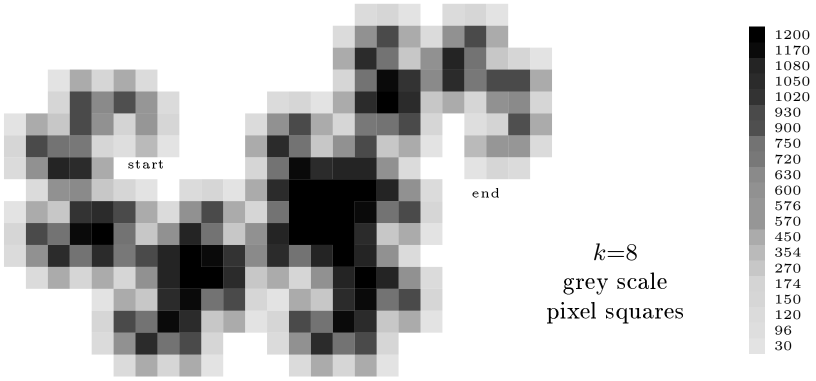




A full-weight straight square is 4 full triangles so total 1200. A full-weight diagonal square is 2 full triangles so total 600. The straight weights have common factor 6 and the diagonal weights have common factor 3. Dividing them out would be full weight 200 in both cases.

In practice, the effect is only a few pixels fuzz at the boundary, and grey down the thin curve start and end. This is the sort of thing general sub-pixel grey-scaling can do, but an exact calculation has the attraction of accurately representing the amount of fractal in each pixel.

For straight alignment, only 21 different net weights occur. The weights near full weight are too close to distinguish by eye in adjacent pixels. The following sample has a non-linear spread of greys to emphasise the pattern.



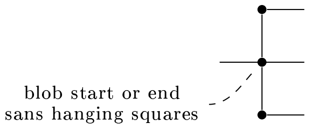
15 Graphs and Trees

15.1 Dragon Graph

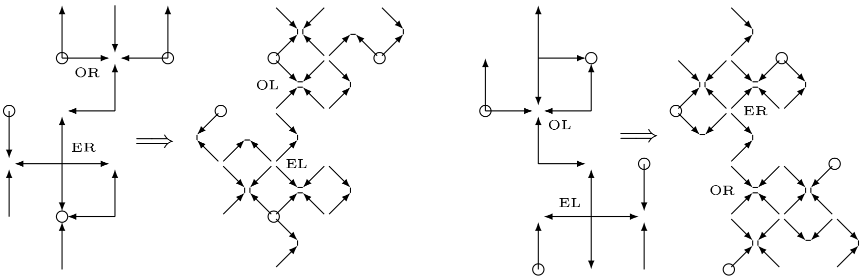
The dragon curve as a graph is, by its construction, a planar unit distance graph and has an Euler path from start to end (traverse all edges once). It is bipartite, like any graph on a square grid, since vertices can be separated into those with coordinates $x+y$ odd or even.

The curve has no Hamiltonian path start to end (visit all vertices once) for $k \geq 4$ since the vertices in hanging squares cannot be visited without repeating the vertex they attach to.

If hanging squares are removed then for $k \geq 5$ there is still no Hamiltonian path since start and end of each blob $\geq$5 then have two degree-2 neighbours.



A path entering or leaving at the centre vertex shown cannot visit both the upper and lower. Just one such blob start or end in the graph would make it non-Hamiltonian but in fact every blob ≥ 5 starts and ends this way, as can be seen by the way the bridge and adjacent segments expand. The following diagram is like figure 60 but more adjacent segments.



These patterns of additional segments begin at OL--EL bridge which is blob 5 to blob 6 and is shown on the right here.

Points circled are on the boundary. They remain so in the expansion and they show segments which are then in turn on the boundary. Taking a subset of those segments suffices for the next expanded forms. The 3-vertex start/end form is seen in the expansions directly at OL and OR. And at EL and ER with the hanging square beside them removed.

The start and end of the curve are degree 1 vertices, other single-visited points are degree 2, and double-visited points are degree 4.

$$\begin{aligned} &\mathit{DegCount}(k,1) = 2\\ &\mathit{DegCount}(k,2) = S_k - 2 \quad = 0,1,3,7,13,23,41,71,121,207,\ldots\\ &\mathit{DegCount}(k,4) = D_k \end{aligned}$$

Counts of edges with vertices of degree 1, 2 or 4 at each end can be made too, using counts of 1, 2 and 3-side boundary squares from (124).

Theorem 85. *The number of edges with degree 1, 2 or 4 vertices at each end are*

$$\mathit{EdgeCount}(k,\text{1,1}) = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{if } k \geq 1 \end{cases}$$

$$\mathit{EdgeCount}(k,\text{1,2}) = \begin{cases} 0 & \text{if } k=0 \\ 2 & \text{if } k \geq 1 \end{cases}$$

$$\begin{aligned} \mathit{EdgeCount}(k,\text{2,2}) &= \begin{cases} 0 & \text{if } k=0,1 \\ \mathit{BQ1}_k + 2k - 5 & \text{if } k \geq 2 \end{cases}\\ &= 0,0,2,6,10,16,26,40,\;\; 62,100,162,264,\ldots \end{aligned}$$

$$\mathit{EdgeCount}(k,\,2{,}4) = \begin{cases} 0 & \textit{if } k = 0,1 \\ 2BQ_k - 2BQ\mathit{1}_k - 4k + 4 & \textit{if } k \geq 2 \end{cases} \tag{366}$$

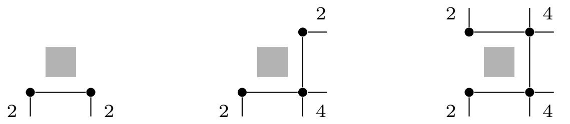
$$= 0,0,0,0,\;\; 4,12,28,60,116,212,380,668,\ldots$$

$$\mathit{EdgeCount}(k,\,4{,}4) = \begin{cases} 0 & \textit{if } k = 0,1 \\ 2^k - 2BQ_k + BQ\mathit{1}_k & \textit{if } k \geq 2 \end{cases}$$

$$= 0,0,0,0,\;\; 0,\;\; 2,\;\; 8,26,\;\; 76,198,480,1114,\ldots$$

Proof. The curve start and end are degree 1 and are connected to each other for k=0 or to a degree 2 vertex for $k \geq 1$.

All other single-visited vertices are degree 2. Per theorem 32, they are where boundary squares touch. All 1-side boundary squares give 2,2 edges and all 2 or 3-side boundary squares with all sides on a blob have double-visited points in between and so do not give 2,2 edges.



1-side,
2,2 edge

2-side on blob,
two 2,4 edges

3-side on blob,
two 2,4 edges

Figure 71: graph edges around boundary squares

2 or 3-side boundary squares with sides not on a blob but instead on a bridge or the curve start or end may give 2,2 edges. In each bridge there is one 2,2 edge, and for $k \geq 5$ the start and end segments are unchanging giving

$$\mathit{EdgeCount}(k,2{,}2) = BQ\mathit{1}_k + \mathit{BridgeCount}_k + 4 \qquad\qquad k \geq 5$$

For $k < 5$, the 2,2 edges can be counted explicitly which is the theorem.

The 2 or 3-side boundary squares with all sides on a blob give two 2,4 edges each. The squares with one side not on a blob are immediately before and after each blob and they would count the 2,4 edge there twice, so subtract $2\mathit{BlobCount}$. The squares with all sides not on a blob are excluded entirely since they are 2,2 edges at the start and end of the curve. So

$$\mathit{EdgeCount}(k,2{,}4) = 2BQ\mathit{2}_k + 2BQ\mathit{3}_k - 2\mathit{BlobCount}_k - 12 \qquad k \geq 5$$

Or 2,4 edges can be calculated by difference from the 2,2 edges. There are $\mathit{DegCount}(k,2) = BQ_k{-}2$ vertices of degree 2. The 2 edges at each such vertex might go to a degree 1, 2 or 4 vertex. Subtracting the edges to degree 1 or 2 is form (366) of the theorem.

$$\mathit{EdgeCount}(k,2{,}4) = 2(BQ_k{-}2) - 2\mathit{EdgeCount}(k,2{,}2) - 2\mathit{EdgeCount}(k,1{,}2)$$

4,4 edges are the remainder out of the total 2^k,

$$\begin{aligned} 2^k = \quad & \mathit{EdgeCount}(k,1{,}1) + \mathit{EdgeCount}(k,1{,}2) \\ & + \mathit{EdgeCount}(k,2{,}2) + \mathit{EdgeCount}(k,2{,}4) + \mathit{EdgeCount}(k,4{,}4) \qquad \square \end{aligned}$$

All edges with a degree 1 or 2 vertex are on the boundary. Some 4,4 edges are on the boundary and some are enclosed. In graph terms, an enclosed edge is part of two 4-cycles. The boundary 4,4 edges are the middle segment of each 3-side boundary square with all sides on a blob as shown in figure 71. So *BQ3* excluding squares on bridges and curve start and end.

$$\begin{aligned} \mathit{EdgeCount}(k, 4{,}4, \mathit{boundary}) &= \begin{cases} 0 & \text{if } k < 5 \\ \mathit{BQ3}_k - 4k + 12 & \text{if } k \geq 5 \end{cases} \\ &= 0, 0, 0, 0, 0, 1, 5, 13, 29, 61, 117, 213, \ldots \end{aligned}$$

The total of these 4,4 edges and the degree 1 and 2 edges are the boundary segments, counting a segment on both left and right boundary just once. Segments on both left and right are the non-blob segments so

$$\begin{aligned} &\mathit{EdgeCount}(k, 1{,}1) + \mathit{EdgeCount}(k, 1{,}2) \\ &+ \mathit{EdgeCount}(k, 2{,}2) + \mathit{EdgeCount}(k, 2{,}4) \\ &+ \mathit{EdgeCount}(k, 4{,}4, \mathit{boundary}) \quad = B_k - \mathit{NonBlobSegments}_k \end{aligned}$$

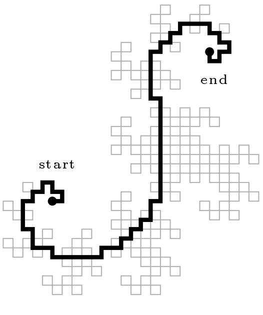
The enclosed 4,4 edges are simply all segments not on the boundary. Again noting that B_k counts both sides of the non-blob segments,

$$\begin{aligned} \mathit{EdgeCount}(k, 4{,}4, \mathit{enclosed}) &= 2^k - (B_k - \mathit{NonBlobSegments}_k) \\ &= \begin{cases} 0 & \text{if } k < 5 \\ 2^k - B_k + 6k - 13 & \text{if } k \geq 5 \end{cases} \\ &= 0, 0, 0, 0, 0, 1, 3, 13, 47, 137, 363, 901, \ldots \end{aligned}$$

Various graph-theoretic topological indices are based on total edges by their vertex degrees. The 4,4 edges are a power 2^k whereas the others go only as the cubic r^k (section 2) so limits for an average index over edges go just as the 2^k many 4,4.

Theorem 86. *The diameter of dragon curve k considered as a graph is uniquely attained from curve start to end crossing each blob straight and stair-step. The length of this path is*

$$\begin{aligned} \mathit{Diameter}_k &= \begin{cases} 1 & \mathit{if}\ k = 0 \\ [3, 4].2^{\left\lfloor \frac{k}{2} \right\rfloor} + k - [4, 3] & \mathit{if}\ k \geq 1 \end{cases} \\ &= 1, 2, 4, 8, 12, 18, 26, 36, 52, 70, 102, 136, 200, 266, \ldots \end{aligned}$$



dragon graph k=9

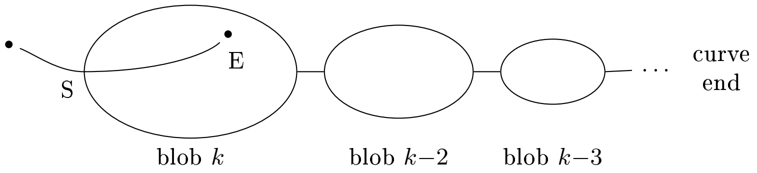
shortest path
start to end

$\mathit{Diameter}_9 = 70$

Proof. The path start to end is similar to *GeomLength* from section 12.1, but taking the blob crossings stair-step so distance $\mathit{Manhattan}(z) = \operatorname{Re} z + \operatorname{Im} z$.

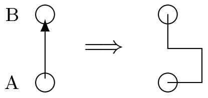
$$\mathit{Diameter}_k = \mathit{NonBlobSegments}_k + \sum_{j=\mathit{BlobList}_k} \mathit{Manhattan}\big(\mathit{BlobDelta}_j\big)$$

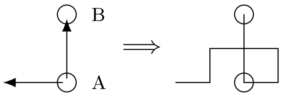
To show this distance is in fact the diameter, consider first a prospective path ending in the middle biggest blob k. It can begin anywhere earlier, including within blob k itself.



S is the start of blob k (at $\mathit{BlobStart}_k$). Path S to E is at most the blob diameter. If the prospective path begins within blob k then the whole prospective path is also at most the blob diameter.

An upper bound for blob diameter can be established from the way line segments expand. Any pair of vertices in a blob has some path between them of length at most the blob diameter. Consider a segment A--B which is part of such a path. After two expansions, those points are distance 2 apart. If there is a another segment on the left side of A then it fills the gap so there are 2 segments to go A to B.





If there is no segment on the left then segment A--B is on the boundary. There is no segment straight below since the boundary never goes straight ahead. But there is a segment to the right since A--B is in a blob and so part of an enclosed unit square.

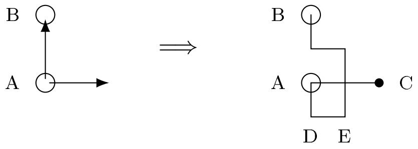


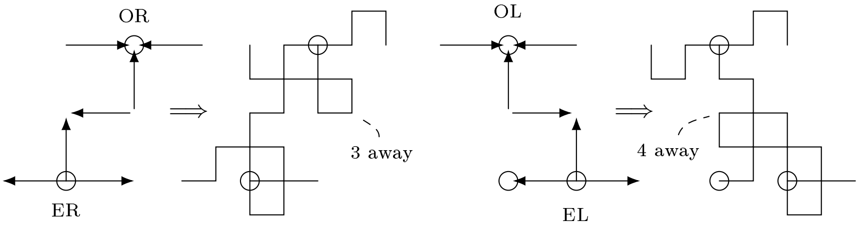
Figure 72

If A is not the endpoint of the path then that path must go C--A--B and a stair-step C to B is 4 segments so still 2× the original. If A and B are both endpoints then they are a fixed 4 apart. The initial values for the diameter upper bounds used below always exceed 4.

If A is an endpoint of some path then the gap A to B is an extra +2. So after two expansions all original points can be reached by at most 2× and at each end +2.

New vertices created in the expansion are 1 or 2 segments away from an original. In figure 72, new vertex D beside an A endpoint is +3 at that end. New vertex E beside an A endpoint can go 4 segments to B which is a net +2 by skipping A. The same if A not an endpoint but C is. So all original and new vertices are separated by at most 2× and at each end +3.

Extra vertices are added to the blob at its start and end when the bridges expand to enclose new unit squares. The following diagram shows how the end types expand twice.



When OR expands there is a new vertex +3 away from the original OR. All blob starts and ends are double-visited, so three is no need to consider the single visited cases of A described above.

When EL expands there is a new vertex +4 away. The crossing diagrams above show that a blob with OR start and EL end occurs (third case) so the diameter of the expanded blob is potentially as much as total +7 segments longer than between originals, but no more than that.

$$\begin{aligned}
\mathit{BlobDiameter}_k &\le 2\mathit{BlobDiameter}_{k-2} + 7 \\
\mathit{BlobDiameterBound}_k &= [\tfrac{43}{32}, \tfrac{59}{32}].2^{\lfloor k/2 \rfloor} - 7 \qquad \text{for } k \ge 10 \qquad (367) \\
&\text{starting } \mathit{BlobDiameter}_{10} = 36 \text{ and } \mathit{BlobDiameter}_{11} = 52
\end{aligned}$$

The shortest path from S to the curve end is the sum of the blob crossings and the non-blob segments through to the curve end. Per *BlobList* there is no blob $k{-}1$ at the curve end so that size is skipped in the sum.

$$\begin{aligned}
\mathit{TailLen}_k = &\sum_{j=k,\,k-2,\,k-3,\,\dots,\,4} \mathit{Manhattan}\big(\mathit{BlobDelta}_j\big) \\
&+ 3\,(k-5) \quad \text{if } k \ge 6 \qquad \text{bridges} \\
&+ \begin{cases} 5 & \text{if } k = 4 \\ 7 & \text{if } k \ge 5 \end{cases} \qquad \text{end segments} \\
= &\,[\tfrac{5}{3}, 2].2^{\lfloor k/2 \rfloor} + \tfrac{1}{2}k - [\tfrac{11}{3}, \tfrac{5}{2}, \tfrac{13}{3}, \tfrac{5}{2}] \\
= &\,5, 8, 12, 17, 27, 34, 54, 67, 109, 132, \dots \qquad k \ge 4
\end{aligned}$$

For k=4 to 9, the blob diameters are calculated explicitly $2, 5, 8, 12, 18, 26$ and each is < the corresponding *TailLen*. For $k \ge 10$, *TailLen* exceeds the blob diameter bound formula (367),

$$\mathit{TailLen}_k - \mathit{BlobDiameter}_k \ge [\tfrac{31}{96}, \tfrac{5}{32}].2^{\lfloor k/2 \rfloor} + \tfrac{1}{2}k + [\tfrac{10}{3}, \tfrac{9}{2}, \tfrac{8}{3}, \tfrac{9}{2}] \;\; > 0$$

So a path to the curve end is longer than anything ending in middle biggest blob k.

If the prospective path ends in one of the later end blobs $k{-}2$ etc then the same calculation applies but *TailLen* does not skip a size in its sum (the $k{-}1$

term) and so is bigger.

If the prospective path ends in one of the earlier start blobs 4 to $k-1$ then the *TailLen* sum is bigger and the blob diameter is smaller.

For the start of the path, the same argument applies and *TailLen* to the curve start does not skip size $k-1$, so a path from the curve start is longer than from anywhere else. $\square$

It can be noted a single blob k crossing length $\mathit{Manhattan}(\mathit{BlobDelta})$ is not enough to exceed the blob diameter bound (367), nor, in general, enough to exceed the actual blob diameter (the blob diameter is not attained by that start to end crossing). But the sum of such blob crossings and the non-blob segments through to the curve end are enough.

The blob diameter bound (367) used starts $k{\ge}10$ to ensure *TailLen* is bigger. The coefficients of $2^{\lfloor k/2 \rfloor}$ in the bound depend on the initial values and they decrease on going up through successive actual blob diameters as initial values.

It would be possible to calculate the eccentricity of blob start S instead of the blob diameter. Its values are smaller than the blob diameter in general, giving a smaller bound. A prospective path starting and ending in blob k still needs a bound for the blob diameter but it would be compared to the whole curve $\mathit{Diameter}_k$ not just $\mathit{TailLen}_k$.

Taking a *Diameter* increment flattens factors 3 and 4 to a single $2^{\lceil k/2 \rceil}$.

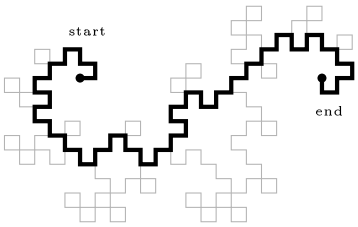
$$\begin{aligned} \mathit{dDiameter}_k &= \mathit{Diameter}_{k+1} - \mathit{Diameter}_k \\ &= \begin{cases} 1 & \text{if } k = 0 \\ 2^{\lceil k/2 \rceil} + [2, 0] & \text{if } k \ge 1 \end{cases} \\ &= 1,\ 2, 4,\ 4, 6,\ 8, 10,\ 16, 18,\ 32, 34,\ 64, 66, \dots \qquad {\scriptstyle k\ge1\ \ 2\times\text{A228693}} \end{aligned}$$

Another longest graph path can be made considering only boundary segments. The effect is to go around the shorter side of each blob skipping hanging squares.

Dragon k=8

Path start to end boundary segments

$\mathit{BoundaryPathLen}_8 = 64$



Theorem 87. *The curve start and end are the points furthest apart going only by boundary segments. For dragon curve level k, they are at distance*

$$\begin{aligned} \mathit{BoundaryPathLen}_k &= \mathit{NonBlobSegments}_k + \sum_{j=\mathit{BlobList}_k} \mathit{BlobLnohang}_j \\ &= \begin{cases} 2^k & \textit{if } k < 4 \\ \frac{1}{8}\mathit{BlobBnohang}_{k+4} + 4k - [\frac{17}{2}, \frac{31}{4}, 9, \frac{31}{4}] & \textit{if } k \ge 4 \end{cases} \qquad (368) \\ &= 1, 2, 4, 8, 12, 20, 28, 44, 64, 96, 148, 236, 376, \dots \end{aligned}$$

$$\begin{aligned}\mathit{gBoundaryPathLen}(x) &= -13\frac{1}{1-x} + 4\frac{1}{(1-x)^2} - \tfrac{1}{2}\frac{1}{1+x} + \tfrac{1}{2}\frac{1+2x}{1-x-2x^3}\\ &\quad + 10 + 5x + 4x^2 + 2x^3\end{aligned}$$

where BlobLnohang and BlobBnohang are blob boundary lengths without hanging squares

$$\begin{aligned}\mathit{BlobLnohang}_k &= \mathit{BlobL}_k - 4\mathit{BlobHQAL}_k\\ &= 0,3,2,\ 7,12,19,\ 34,\ 63,100,171,\ldots \qquad k \ge 4\\ \mathit{BlobBnohang}_k &= \mathit{BlobB}_k - 4\mathit{BlobHQA}_k \qquad\qquad (369)\\ &= 0,6,8,14,36,62,104,190,324,542,\ldots \qquad k \ge 4\end{aligned}$$

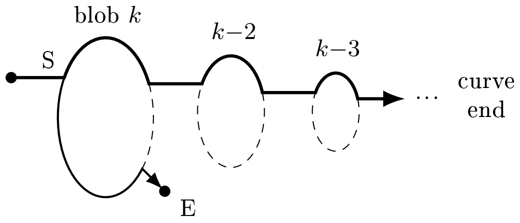
Proof. The path from start to end must traverse all the non-blob segments. For each blob, it can go around either left or right. The left is always shorter since some explicit calculation for $k < 9$ and then some recurrence or generating function manipulations gives a difference

$$\begin{aligned}\mathit{BlobRnohang}_k &= \mathit{BlobR}_k - 4\mathit{BlobHQAR}_k\\ &= 0,3,6,7,24,43,70,127,224,371,\ldots \qquad k \ge 4\end{aligned}$$

$$\begin{aligned}\mathit{BlobRnohang}_k - \mathit{BlobLnohang}_k &= \begin{cases} 0,0,4,0,12 & k = 4 \text{ to } 8\\ 4\mathit{BlobR}_{k-5} + [16,8,8,8] & k \ge 9\end{cases}\\ &\ge 0\end{aligned}$$

The *BlobBnohang* form (368) for *BoundaryPathLen* is reached by further recurrence or generating function manipulations to sum *BlobLnohang*.

Suppose a prospective path ends in the middle biggest blob k. It can start anywhere earlier, including somewhere on that blob.



S is the start of blob k. The distance from there to the end of the curve is

$$\begin{aligned}\mathit{BoundaryTail}_k &= \mathit{BlobLnohang}_k + \sum_{j=4}^{k-2}\big(\mathit{BlobLnohang}_j + 3\big)\\ &\quad + \begin{cases} 5 & \text{if } k{=}4\\ 7 & \text{if } k \ge 5\end{cases}\\ &= 5,10,12,23,33,50,80,131,205,342,\ldots \qquad k \ge 4\end{aligned}$$

E is the end of the prospective path. It goes around the blob boundary a distance at most $\frac{1}{2}\mathit{BlobBnohang}_k$ and then might go into hanging squares. For $k{=}4$ and $k{=}5$, it can be verified explicitly that *BoundaryTail* exceeds the maximum E. For $k \ge 6$, the maximum extra into a hanging quad is +5. The difference between *BoundaryTail* and that distance is positive by yet further recurrence or generating function manipulations,

$$\begin{aligned}
&\mathit{BoundaryTail}_k - \left(\tfrac{1}{2}\mathit{BlobBnohang}_k + 5\right) \\
&= \frac{1}{4}\left(\begin{array}{l} dJA_{k-3} + 18dJA_{k-4} - 10dJA_{k-5} \\ + 8(k-6) \ + [15,21,11,17] \end{array}\right) \qquad\qquad k \ge 6 \\
&> 0 \qquad \text{since } dJA \text{ non-decreasing}
\end{aligned}$$

Then similar to the proof above a path ending in a later blob also has the curve end further than anywhere in the blob, and likewise at the start of a path the curve start is further. ☐

If the hanging squares are included in the boundary path, so a path is forced to go all the way around the boundary, not cutting across hanging attachment points, then a similar calculation gives curve start and end furthest apart for such paths.

15.1.1 Dragon Graph Cheeger Constant

For a graph with vertex set V, the Cheeger constant is defined as

$$\min \frac{|\text{edges } A \text{ to } V\backslash A|}{|A|} \qquad \text{for } A \subset V \text{ and } |A| \le |\tfrac{1}{2}V|$$

A small value represents a narrow bottleneck, expressing how few edges there are connecting some vertex subset A to the rest of the graph (vertices $V\backslash A$), as a fraction of the vertices in A. A is at most half the vertices so the ratio is taken over the vertices on the smaller side of the bottleneck.

Theorem 88. *The Cheeger constant of dragon curve k as a graph is attained at the edge preceding middle biggest blob k, or for k≤3 in the middle of the curve.*

$$\mathit{Cheeger}_k = \begin{cases} 1, 1, \frac{1}{2}, \frac{1}{4} & \textit{for } k = 0 \text{ to } 3 \\ \frac{1}{\mathit{BlobPrecedingP}_k} & \textit{for } k \ge 4 \end{cases}$$

where BlobPrecedingP$_k$ *is the number of vertices (distinct points) before blob k, and not including any of that blob*

$$\begin{aligned}
\mathit{BlobPrecedingP}_k &= 7 + \sum_{j=4}^{k-1} \left(\mathit{BlobP}_j + 2\right) \qquad\qquad k \ge 4 \\
&= \tfrac{1}{5}2^k + \tfrac{1}{8}\mathit{BlobB}_{k+2} + \tfrac{1}{20}[26,37,54,43] \qquad (370) \\
&= 7, 13, 24, 43, 79, 149, 284, 543, 1043, 2017, \ldots \qquad k \ge 4
\end{aligned}$$

$$\begin{aligned}
\mathit{gBlobPrecedingP}(x) &= x^4 \frac{7}{1-x} + \frac{1}{1-x}\left(x\,\mathit{gBlobP}(x) + x^5 \frac{2}{1-x}\right) \\
&= \tfrac{5}{4}\frac{1}{1-x} - \tfrac{1}{10}\frac{7+4x}{1+x^2} + \tfrac{1}{5}\frac{1}{1-2x} + \tfrac{1}{4}\frac{1+2x+2x^2}{1-x-2x^3} \\
&\quad - (1 + 2x + 4x^2 + 5x^3)
\end{aligned}$$

Proof. For $k \le 3$, the curve is a straight line so the worst bottleneck is its middle.

For $k \ge 4$, the edge before the middle biggest blob has $\mathit{BlobPrecedingP}_k$ vertices preceding. This is the sum of the preceding blobs plus 2 points on the bridge after each, and 7 points at the start of the curve. Some recurrence or generating function manipulations give (370).

The total vertices are the distinct visited points P_k from (175). Some recurrence or generating function manipulations give the following difference, showing $\mathit{BlobPrecedingP}_k < \frac{1}{2} P_k$ so the points preceding are the smaller subset.

$$\tfrac{1}{2} P_k - \mathit{BlobPrecedingP}_k \;=\; \tfrac{1}{20} 2^k + \tfrac{1}{2} dJA_{k-1} - [\tfrac{3}{10}, \tfrac{3}{5}, \tfrac{17}{10}, \tfrac{7}{5}] \;\; > 0 \tag{371}$$

The end of the curve is a reversal of the start. Its blobs go to $k-2$ so the points after the middle biggest blob are fewer than those before it. So $\mathit{BlobPrecedingP}_k$ is the maximum for any 1-edge bottleneck.

A bottleneck of ≥ 2 edges would have minimum Cheeger constant $2/(\frac{1}{2} P_k)$ $= 1/(\frac{1}{4} P_k)$. But $\mathit{BlobPrecedingP}_k$ is bigger than $\frac{1}{4} P_k$ since some more recurrence or generating function manipulation gives a difference with all positive terms.

$$\begin{aligned} \mathit{BlobPrecedingP}_k - \tfrac{1}{4} P_k \;&=\; \tfrac{3}{40} 2^k + \tfrac{1}{4} dJA_k + \tfrac{3}{8} dJA_{k+1} + [\tfrac{17}{40}, \tfrac{29}{40}, \tfrac{73}{40}, \tfrac{61}{40}] \qquad (372) \\ &> \; 0 \end{aligned}$$

☐

The difference identities (371),(372) use dJA only as a convenient way to express the cubic recurrence terms and see the result is positive.

At (370), $\mathit{BlobPrecedingP}$ grows as $\frac{1}{5} 2^k$. P_k grows as $\frac{1}{2} 2^k$, for a ratio approaching $\frac{2}{5}$. That ratio is approached from above since the cubic and periodic parts of the following identity are positive. In the proof it's enough that the ratio is always between $\frac{1}{4}$ and $\frac{1}{2}$.

$$\begin{aligned} \mathit{BlobPrecedingP}_k \;&= \tfrac{2}{5} P_k + \tfrac{1}{20} \delta_k \; + [\tfrac{7}{20}, \tfrac{13}{20}, \tfrac{7}{4}, \tfrac{29}{20}] && k \ge 4 \\ \text{where } \delta_k &= \delta_{k-1} + 2\delta_{k-3} \text{ starting } \delta_{4,5,6} = 5, 15, 13 \\ &= 5, 15, 13, 23, 53, 79, 125, 231, \ldots && k \ge 4 \end{aligned}$$

Variations on the Cheeger constant can be made by weighting the vertices. An easy possibility for the dragon curve is to count double-visited points as double weight, being the number of visits to each point. The worst location is still the segment before the middle biggest blob with weight BlobN_k preceding that location. The ratio BlobN_k to total $2^k{+}1$ approaches $\frac{2}{5}$ per its $\frac{2}{5} 2^k$ in (302), the same ratio as $\mathit{BlobPrecedingP}_k$ to P_k.

15.2 Twindragon Graph

Theorem 89. *For twindragon k as a graph, the length of the shortest path from curve start to end is*

$$\begin{aligned} \mathit{TEndLength}_k &= \begin{cases} 2 & \text{if } k{=}0 \\ \tfrac{1}{3}\Big([10, 14]\, 2^{\left\lfloor \frac{k}{2} \right\rfloor} + [16, 11, 14, 13] \Big) + k & \text{if } k \ge 1 \end{cases} \qquad (373) \\ &= 2, 2, 4, 8, 12, 20, 28, 40, 56, 80, \ldots \end{aligned}$$

Proof. The twindragon comprises two dragon curves back-to-back.

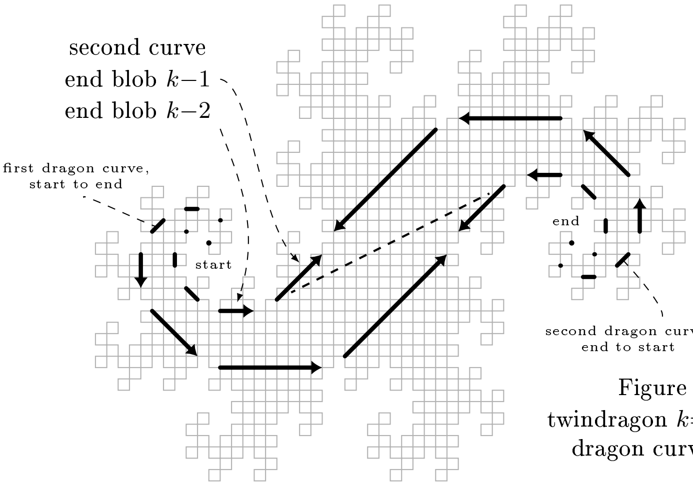


Figure 73:
twindragon $k{=}9$ as two
dragon curves $k{+}1$

At the start of the twindragon, the left boundary is the end of the second curve. The bridges between its blobs are on the boundary so the shortest path is to go around across those blobs straight or stair-step as per diameter theorem 86.

Per the expansions in that theorem, the bridges are directed to the left side of the blobs and these are the inward side of the spiralling end here. So the blob endpoints are the furthest outwards among the bridge points.

Then at the start of end blob $k{-}1$ of the second curve, a direct stepped path can be taken across the twindragon middle to the corresponding point at the end of the twindragon in the first curve. This is shown dashed in figure 73. This path can stay entirely between the diagonals of end blob $k{-}1$ and its following middle biggest blob $k{+}1$. Since the two dragon curves mesh perfectly there is a full grid of segments in between.

$$\begin{aligned} \mathit{TMiddle}_k &= \big(b^{k+1} {-} i\mathit{BlobStart}_{k-1}\big) - \big(i\mathit{BlobStart}_{k-1}\big) && \text{across middle} \\ &= 2{-}4i,\ 6{-}2i,\ 8{+}4i,\ 4{+}14i,\ {-}10{+}20i, \dots && k{\ge}5 \end{aligned}$$

For $k \ge 6$, there is a blob $k{-}2 \ge 4$ and a fixed 7 segments at the end of dragon curve $k{+}1$. The path through the twindragon from start to end length is then

$$\begin{aligned} \mathit{TEndLength}_k = 2\left(7 + \sum_{j=4}^{k-2} \mathit{Manhattan}\big(\mathit{BlobDelta}_j\big) + 3\right) \qquad & k{\ge}6 \\ + \mathit{Manhattan}\big(\mathit{TMiddle}_k\big) \end{aligned}$$

For $k < 6$, the shortest path is calculated explicitly and is per (373) too. □

$\mathit{TEndLength}_k$ is shorter than the dragon $\mathit{Diameter}_{k+1}$ start to end since the second back-to-back dragon gives opportunities for a shorter path in the twindragon. This is achieved by following the second dragon curve (its end) through to the middle as shown above. The shortening is

$$\begin{aligned} &\mathit{Diameter}_{k+1} - \mathit{TEndLength}_k \\ &\qquad = \begin{cases} 0 & \text{if } k{=}0 \\ \frac{2}{3}\Big(2^{\lceil k/2 \rceil} + [5,1,4,2]\Big) & \text{if } k \ge 1 \end{cases} \end{aligned}$$

$$= 0,\ 2,4,\ 4,6,\ 6,8,\ 12,14,\ 22,24,\ 44,46,\ 86,88,\dots$$

even 2×A023105, odd A052953

When the twindragon is scaled by $1/\sqrt{2}^{k+1}$ for start to end a unit length, the odd and even cases of *TEndLength* do not converge to the same value. But they do if distances are taken geometrically, so 45° along the blob diagonals and directly across the middle by its slope.

$$\begin{aligned} \mathit{TGeomLength}_k &= 2\left(7 + \sum_{j=4}^{k-2} \left|\mathit{BlobDelta}_j\right| + 3\right) + \left|\mathit{TMiddle}_k\right| \qquad k \ge 6 \\ &= \quad (\tfrac{2}{3}+\tfrac{1}{3}\sqrt{2})\,\sqrt{2}^{\,k+1} \ + (4-\tfrac{3}{2}\sqrt{2})\,k \\ &\quad + \tfrac{1}{6}\left[-52+20\sqrt{2},\ -68+29\sqrt{2},\ -56+22\sqrt{2},\ -64+31\sqrt{2}\,\right] \\ &\quad + \tfrac{1}{3}\sqrt{5.2^{k+1} - [44,40,40,80]\,2^{\lfloor k/2 \rfloor} + [52,20,40,80]} \end{aligned}$$

$$\frac{\mathit{TGeomLength}_k}{\sqrt{2}^{k+1}} \to \tfrac{1}{3}\left(2+\sqrt{2}+\sqrt{5}\right) \quad = 1.883427\dots \tag{374}$$

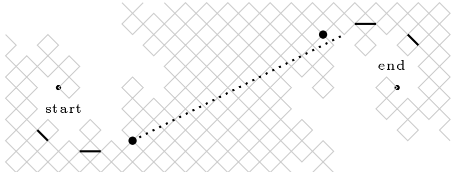
This can be compared to the corresponding dragon curve *GeomLength* limit $1+\sqrt{2} = 2.414\dots$ from section 12.1. The twindragon is shorter by factor

$$\frac{\mathit{TGeomLength}_k}{\mathit{GeomLength}_{k+1}} \to \tfrac{1}{3}\left(\sqrt{2}-\sqrt{5}+\sqrt{10}\right) \quad = 0.780141\dots$$

When rotated to the twindragon endpoints horizontal as in figure 73, the middle part has slope $x = 2y$ except when $k \equiv 0 \bmod 4$ where it is one less unit square at 45° so $x = 2y - \sqrt{2}$. In both cases, this gives $\sqrt{5}$ in the limit (374).

$$\begin{aligned} y_k &= \operatorname{Im}\ \mathit{TMiddle}_k\,.\overline{\omega}_8^{k+1} &&= \tfrac{1}{3}\sqrt{2}^{k+1} - [\tfrac{1}{3}\sqrt{2},\ \tfrac{2}{3},\ \tfrac{1}{3}\sqrt{2},\ \tfrac{4}{3}] \\ x_k &= \operatorname{Re}\ \mathit{TMiddle}_k\,.\overline{\omega}_8^{k+1} &&= 2y_k - [\sqrt{2}, 0, 0, 0] \qquad \text{for } k \ge 1 \end{aligned}$$

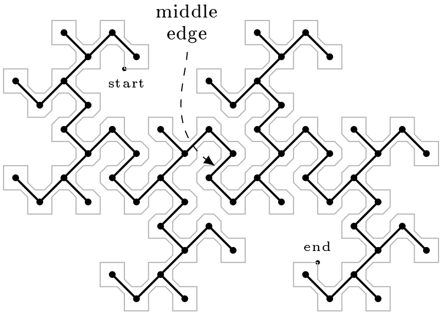
$$\left|\mathit{TMiddle}_k\right| = \sqrt{\begin{cases} 5y_k^2 + 2\sqrt{2}y_k - 2 & k \equiv 0 \bmod 4 \\ 5y_k^2 & k \equiv 1,2,3 \bmod 4 \end{cases}}$$



k=8
middle crossing
x one unit square
left of slope 2:1
shown dotted

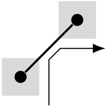
15.3 Twindragon Area Tree

When the corners of the twindragon curve are chamfered off, the unit squares enclosed inside the curve are connected through the resulting gaps. Call this an area tree.



k=6
twindragon
area tree

An equivalent definition is to connect unit squares which are on the left of consecutive curve segments. When the curve turns to the right the unit squares on the left of the segments are distinct. A turn is always left or right (never straight ahead) so those connections are through corners of the squares

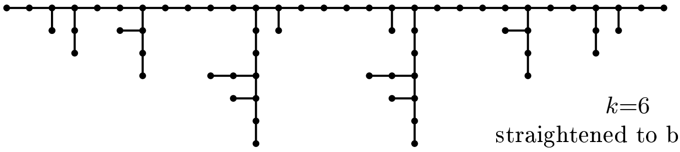


edge between unit squares
on left sides of consecutive
curve segments

Mandelbrot[34] conceives these area connections as rivers. The curve follows the riverbank upstream until reaching a source and then goes back down along the other side of the river and branches. For a closed curve like the twindragon, the squares inside the curve form entirely inland waterways. For area enclosed on the outside of a curve (or any unclosed curve), the rivers flow eventually to the "sea" outside.

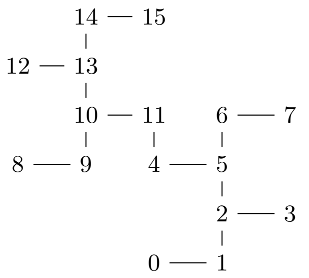
Nekrashevych[37] considers the tree as adjacencies between fractal tiles and as a Schreier graph.

The tree edges correspond to the interior walls of the twindragon midpoint labyrinth shown in figure 44. Vertices are degree 1, 2 or 3 and are quite sparse when straightened out.

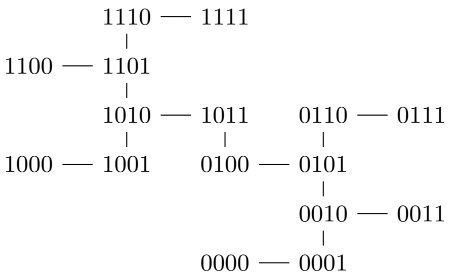


k=6
straightened to branches
off the diameter

As from section 11.1, the unit squares inside the twindragon are diamonds at 135° corresponding to points in complex base i+1. Vertices of the twindragon area tree can be numbered that way, and in that orientation.



k=4 twindragon area tree,
complex i+1 numbering



binary

Theorem 90. *Label the vertices of twindragon area tree k with the point numbers n of complex base i+1. A horizontal edge is between a given n and its least significant bit toggled,*

$$\underbrace{\boxed{\quad h\quad}\boxed{0} \longleftrightarrow \boxed{\quad h\quad}\boxed{1}}_{\textit{edge, horizontal}} \qquad k \geq 1$$

A vertical edge is between n points with low bits of the following form,

$$\begin{array}{rl} \textit{upper} & \boxed{\quad h\quad}\boxed{10}\boxed{11\ldots11} \qquad k \geq 2 \\ & \qquad\qquad\quad \updownarrow \textit{ edge, vertical} \\ \textit{lower} & \boxed{\quad h\quad}\boxed{01}\underbrace{\boxed{00\ldots00}}_{\textit{zero or more bits}} \end{array}$$

Proof. In twindragon k, suppose the diamonds at corners are complex base point numbers s_k start, e_k end, and a_k, c_k opposite corners. The twindragon consists of two copies of the previous level k−1

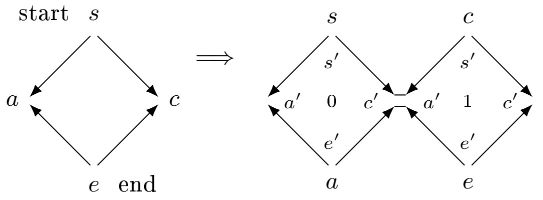


Figure 74:
corner
expansion

So the respective points are given in terms of k−1, starting from a single zero point $s_0 = e_0 = a_0 = c_0 = 0$ for k=0. For c_k and e_k, the sub-part corners are in the right half so add 2^{k-1}.

$$\begin{aligned} s_k = s_{k-1} \qquad &= 0 \\ a_k = e_{k-1} \qquad &= \begin{cases} 0 & \text{if } k{=}0 \\ 2^{k-1}-1 \quad = 011\cdots111 \quad k{-}1 \text{ ones} & \text{if } k \geq 1 \end{cases} \\ c_k = s_{k-1} + 2^{k-1} &= \begin{cases} 0 & \text{if } k{=}0 \\ 2^{k-1} \qquad\;\; = 100\cdots000 & \text{if } k \geq 1 \end{cases} \end{aligned}$$

$$e_k = e_{k-1} + 2^{k-1} \;=\; 2^k - 1 \qquad = 111\cdots111 \quad k \text{ ones}$$

The connection between the two twindragon sub-parts is $0.c'$ to $1.a'$. In k=1, this is a horizontal edge 0 to 1

$$\begin{aligned} c_0 \quad &= 0 && \text{left} \\ a_0 + 2^0 &= 1 && \text{right} \end{aligned}$$

In $k \geq 2$, this connection is *lower* $= 0.c'$ to *upper* $= 1.a'$

$$\begin{aligned} a_{k-1} + 2^{k-1} &= 2^{k-1} + 2^{k-2} - 1 && = 1011\ldots11 && = \textit{upper} \\ c_{k-1} &= 2^{k-2} && = 0100\ldots00 && = \textit{lower} \end{aligned}$$

lower to *upper* edge is vertical since writing these bit patterns in complex base b=i+1 has difference

$$\textit{upper} - \textit{lower} \;=\; \left(b^{k-1} + \sum_{j=0}^{k-3} b^j \right) - b^{k-2} \;=\; i \qquad\qquad k \geq 2$$

When the twindragon replicates all the existing edges are replicated so there is an arbitrary run of high bits on the edge forms. ☐

These bit patterns can be used to construct the tree for computer calculation, including drawing it in the complex base shape by the edge directions. For a whole tree it's probably most efficient to make edges *upper* to *lower* by a loop over bit patterns. If going by n or an isolated part of the tree then some bit-twiddling on n can identify when it has an edge to an upper and/or lower vertex. Horizontal edges are always simply the low bit toggled.

```
TAVertexToLower($n$)
    $mask$ = BITXOR($n, n$+1)            lowest 0 and all bits below
    **if** BITAND($n, mask$+1) $\neq$ 0       bit above lowest 0
    **then** $n$ is an *upper* and has edge downwards to
        $lower = n - mask$

TAVertexToUpper($n$)
    $mask$ = BITXOR($n, n$−1)            lowest 1 and all bits below
    **if** BITAND($n, mask$+1) = 0        bit above lowest 1
    **then** $n$ is a *lower* and has edge upwards to
        $upper = n + mask$
```

The direction upper or lower from n can also be a parameter 0 = go up or 1 = go down. A possible low run of that bit is skipped and the next run (opposite bit) must be only a single bit long. *mask* is the same as above but applied with an XOR to toggle the low run and next two bits.

```
TAVertexToOther($n$, $direction$ = 0 or 1)
    $transitions$ = BITXOR($n$, 2$n$+$direction$)
    $mask$ = BITXOR($transitions$, $transitions$−1)
```

```
if BITAND($transitions$, $mask$+1) $\neq 0$
then there is an edge to
    BITXOR($n$, $2mask$+1)
```

Predicates for when a vertex n has an upper or lower neighbour follow from the bit patterns as simply testing bit above lowest 1-bit or 0-bit. These are the curve turn sequence,

$$\begin{aligned} \mathit{TAVertexToUpperPred}(n) &= \mathit{TurnLpred}(n) \\ \mathit{TAVertexToLowerPred}(n) &= \mathit{TurnRpred}(n{+}1) \end{aligned}$$

These bit operations are all for the tree continued infinitely. For a tree of given k, the *TAVertexToUpper* would be restricted to an $n < 2^k$.

Theorem 91. *The number of degree 0, 1, 2 or 3 vertices in twindragon area tree k are*

$$\mathit{TADegCount}(k,0) = \begin{cases} 1 & \textit{if } k = 0 \\ 0 & \textit{if } k \geq 1 \end{cases} \qquad \text{A000007}$$

$$\mathit{TADegCount}(k,1) = \begin{cases} 0,2,2 & \textit{if } k = 0 \,\mathrm{to}\, 2 \\ 2^{k-2}+2 & \textit{if } k \geq 3 \end{cases} \tag{375}$$

$$= 0,2,2,4,6,10,18,34,66,130,\ldots$$

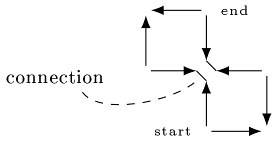
$$\mathit{TADegCount}(k,2) = \begin{cases} 0,0,2 & \textit{if } k = 0 \,\mathrm{to}\, 2 \\ 2^{k-1}-2 & \textit{if } k \geq 3 \end{cases}$$

$$= 0,0,2,2,6,14,30,62,126,254,\ldots$$

$$\mathit{TADegCount}(k,3) = \begin{cases} 0,0,0 & \textit{if } k = 0 \,\mathrm{to}\, 2 \\ 2^{k-2} & \textit{if } k \geq 3 \end{cases} \tag{376}$$

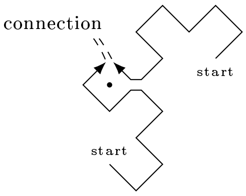
$$= 0,0,0,2,4,8,16,32,64,128,\ldots$$

Proof. Twindragon level k is two twindragons level $k{-}1$ meeting at a corner



Twindragon k
from two $k{-}1$
connected at corner

The unit square at the corner of the sub-parts is formed by two dragon curve ends meeting. For $k \geq 3$, those ends are a U shape. This is seen in k=3 ($JA_3 = 1$) and that shape is maintained by each expansion thereafter (the right side unfold hanging square from (362)).



Twindragon corner
connection square
for $k{-}1 \geq 3$

So for $k{-}1 \geq 3$ the square which will connect to make twindragon k has just one edge leaving it. The tree vertices by degree are two copies of the $k{-}1$ parts except the connection changes two degree-1 squares to degree-2 each.

$$\begin{aligned} \mathit{TADegCount}(k,1) &= 2\,\mathit{TADegCount}(k{-}1,1)\ -2 \qquad k \geq 4 \\ \mathit{TADegCount}(k,2) &= 2\,\mathit{TADegCount}(k{-}1,2)\ +2 \\ \mathit{TADegCount}(k,3) &= 2\,\mathit{TADegCount}(k{-}1,3) \end{aligned} \qquad \square$$

Second Proof of Theorem 91. Vertex degrees can also be counted from the bit patterns of theorem 90.

For $k \geq 1$, every vertex can toggle its low bit for the left to right edge so degree ≥ 1. n is only one of these left or right so degree cannot be 4, only at most 3.

Degree-3 vertices are those n which are both *upper* and *lower* forms. An *upper* ending 1011…11 is also a *lower* only when it ends 01 so three bits 101. Similarly *lower* ending 0100…00 is also an *upper* only when it ends 10 so three bits 010. The bits above these are arbitrary so

$$\mathit{TADegPred}(k,n,3) = \begin{cases} 1 & \text{if } k \geq 3 \text{ and } n \equiv 2 \,\mathrm{or}\, 5 \bmod 8 \\ 0 & \text{otherwise} \end{cases} \qquad n\ \text{A047617} \quad (377)$$

$$\mathit{TADegCount}(k,3) = 2.2^{k-3} \qquad k \geq 3$$

Degree-2 vertices are *upper* but not *lower* or vice-versa. Count *upper* and subtract those which are both *upper* and *lower* (the degree-3 vertices). The count of *lower* less both likewise.

$$\mathit{TADegCount}(k,2) = 2\left(\left(\sum_{j=0}^{k-2} 2^j\right) - \mathit{TADegCount}(k,3)\right)$$

Degree-1 vertices are the remainder of the total 2^k vertices. Or to count directly, they are those n which are neither *upper* nor *lower*. For $k \geq 3$, an n entirely 11…11 or 011…11 is neither *upper* nor *lower*. Otherwise n with one or more trailing 1-bits must end …0011…11 to avoid being *upper*, and it cannot end a single …001 or that would be *lower*. The same for trailing 0-bits to be neither *upper* nor *lower*, giving

$$\mathit{TADegCount}(k,1) = 2\left(2 + \sum_{j=0}^{k-4} 2^j\right) \qquad k \geq 3 \qquad \square$$

A yet further approach for $\mathit{TADegCount}(k,1)$ is that a degree-1 vertex has all sides of the twindragon diamond consecutive and so is on the left of a sequence of 3 left turns. There are $\mathit{A3left}_{k+1}$ of these (section 4.1) in each of the two dragons making up the twindragon, and for $k{\geq}1$ the twindragon start and end are each a further 3 left turns so

$$\mathit{TADegCount}(k,1) = 2\,\mathit{A3left}_{k+1} + 2 \qquad k \geq 1$$

The total of all degrees is twice total edges, as for any graph. The twindragon has 2^k unit squares inside so the area tree has $2^k{-}1$ edges.

$$2^k{-}1 \;=\; \tfrac{1}{2}\sum_{d=0}^{3} d\,.\,\mathit{TADegCount}(k,d)$$

For $k \ge 4$, the degree-3 vertices are copies of the previous level tree, no new degree-3 arise.

The degree of a given vertex n follows from the bit patterns too. An *upper* is a 1-bit above lowest 0. A *lower* is a 0-bit above lowest 1. But in both cases the bit above cannot be outside k bits for a level k tree.

$$\begin{aligned}\mathit{TADegree}(k,n) = {}& \big(1 \text{ if } k \ge 1\big)\\ &+ \big(\mathit{BitAboveLowestZero}(n) \text{ if } k \ge 1\big)\\ &+ \big(1{-}\mathit{BitAboveLowestOne}(n) \text{ if } n{\ne}0 \text{ and } n{\ne}2^{k-1}\big)\end{aligned}$$

For the tree continued infinitely there is no restriction,

$$\begin{aligned}\mathit{TADegree}(\infty,n) &= \begin{cases}1 & \text{if } n=0\\ \begin{aligned}&2+\mathit{BitAboveLowestZero}(n)\\ &\quad-\mathit{BitAboveLowestOne}(n)\end{aligned} & \text{if } n \ge 1\end{cases}\\ &= 1,2,3,1,2,3,2,1,2,2,3,2,1,3,2,1,2,2,3,1,2,3,2,\dots\end{aligned}$$

This can be written as an increment (using *Zero* in terms of *One* from (6))

$$\mathit{TADegree}(\infty,n) = \begin{cases}1 & \text{if } n=0\\ 2+\mathit{dBitAboveLowest}(n) & \text{if } n \ge 1\end{cases}$$

$$\begin{aligned}\mathit{dBitAboveLowest}(n) &= \mathit{BitAboveLowestZero}(n) - \mathit{BitAboveLowestOne}(n)\\ &= \mathit{BitAboveLowestOne}(n{+}1) - \mathit{BitAboveLowestOne}(n)\\ &= 0,1,-1,0,1,0,-1,0,0,1,0,-1,1,0,-1,0,0,1,-1,0,\dots \qquad n{\ge}1\end{aligned}$$

Cases for *dBitAboveLowest* can be written out like *sBitAboveLowest* at (22)

$$\begin{aligned}\mathit{dBitAboveLowest}(n) &= \begin{cases}-\mathit{BitAboveLowestOne}(n) & n{\equiv}0 \bmod 4\\ \mathit{BitAboveLowestOne}(n{+}1) & n{\equiv}1 \bmod 4\\ -\mathit{BitAboveLowestOne}(n) \quad +1 & n{\equiv}2 \bmod 4\\ \mathit{BitAboveLowestOne}(n{+}1) - 1 & n{\equiv}3 \bmod 4\end{cases}\\ &= (-1)^{\mathit{LowestBit}}(n)\,.\begin{pmatrix}\mathit{SecondLowestBit}(n)\\ \quad -\mathit{BitAboveLowestOne}\big(n+\mathit{LowestBit}(n)\big)\end{pmatrix}\end{aligned}$$

At (375),(376), count of degree-1s and degree-3s differ by just 2. Per (377), degree-3s are every $n \equiv 2,5 \bmod 8$, and it can be seen in $\mathit{TAdegree}(\infty,n)$ there is a degree-1 between each.

degree-3 at $n \equiv 2,5 \bmod 8$

			↓			↓					↓			↓					↓			↓		
$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	…
$\mathit{TAdegree}(\infty,n) =$	1	2	3	1	2	3	2	1	2	2	3	2	1	3	2	1	2	2	3	1	2	3	2	…
	↑			↑				↑					↑			↑				↑				

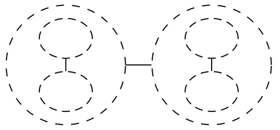
one degree-1 between each degree-3

This can be seen firstly by $n \equiv 1 \bmod 8 =$ binary 001 and $n \equiv 6 \bmod 8 =$

binary 110 are always $\mathit{dBitAboveLowest} = 0$ so degree-2. Then at $n \equiv 3 \bmod 4$, further bits to the lowest zero are some $x011\ldots11$. And increments $n{+}1 \equiv 4 \bmod 4 = x100\ldots00$. They have $\mathit{dBitAboveLowest} = x{-}1$ and $-x$ respectively so one is 0 and the other -1, hence exactly one is -1 for degree-1.

There is a degree-1 at $n{=}0$, and for finite $k \geq 1$ there is another there is another at the end $n = 2^k{-}1$, so 2 more degree-1s than degree-3s.

The twindragon is symmetric in 180° rotation so the squares connected by the new middle edge are at equivalent positions in each half. So the tree is symmetric across its centre edge and likewise each half is symmetric across its centre edge, etc, all the way down to a single vertex.



halves identical across centre edge,
each half likewise identical across centre edge

A tree with this recursive isomorphic halves property always has 2^k vertices. Various such trees can be made by choosing which vertex of each half to connect. The connection can be between the same vertex in each half, like the twindragon area tree has, or between any two of equal eccentricity.

A straight-line path of 2^k vertices is trivially such a tree and is the only such tree for $k \leq 2$. For $k{=}3$, there is the 8-path and one non-path. The twindragon area tree is the non-path. For $k \geq 4$, the twindragon area tree is one among several trees.

Theorem 92. *In twindragon area tree $k \geq 1$, the only automorphism is a 180° rotation. Single vertex $k{=}0$ trivially has no automorphisms.*

Proof. k has an automorphism swapping the two $k{-}1$ halves. Any other automorphism would be swapping branches within a half, and that would be so also in $k{-}1$. So by induction there are no such.

The automorphism swapping halves in $k{-}1$ is lost because $k{-}1$ is bicentral (for $k{-}1 \geq 1$) so branches at the connection are unequal sizes. This holds for all trees of isomorphic halves connected at same vertex as described above. ☐

The connection argument for the degree counts above can also give counts of edges which have vertices of degree 1, 2 or 3 at each end. Twindragon $k \geq 4$ has corner square degree-1 as above, and also the square connected to that corner is degree-2. When the degree-1 of each half are linked their adjacent edges change from 1,2 to 2,2 and the new edge is 2,2 also.

$$\begin{aligned}
\mathit{TAEdgeCount}(k,\, 1{,}2) &= 2\,\mathit{TAEdgeCount}(k{-}1,\, 1{,}2) \;-\; 2 \qquad k \geq 5 \\
\mathit{TAEdgeCount}(k,\, 2{,}2) &= 2\,\mathit{TAEdgeCount}(k{-}1,\, 2{,}2) \;+\; 3 \\
\mathit{TAEdgeCount}(k,\, \mathit{other}) &= 2\,\mathit{TAEdgeCount}(k{-}1,\, \mathit{other})
\end{aligned}$$

With initial counts calculated explicitly,

$$\mathit{TAEdgeCount}(k,\, 1{,}1) = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mathit{TAEdgeCount}(k,\,1{,}2) &= \begin{cases} 0,0,2,2 & \text{if } k = 0 \,\text{to}\, 3 \\ 2^{k-3}+2 & \text{if } k \ge 4 \end{cases} \\
&= 0,0,2,2,4,6,10,18,34,66,130,\ldots \\
\mathit{TAEdgeCount}(k,\,2{,}2) &= \begin{cases} 0,0,1,0 & \text{if } k = 0 \,\text{to}\, 3 \\ 2^{k-2}-3 & \text{if } k \ge 4 \end{cases} \\
&= 0,0,1,0,1,5,13,29,61,125,253,\ldots \\
\mathit{TAEdgeCount}(k,\,1{,}3) &= \begin{cases} 0,0,0,2 & \text{if } k = 0 \,\text{to}\, 3 \\ 2^{k-3} & \text{if } k \ge 4 \end{cases} \\
&= 0,0,0,2,2,4,8,16,32,64,128,\ldots \\
\mathit{TAEdgeCount}(k,\,2{,}3) &= \begin{cases} 0,0,0,2 & \text{if } k = 0 \,\text{to}\, 3 \\ 3.2^{k-3} & \text{if } k \ge 4 \end{cases} \\
&= 0,0,0,2,6,12,24,48,96,192,384,\ldots \\
\mathit{TAEdgeCount}(k,\,3{,}3) &= \begin{cases} 0,0,0 & \text{if } k = 0 \,\text{to}\, 2 \\ 2^{k-3} & \text{if } k \ge 3 \end{cases} \\
&= 0,0,0,1,2,4,8,16,32,64,128,\ldots
\end{aligned}$$

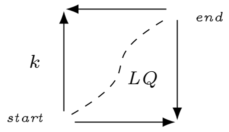
Various graph-theoretic topological indices are based on sums over edges and their vertex degrees. Notice all the edge types (except the solitary 1,1 in k=1) go as a power 2^k so all contribute to a limit if taking a mean index over number of edges.

As an example, the second Zagreb index M_2 of Gutman and Trinajstić [25] is product of vertex degrees at the ends of each edge.

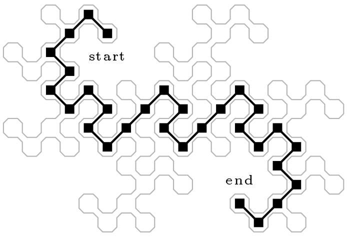
$$\begin{aligned}
\mathit{ZagrebM2}(\mathit{graph}) &= \sum_{\mathit{edges}} \mathit{degree}_1 \,.\, \mathit{degree}_2 \\
\mathit{TAZagrebM2}_k &= \sum_{d_1,d_2=1,2,3} d_1.d_2.\,\mathit{TAEdgeCount}(k,\,d_1{,}d_2) \\
&= \begin{cases} 0,1,8,31 & \text{if } k = 0 \,\text{to}\, 3 \\ 5.2^k-8 & \text{if } k \ge 4 \end{cases} \\
&= 0,1,8,31,72,152,312,632,1272,\ldots \qquad k{\ge}4 \;\; \text{A154252}
\end{aligned}$$

15.3.1 Twindragon Area Tree Diameter and Wiener Index

Theorem 93. *The graph diameter of twindragon area tree k is uniquely attained between the squares at curve start and end. They are connected along the left boundary squares of the two k+1 back-to-back dragons forming the twindragon.*

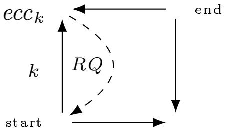


$$\begin{aligned}
\mathit{TAdiameter}_k &= LQ_{k+1} - 1 \\
&= 0,1,3,5,9,17,29,49,85,145,\ldots
\end{aligned}$$



k=6
diameter squares
between dragons
$\mathit{TAdiameter}_6 = 29$

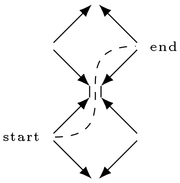
Proof. It will also be shown that the graph eccentricity (longest path beginning there) of the square at the opposite corner is achieved by the right boundary squares going to the start square.



$$\begin{aligned} ecc_k &= RQ_k{-}1 \\ &= 0,1,2,4,8,14,24,42,72,122,\ldots \end{aligned}$$

$k{\geq}1$ $\frac{1}{2}$ A227036

For k=0, the twindragon is a single square and the diameter and this eccentricity are both 0. Suppose then the theorem and eccentricity are true of some k. For k+1, the expansion of the twindragon component sides are

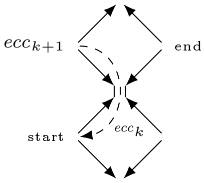


$$\begin{aligned} \mathit{TAdiameter}_{k+1} &= 2ecc_k + 1 \\ &= LQ_{k+2}{-}1 \end{aligned}$$

Any tree has every vertex *eccentricity* $\geq \lceil \mathit{diameter}/2 \rceil$. When two copies of a tree are joined by a new edge the new diameter is always across that connection since the eccentricities of the two vertices connected give a path through there $\geq 2\lceil d/2 \rceil + 1 > d$.

For $\mathit{TAdiameter}_{k+1}$, this is $2ecc_k + 1$ going twindragon start to end. The four k dragon curves which have these squares on their left are the left side of a k+2 curve and therefore $\mathit{TAdiameter}_{k+1} = LQ_{k+2} - 1$. Or also $2RQ_k$ as from figure 19 and identity (113).

For the eccentricity in level k+1,



$$\begin{aligned} ecc_{k+1} &= \max(\mathit{TAdiameter}_k,\ LQ_k + ecc_k) \\ &= \max(LQ_{k+1}{-}1,\ LQ_k{+}RQ_k{-}1) \\ &= \max(LQ_{k+1}{-}1,\ RQ_{k+1}{-}1) \\ &= RQ_{k+1} - 1 \end{aligned}$$

Corner ecc_{k+1} is dragons pointing away so the start point of level k. If its eccentricity is achieved within the upper half then it is $\mathit{TAdiameter}_k$. If the eccentricity goes into the bottom half then it follows left side k to the connection and the eccentricity there is ecc_k is to the start square. The latter is greatest since its two parts make a dragon right RQ_{k+1} and this is $> LQ_{k+1}$ since $RQ_{k+1} - LQ_{k+1} = dJA_{k+2} > 0$ from (166). ☐

LQ_k is even for $k \geq 1$, since it is for $k = 1, 2, 3$ and its recurrence then maintains that. So *TAdiameter*$_k$ is odd for $k \geq 1$ making the area tree bicentral after the single-vertex tree of k=0.

The height of a rooted tree is the eccentricity of the root. For the twindragon area tree, *TAdiameter* is attained by a path from the start, so *TAdiameter* is also the height.

The Wiener index is a measure of total distance between pairs of vertices in a graph.

$$\mathit{Wiener\ index} \;=\; \tfrac{1}{2} \sum_{\text{vertices } u,v} \mathit{distance}(u,v)$$

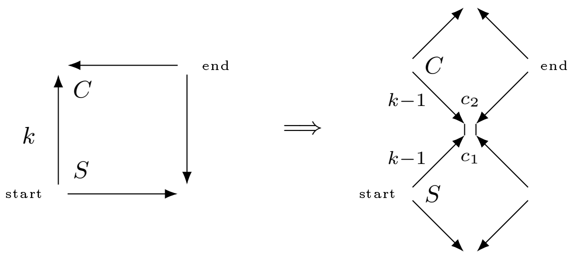
Factor $\frac{1}{2}$ has the effect of taking distance between a pair u, v in just one direction, not also its reverse v, u.

Theorem 94. *The Wiener index of twindragon area tree k is*

$$\begin{aligned} \mathit{TAW}_k &= 10\,\mathit{TAW}_{k-1} - 32\,\mathit{TAW}_{k-2} + 160\,\mathit{TAW}_{k-3} && (378)\\ &\quad - 768\,\mathit{TAW}_{k-4} + 1024\,\mathit{TAW}_{k-5} \\ &= \left(\tfrac{1}{136} LQ_k + \tfrac{19}{136} LQ_{k+1} + \tfrac{9}{272} LQ_{k+2} - \tfrac{1}{8}\right) 4^k \;-\; \tfrac{3}{34} 2^k && (379)\\ &= 0,\ 1,\ 10,\ 68,\ 488,\ 3536,\ 23968, \ldots \end{aligned}$$

$$\begin{aligned} g\mathit{TAW}(x) \;&=\; \frac{x - 32x^4}{(1{-}2x)\,(1{-}4x)\,(1{-}4x{-}128x^3)} \\ &= -\tfrac{3}{34}\frac{1}{1{-}2x} \;-\; \tfrac{1}{8}\frac{1}{1{-}4x} \;+\; \tfrac{1}{136}\,\frac{29 + 112x + 768x^2}{1 - 4x - 128x^3} \end{aligned}$$

Proof. Twindragon area tree k comprises two $k{-}1$ trees connected across a middle edge c_1, c_2.



Let *TAwS* be the sum of distances from start vertex S to all other vertices, and let *TAwC* be the sum of distances from corner connection vertex C to all other vertices.

$$\begin{aligned} \mathit{TAwS}_k &= \textstyle\sum_v \mathit{distance}(S,v) & \qquad \mathit{TAwC}_k &= \textstyle\sum_v \mathit{distance}(C,v) \\ &= 0, 1, 6, 22, 80, 296, \ldots & &= 0, 1, 4, 18, 72, 248, \ldots \end{aligned} \qquad (380)$$

TAwS can be calculated from the two $k{-}1$ sub-trees. For S to vertices in the lower half, the total distance is TAwS_{k-1}. For S to vertices in the upper half, take first the distance from S to c_2, which is RQ_{k-1} as from the diameter in theorem 93. There are 2^{k-1} vertices in the upper half, so that factor on this distance. Then c_2 is a C type corner so $TAwC_{k-1}$ from c_2 to the upper vertices.

Similarly $TAwC$, except the distance C to c_1 is LQ_{k-1}

$$\begin{aligned} TAwS_k &= TAwS_{k-1} + 2^{k-1}RQ_{k-1} + TAwC_{k-1} && (381)\\ TAwC_k &= TAwS_{k-1} + 2^{k-1}LQ_{k-1} + TAwC_{k-1} && (382)\\ &\text{starting } TAwS_0 = 0,\ \ TAwC_0 = 0 \end{aligned}$$

The right sides are a sum $TAwSC = TAwS + TAwC$, and (381)+(382) is a recurrence for that sum

$$\begin{aligned} TAwC_k &= TAwSC_{k-1} + 2^{k-1}LQ_{k-1}\\ TAwSC_k &= 2\,TAwSC_{k-1} + 2^{k-1}BQ_{k-1} \ \ = 2^{k-1}\sum_{j=0}^{k-1} BQ_j\\ &= 2^{k-2}\big(BQ_{k+2} - 5\big)\\ &= 0, 2, 10, 40, 152, 544, 1888, \dots \end{aligned}$$

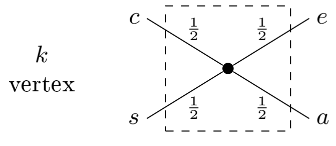
The Wiener index can then be calculated from $TAwC$ of the two tree halves. Distance between vertex pairs both in the upper half is Wiener TAW_{k-1}, and the same when both in the lower half. For one vertex in the lower half and one in the upper, there is distance $TAwC_{k-1}$ to go from lower vertices to c_1, multiplied by 2^{k-1} upper vertices which each one then goes to. The same upper vertices to c_2. Then add 4^{k-1} total paths across edge c_1, c_2.

$$\begin{aligned} TAW_k &= 2\,TAW_{k-1} + 2.2^{k-1}\,TAwC_{k-1} + 4^{k-1}\\ &= 2^{k-1}(2^k{-}1) + 2^k\sum_{j=0}^{k-1} TAwC_{k-1} \end{aligned}$$

This cumulative $TAwC$ is cumulative LQ plus double-cumulative BQ. Some recurrence or generating function manipulations give (378) and (379). For the generating functions, a term like 2^kLQ_k becomes $gLQ(2x)$. The final factor 4^k is substitute $4x$ into the dragon cubic so $1 + 4x + 2(4x)^3$ in the $gTAW$ denominator. ☐

Second Proof of Theorem 94. The Wiener index can also be calculated bottom-up by considering traversals of edges at vertices.

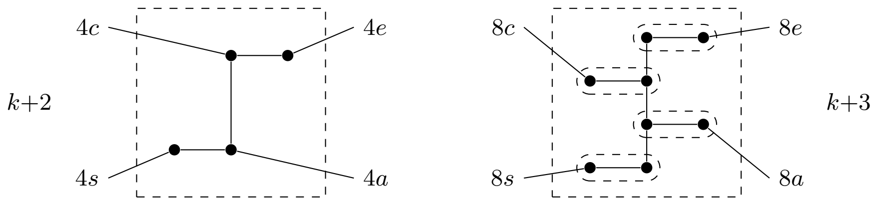
Take each vertex and possible edges in directions s, a, c, e in the manner of figure 74. Let s, a, c, e be the number of vertices in the sub-tree on the other side of each such edge respectively. Vertices are at most degree-3 so at least one of these counts is 0 for no other vertices and no edge there.



$$TAW_k = \tfrac{1}{2}\sum_{\substack{\text{each}\\\text{vertex}}}\ \sum_{\substack{t\,=\\ s,a,c,e}} t\,(2^k - t) \qquad (383)$$

The Wiener index is sum of crossings of each edge. The number of paths crossing an edge is product of number of vertices on each side. For example e on one side and everything else $2^k{-}e$ on the other. Summing over all edges at each vertex counts edges twice (the vertex at each end) so $\frac{1}{2}$ at (383).

On expanding 2 and 3 times each vertex becomes



Crossings in $k{+}2$ are TAW_{k+2}. Expanding $k{+}2$ to $k{+}3$ replaces each vertex by a pair. The edges between the pairs are the same crossings as in $k{+}2$, but 2× vertices so products 4×. The pair edges are 1 vertex in from the outside so their crossings

$$\begin{aligned} &\sum_{\text{vertices}} \sum_{t=s,a,c,e} \big(8t{+}1\big)\,\big(2^{k+3} - 8t{+}1\big) \\ &= 16.4^k + 12.2^k \ + 64 \sum_{\text{vertices}} \sum_{t=s,a,c,e} t\,(2^k{-}t) \end{aligned} \tag{384}$$

(384) is since $s{+}a{+}c{+}e = 2^k{-}1$. The double sum part of (384) is TAW_k from (383), so

$$TAW_{k+3} \ = 4\,TAW_{k+2} + 128\,TAW_k + 16.4^k + 12.2^k \qquad \square$$

It's necessary to go to $k{+}2$ and $k{+}3$ to match up crossings of the extra edges on expansion to make a recurrence. The outer edge crossings correspond to k, as do the edges immediately inwards such as the pairs. But the middle vertical is products like $(c{+}e).(s{+}a)$ and $k{+}2, k{+}3$ have extra edges in the s and e direction which are not isolated by the sum in k. Expanding to $k{+}1$ can give the middle edge, but not the extras.

The expansion of each vertex to a horizontal pair allows the crossings of horizontal and vertical edges to be counted separately. $k{-}1$ connections become the verticals in k, but in k there are 2× vertices each side so 4× crossings. The horizontal crossings are then by difference from total TAW_k.

$$\begin{aligned} TAW_k &= TAWhoriz_k + TAWvert_k \\ TAWvert_k &= \begin{cases} 0 & \text{if } k{=}0 \\ 4\,TAW_{k-1} & \text{if } k \ge 1 \end{cases} \\ &= 0, 0, 4, 40, 272, 1952, 14144, \ldots \\ TAWhoriz_k &= \begin{cases} 0, 1, 6 & \text{if } k = 0, 1, 2 \\ 128\,TAW_{k-2} + 4^{k-1} + 3.2^{k-1} & \text{if } k \ge 3 \end{cases} \\ &= 0, 1, 6, 28, 216, 1584, 9824, \ldots \end{aligned}$$

There are 2^{k-1} horizontals and $2^{k-1}-1$ verticals, for $k{\ge}1$, but the vertical crossings are greater, as can be seen by initial $TAWvhdiff_{3,4,5} > 0$ and then recurrence manipulations to reach (385).

$$\begin{aligned} TAWvhdiff_k &= TAWvert_k - TAWhoriz_k \\ &= 4\,TAWvhdiff_k + 128\,TAWvhdiff_k + 4^{k-1} + 9.2^{k-1} \qquad k{\ge}4 \end{aligned} \tag{385}$$

$$= 0, -1, -2, 12, 56, 368, 4320, \ldots$$

In terms of just tree connectivity, vertical and horizontal crossings can be counted working inward from leaves. The replications mean that leaf vertices have horizontal edges, degree-2 have 1 vertical and 1 horizontal, and degree-3 have 2 vertical, 1 horizontal. Going inward from the leaves reaches outermost degree-3s with at least 2 edge directions determined, giving the third.

The Wiener index divided by number of pairs is a mean distance between vertices. Such a mean is usually taken over vertex pairs in one direction (like the Wiener index) and excluding a vertex to itself, so number of pairs is binomial $\binom{2^k}{2} = \frac{1}{2}2^k(2^k-1)$. The mean can be expressed as a fraction of *TAdiameter*.

TAW at (379) is written in terms of LQ with a view to dividing *TAdiameter* which grows as LQ_{k+1}, and ratios $LQ_{k+1}/LQ_k \to r$ etc as from section 2. The result is limit mean distance between vertices of just over $\frac{2}{5}$ the diameter.

$$\begin{aligned}\frac{TAW_k}{\frac{1}{2}(4^k-2^k)\,.\,\mathit{TAdiameter}_k} &\to \tfrac{1}{68}\frac{1}{r} + \tfrac{19}{68} + \tfrac{9}{136}r &&= 0.400294\ldots \\ &= \tfrac{7}{136}r^4 - \tfrac{6}{17}\frac{1}{r^5} && \text{mean}\end{aligned}$$

Gutman, Furtula and Petrović [24] consider a terminal Wiener index which is distances between pairs of terminal vertices (ie. leaf nodes, degree 1).

Theorem 95. *The terminal Wiener index of twindragon area tree k is, in terms of the full Wiener index,*

$$TATW_k = \begin{cases} 0, 1, 3, 22 & \text{if } k = 0 \text{ to } 3 \\ \frac{1}{16}TAW_k + \frac{7}{32}4^k + \frac{15}{32}2^k - 1 & \text{if } k \ge 4 \end{cases} \tag{386}$$
$$= 0, 1, 3, 22, 93, 459, 2423, 13807, \ldots$$

Proof. Make a calculation similar to TAW theorem 94 above. c_1 and c_2 are terminal vertices for $k{-}1 \ge 3$ and on joining are no longer terminals so adjust to exclude them.

$$\mathit{TAtwS}_k = \sum_{\text{leaf } v} \mathit{distance}(S, v) \qquad \mathit{TAtwC}_k = \sum_{\text{leaf } v} \mathit{distance}(C, v)$$
$$= 0, 1, 3, 12, 33, 101, \ldots \qquad = 0, 1, 3, 10, 31, 89, \ldots$$

$$\begin{aligned}\mathit{TAtwS}_k = {} & \mathit{TAtwS}_{k-1} - (RQ_{k-1} - 1) && \text{lower except } c_1 \quad k{\ge}4 \\ & + \big(\mathit{TADegCount}(k{-}1, 1) - 1\big).RQ_{k-1} && \text{S to } c_2 \\ & + \mathit{TAtwC}_{k-1} && c_2 \text{ into upper} \\ \mathit{TAtwC}_k = {} & \mathit{TAtwS}_{k-1} - (LQ_{k-1} - 1) && \text{upper except } c_2 \quad k{\ge}4 \\ & + \big(\mathit{TADegCount}(k{-}1, 1) - 1\big).LQ_{k-1} && \text{C to } c_1 \\ & + \mathit{TAtwC}_{k-1} && c_1 \text{ into lower}\end{aligned}$$

starting $\mathit{TAtwS}_3 = 12$, $\mathit{TAtwC}_3 = 10$

$$\begin{aligned}\mathit{TAtwSC}_k &= \mathit{TAtwS}_k + \mathit{TAtwC}_k \\ &= 2\,\mathit{TAtwSC}_{k-1} + 2^{k-3}BQ_{k-1} + 2 \qquad k \ge 4\end{aligned}$$

$$= 24.2^{k-3} - 2 + 2^{k-3} \sum_{j=3}^{k-1} BQ_j$$

$$\begin{aligned}
\mathit{TAtwC}_k &= \mathit{TAtwSC}_{k-1} + 2^{k-3} LQ_{k-1} + 1 && k \geq 4 \\
\mathit{TATW}_k &= 2\big(\mathit{TATW}_{k-1} - \mathit{TAtwC}_{k-1}\big) && \text{halves except } c_1{,}c_2 \\
&\quad + 2\big(\mathit{TADegCount}(k{-}1,1) - 1\big) . \mathit{TAtwC}_{k-1} && c_1, c_2 \text{ into halves} \\
&\quad + \big(\mathit{TADegCount}(k{-}1,1) - 1\big)^2 && \text{across } c_1 \text{ to } c_2 \\
&= 2.4^{k-3} + (2k{+}15)\,2^{k-3} + 2^k \sum_{j=3}^{k-1} \mathit{TAtwC}_{k-1} && k \geq 4 \qquad \square
\end{aligned}$$

TATW term $\frac{1}{16}\mathit{TAW}$ in (386) arises essentially from the number of terminal vertices $\mathit{TADegCount}(k,1)$ being $\frac{1}{4}$ of the total 2^k (plus a constant).

The mean distance between distinct pairs of terminal vertices as a fraction of the diameter has the same limit as the full *TAW*.

$$\frac{\mathit{TATW}_k}{\binom{\mathit{TADegCount}(k,1)}{2} . \mathit{TAdiameter}_k} \to \tfrac{1}{68}\frac{1}{r} + \tfrac{19}{68} + \tfrac{9}{136}r \qquad \text{same as } \mathit{TAW}$$

15.3.2 Twindragon Area Tree Parent, Depth, Width

Theorem 96. *Label the vertices of twindragon area tree k with the point numbers n of complex base i+1. The parent of vertex $n \geq 1$ is in the direction given by the following state machine on bits of n high to low.*

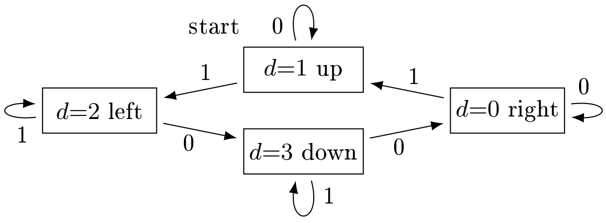


Figure 75: *TAparentDir*(n), bits of n high to low

$$\begin{aligned}
\mathit{TAparentDir}(n) &= \textit{final state of bits high to low} \\
&= 2, 3, 2, 0, 3, 3, 2, 0, 1, 0, 3, 0, 3, 3, 2, 0, 1, \ldots \qquad n{\geq}1
\end{aligned}$$

$$\mathit{TAstepDir}(n, \mathit{dir}) = \begin{cases} \mathit{TAVertexUpper}(n) & \textit{if dir} = 1 \\ \mathit{TAVertexLower}(n) & \textit{if dir} = 3 \\ \mathit{BITXOR}(n,1) & \textit{if dir} = 0 \textit{ and n even} \\ \mathit{BITXOR}(n,1) & \textit{if dir} = 2 \textit{ and n odd} \end{cases}$$

$$\begin{aligned}
\mathit{TAparent}(n) &= \mathit{TAstepDir}(n, \mathit{TAparentDir}(n)) \\
&= 0, 1, 2, 5, 2, 5, 6, 9, 10, 11, 4, 13, 10, 13, 14, 17, 18, \ldots \qquad n{\geq}1
\end{aligned}$$

The initial state is d=1. Since $n \geq 1$ there is always a high 1-bit so always a transition from there to d=2 (left). If n=1 then that is the only transition.

Proof. In the corners of theorem 90, the parent node is towards the start s. The expansion of figure 74 shows how aiming towards a given corner of k becomes a corner of $k{-}1$ according as the top bit of n.

For example, if the high bit of n is 1 then aiming for s which is in the 0 half means go towards a' and from there *lower* step to c' and from there to s. For finding the parent, it's enough to know that aiming for s means in part 1 new state towards a and then *lower*.

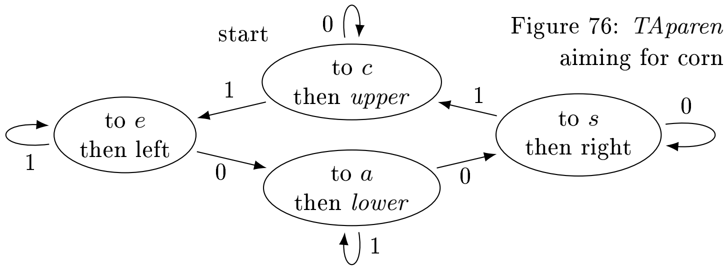


Figure 76: *TAparentDir*(n) aiming for corners

s or a are in the 0 low half of the tree. A 1-bit means n is in the 1 high half of the tree and it's necessary to go towards the a in that high half, or if already there then go *lower* to the 0 half. Similarly conversely when in the 0 half and aiming for e or c which are in the 1 half.

When already in the 0 half, aiming for a becomes aiming for e on expansion. If already at e then its only edge is to the left. Similarly in the 1 half aiming c becomes s on expansion and its edge is to the right.

In terms of the bit patterns, for $n = 1xxxxx$ the way to zero its high 1-bit is to go towards corner $a = 101111$ and take the *lower* edge down to $c = 010000$. Working down the bits of n, if a bit is not already the desired 0 or 1 for such an aim then it is to be set or cleared by going towards the necessary *upper* or *lower* flip, or at the lowest bit by horizontal edge flip. ☐

The *TAparentDir* state machine has a similar structure to *dir* mod 4 in figure 9, but +1 over *dir*, and states 0,3 of *TAparentDir* have out transitions flipped 0↔1. The effect of those changes is that the runs of 1-bits which *dir* identifies become like

dir(n)	11111	0 0000	11111	0 0000	11111
TAparentDir(n)	11111	0 1111	00000	1 0000	11111

bit flip

High 1-bits are unchanged, then 0 goes to state 3 in *TAparentDir* so bit flip there and state 0 which is the 0-bits, until state 1 unchanged again. The flip or not changes after each 10 of *dir*,

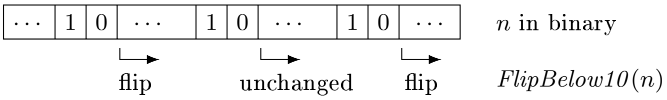
$$\mathit{TAparentDir}\big(\mathit{FlipBelow10}(n)\big) - 1 \equiv \mathit{dir}(n) \mod 4$$

$$\mathit{FlipBelow10}(n) = \text{at each 10 pair in } n \text{ flip output bits below}$$

$$= 0, 1, 2, 3, 5, 4, 6, 7, 11, 10, 9, 8, 13, 12, 14, 15, 23, 22, 21, \ldots$$

$$\text{binary} = 0, 1, 10, 11, 101, 100, 110, 111, 1011, 1010, \ldots$$

For *FlipBelow10*, pairs 10 are found in n without any flips. The output begins as n and is modified by 0↔1 flips below each. Those flips accumulate, so after two 10 there are two flips so unchanged, etc.



The 10 pairs located are like *FlipAt10* from theorem 54, but there each 10 is flipped too. *FlipBelow10* is also similar to the negations for *point* in figure 12, but there below 01 pairs.

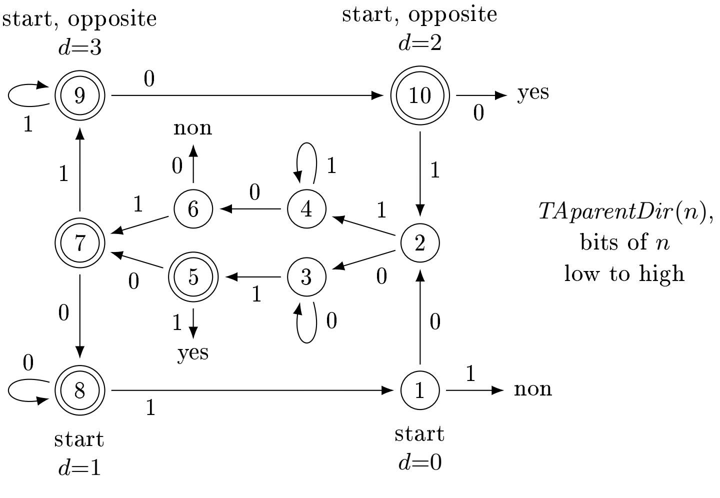
The converse, from *TAparentDir* runs to those of *dir* is a similar flip below 10, but when a flip is applied the next lower 10 is identified in those changed bits (not the original n).

$$\begin{aligned} \mathit{dir}\big(\mathit{UnFlipBelow10}(n)\big)+1 &\equiv \mathit{TAparentDir}(n) \mod 4 \\ \mathit{UnFlipBelow10}(n) &= \text{at each 10 pair in } n \text{ flip bits below} \\ &= 0,1,2,3,5,4,6,7,11,10,9,8,13,12,14,15,23,22,20,\dots \\ \text{binary} &= 0,1,10,11,101,100,110,111,1011,1010,\dots \end{aligned}$$

The first few values of *FlipBelow10* and *UnFlipBelow10* are the same but further values are not. Below the highest 10 pair in n, if there is any 10 or 01 then further bits below there are different flipped or not. The first difference is at n=18 binary 10010 which is 10 immediately followed by 01 and a 0 bit below. The 01 is flipped by both, but the 0 is flipped by *FlipBelow10* and unchanged by *UnFlipBelow10*.

In general, to be the same means below the highest 10 (if any) only a run of all 0s or all 1s until the lowest 2 bits. For $k \geq 2$ bits $2^{k-1} \leq n < 2^k$, there are just $4(k-2)$ many n with $\mathit{FlipBelow10}(n) = \mathit{UnFlipBelow10}(n)$.

The *TAparentDir* state machine in figure 75 is bits high to low. Some usual state machine manipulations can take bits low to high instead.



The start state is 1 to test for $\mathit{TAparentDir}(n) = 0$ (right), or state 8 to test for $\mathit{TAparentDir}(n) = 1$ (up). In both cases, an n is accepted on ever reaching "yes", or ending at any of the double-circle accepting states. Reaching "non" or ending in any non-accepting state is an n not of the respective parent direction.

For d=0, an odd n goes immediately to "non". This is simply that an odd n has no edge to the right at all.

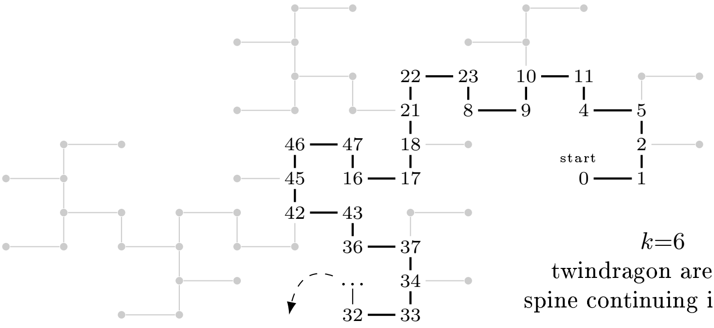
Directions 2 and 3 are the same state machine, but starting states 10 or 9 respectively and flipping the sense of accepting and non-accepting everywhere.

High 0-bits on n do not change the results, since each state reached by a 0 has the same accepting-ness or not as its predecessor. Geometrically this is simply the first half, quarter, etc, within a bigger tree.

States 8 and 10 are only reached by 0-bits and are preceded by accepting states. Since $n{\neq}0$, if n is written without high 0-bits then states 8 and 10 are never final states and their accepting-ness doesn't matter.

In the 0 half of the expansion in figure 74, the connection to the high half is at c'. The same aiming-for procedure as above can be applied to go towards c then up. This goes down the tree continued infinitely. Starting from $n{=}0$ it steps along the infinite spine. Starting from other n goes first to the spine then down.

$$\begin{aligned}\mathit{TAtospineDir}(n) &= \text{final state of figure 75 starting at } d{=}0\\ &= 0,1,1,2,1,2,3,2,1,2,3,2,0,3,3,2,1,2,\dots\\ \mathit{TAtospine}(n) &= \mathit{TAstepDir}(n,\, \mathit{TAtospineDir}(n))\\ &= 1,2,5,2,11,4,5,6,23,8,9,10,13,10,13,14,47,16,\dots\\ \mathit{TAspine}(m) &= \mathit{TAtospine}(\,\mathit{TAtospine}(\dots(0))) \qquad m \text{ times}\\ &= 0,1,2,5,4,11,10,9,8,23,22,21,18,17,16,47,46,45,\dots\end{aligned}$$



$k{=}6$
twindragon area tree
spine continuing infinitely

The depth of a vertex is its distance to the root. The root itself is depth 0. The aiming-for corner procedure for parent direction gives the depth of vertex n by summing distances across preceding trees.

Theorem 97. *The depth of vertex n in the twindragon area tree is given by sums RQ followed by run of LQ according to the following bit runs of n,*

$$\begin{array}{rl} & \text{high} \hspace{12em} \text{low}\\ n \;= & \begin{array}{|c|c|c|c|c}\hline \underbrace{\text{111...11}} & \text{011...11} & \underbrace{\text{000...00}} & \text{100...00} & \underbrace{\text{11...}}\\ \hline\end{array}\end{array}$$

$$\mathit{TAdepth}(n) = \begin{array}{l} RQ_k{+}LQ_{k-1}\\ \quad +\cdots\\ \quad +LQ_{k-l}\end{array} \quad + \begin{array}{l} RQ_p{+}LQ_{p-1}\\ \quad +\cdots\\ \quad +LQ_{p-q}\end{array} \quad + \cdots \tag{387}$$

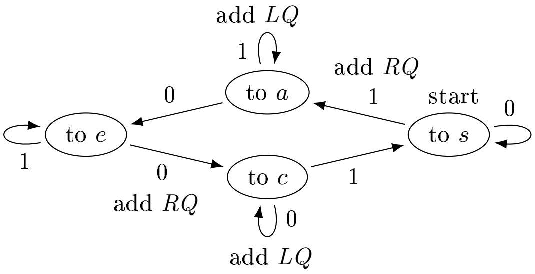
$$= 0, 1, 2, 3, 4, 3, 4, 5, 8, 7, 6, 5, 8, 7, 8, 9, 14, \ldots$$

These are the runs of 1*-bits in* $\mathit{UnFlipBelow10}(n)$*,*

$$\mathit{UnFlipBelow10}(n) \;=\; \underbrace{\boxed{111\ldots11}}\boxed{000\ldots00}\underbrace{\boxed{111\ldots11}}\boxed{000\ldots00}\underbrace{\boxed{11\ldots}}$$

The index k etc of the RQ and LQ terms are the bit position of the respective bits of each run, counting upwards from 0 for the least significant bit of n.

Proof. In the manner of *TAparent*, the distance to the start s follows by the state machine of figure 76.



On expansion, when the target corner is in the opposite half of the tree the distance across that other half is added. In the manner of *TAdiameter* theorem 93, this is either RQ or LQ following the boundary squares on the right or left side of the dragon sub-curve.

The positions where RQ or LQ distances are added are then the n runs of the theorem, and per the *dir* to *TAparentDir* correspondence these runs are the bits of *UnFlipBelow10*. ☐

$\mathit{UnFlipBelow10}(n)$ itself has a geometric interpretation as the total sizes of all power-of-2 sub-trees traversed to reach n.

Let $\mathit{WidthS}(k, d)$ be the number of vertices at depth d. Let $\mathit{WidthC}(k, d)$ be the number of vertices at depth d from the corner connection, in the manner of *ecc* from theorem 93. Then across the middle connection between $k{-}1$ sub-trees have mutual recurrences

$$\mathit{WidthS}(k, d) = \mathit{WidthS}(k{-}1, d) + \mathit{WidthC}(k{-}1,\ d{-}RQ_{k-1}) \tag{388}$$
$$\mathit{WidthC}(k, d) = \mathit{WidthS}(k{-}1, d) + \mathit{WidthC}(k{-}1,\ d{-}LQ_{k-1}) \tag{389}$$

starting

$$\mathit{WidthS}(k, 0) = \mathit{WidthC}(k, 0) = 1$$
$$\mathit{WidthS}(k, d) = \mathit{WidthC}(k, d) = 0 \quad \text{if } d < 0$$

$$\mathit{WidthS}(0, d) = 1$$
$$\mathit{WidthS}(1, d) = 1, 1$$
$$\mathit{WidthS}(2, d) = 1, 1, 1, 1$$
$$\mathit{WidthS}(3, d) = 1, 1, 1, 2, 2, 1$$
$$\mathit{WidthS}(4, d) = 1, 1, 1, 2, 2, 2, 1, 2, 3, 1$$

$$WidthS(5,d) = 1,1,1,2,2,2,1,2,3,2,1,1,2,3,2,2,3,1$$

The two terms of (388),(389) are vertices from the first and second $k{-}1$ sub-parts. For the second sub-part, the depth d is reduced by the distance to the connection point and is then $WidthC$.

The sum of widths at all depths is the total 2^k vertices

$$2^k \;=\; \sum_{d=0}^{TAdiameter_k} WidthS(k,d) \;=\; \sum_{d=0}^{ecc_k} WidthC(k,d)$$

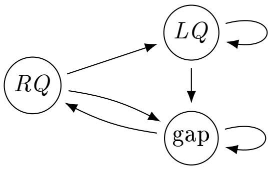
The maximum width is unbounded with increasing k since there are 2^k vertices which must fit somewhere in $TAdiameter_k = LQ_{k+1}{-}1$ many depths and the latter grows only as the cubic r^k.

The width at d is the number of solutions to $TAdepth(n) = d$, so from (387)

$$d = \; \underset{k-1 \geq w}{RQ_w} \overbrace{+LQ_{w-1}+\cdots+LQ_x}^{\text{zero or more}} \quad \underset{\substack{\uparrow \\ \text{index gap} \geq 1}}{+} RQ_y + LQ_{y-1} + \cdots + LQ_z \quad \underset{\substack{\uparrow \\ \text{index gap} \geq 1}}{+} \cdots$$

These runs are in the $WidthS$ recurrence (388) too. An RQ subtraction from d goes to C and $WidthC$ can stay there for a run of LQ subtractions.

A combinatorial interpretation of $WidthS$ is the number of ways to write d as sums of RQ and LQ terms in such runs.



The index positions are significant for counting combinations. At the low end $RQ_1{=}2$ and $LQ_2{=}2$ but their different positions are distinct ways to add 2 (runs and gaps permitting). Likewise $RQ_0{=}1$ and $LQ_1{=}1$. For example depth $d{=}8$ has combinations

$$\begin{aligned} &WidthS(4,8) = 3 \text{ ways} \\ &\quad 8 = RQ_3 + LQ_2 + LQ_1 && = 5+2+1+\text{gap} \\ &\quad 8 = RQ_3 + LQ_2 \qquad\quad + RQ_0 && = 5+2+\text{gap}+1 \\ &\quad 8 = RQ_3 \qquad\quad + RQ_1 + LQ_0 && = 5+\text{gap}+2+1 \end{aligned}$$

$RQ_0 = LQ_0 = 1$ are also the same but the runs and gaps mean they never make distinct forms. RQ_0 only occurs when the position above it (index 1) is a gap, whereas LQ_0 only occurs when not a gap.

As a remark, all values of RQ and LQ are distinct except for 1 and 2 noted. Each LQ_k falls in between RQ_k and its preceding RQ_{k-1},

$$\begin{aligned} &RQ_k > LQ_k > RQ_{k-1} && \text{for } k \geq 3 \\ &\text{since } RQ_k - LQ_k = dJA_{k+1} > 0 && \text{from (166)} \end{aligned}$$

$$\text{and} \quad LQ_k - RQ_{k-1} = dJA_{k-1} > 0 \qquad \text{from (164),(165)}$$

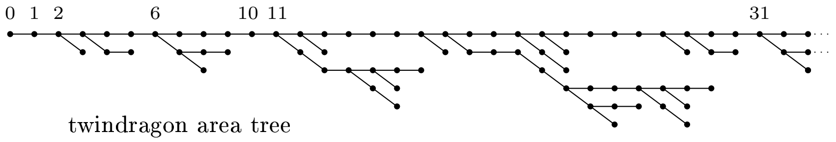
The same run forms apply for $WidthC$ except it starts in an LQ run already. So start LQ_{k-1} and further LQ terms then a gap etc, or gap immediately with no high LQ_{k-1} at all.

Repeatedly expanding (388) is $WidthS$ as a sum of $WidthC$, where depth < 0 is taken to have width 0.

$$WidthS(k,d) = \sum_{j=0}^{k-1} WidthC(j,\ d{-}RQ_j) \qquad d \ge 1 \tag{390}$$

For a given d, these $WidthC$ terms are 0 when j big enough that $d{-}RQ_j < 0$. So when k is big enough $WidthS(k,d)$ does not change with further increases in k. This is the width of a twindragon area tree continued infinitely. ($WidthC$ treated similarly would be the same as $WidthS$ since (389) becomes only its $WidthS$ term when k big enough that $d{-}LQ_{k-1} < 0$.)

$$\begin{aligned} WidthS(\infty,d) &= WidthS(k,d) \qquad \text{for } k \text{ where } RQ_k > d \qquad (391)\\ &= 1,1,1,2,2,2,1,2,3,2,1,1,2,3,2,3,4,2,2,2,\ldots \end{aligned}$$



twindragon area tree
continued infinitely

The depths $0,1,2,6$, etc shown are a single vertex tree width 1.

Theorem 98. *In the twindragon area tree extended infinitely, depths with just a single vertex* $WidthS(\infty,d) = 1$ *are at* d *equal to*

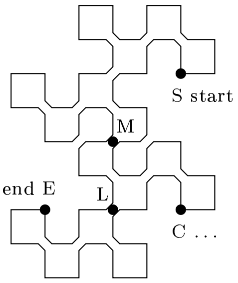
$$DOne(m) = \begin{cases} 0 & \textit{if } m{=}0 \\ JA_{m+3} & \textit{if } m{\equiv}0 \bmod 3 \textit{ and } m > 0 \\ JA_{m+3} - 1 & \textit{if } m{\equiv}1 \bmod 3 \\ JA_{m+2} & \textit{if } m{\equiv}2 \bmod 3 \end{cases}$$

$$= 0,\ 1,2,\ 6,\ 10,11,\ 31,\ 53,54,\ 154,\ 262,263,\ 755,\ 1281,1282,\ldots$$

$$gDOne(x) = -1 - \tfrac{1}{2}\frac{1}{1{-}x} - \frac{x}{1{-}x^3} + \tfrac{1}{2}\frac{3+2x+2x^2}{1{-}x{-}2x^3} - \frac{x^2}{1{-}7x^3+12x^6-8x^9}$$

Terms $m \equiv 1,2 \bmod 3$ are consecutive depths. The index at $m{\equiv}2$ drops to be the same as at $m{\equiv}1$ and without the constant -1.

Proof. The theorem can be verified explicitly for depths up to $d < 25 = RQ_6$. Then by induction suppose it is true of depths $d < RQ_{k-1}$ with $k \ge 7$ and consider further depths in the range $RQ_{k-1} \le d < RQ_k$. Per (391) this range is where $WidthS(\infty,d) = WidthS(k,d)$, and in sum (390) term $WidthC(k{-}1,d)$ is the highest. The following diagram illustrates sub-parts.



Twindragon k
start S to end E

S to C is RQ_k and S to M is RQ_{k-1} as from theorem 93 diameter proof. So depths $RQ_{k-1} \le d < RQ_k$ are squares between M and C.

The tree half above M partly overlaps below M. The upper half extends from S to depth $\mathit{TAdiameter}_{k-1} = LQ_k{-}1$. This is only a partial overlap since it is $< RQ_k$. Must have d exceeding this overlap or it would be width ≥ 2.

$$d > LQ_k \qquad \text{so that } \mathit{WidthS}(k{-}1, d) = 0$$

Point M is a corner position of the lower half. S to M is RQ_{k-1} so depth $\mathit{drem} = d - RQ_{k-1}$ down from M and now seeking $\mathit{WidthC}(k{-}1, \mathit{drem}) = 1$ with drem in the range

$$RQ_k - RQ_{k-1} = LQ_{k-1} > \ \mathit{drem} \ge \ LQ_k - RQ_{k-1} = dJA_{k-1}$$

The tree quarter below L overlaps some of M to C. M to L is LQ_{k-2} and the quarter extends down $\mathit{ecc}_{k-2} = RQ_{k-2} - 1$. This is past C since

$$LQ_{k-2} + RQ_{k-2} = RQ_{k-1} > LQ_{k-1} \quad \mathit{drem} \text{ limit}$$

So drem must be smaller than the L point LQ_{k-2}. This leaves the quarter M to C and restricted range of drem. M is a start corner in that quarter so now seeking

$$\mathit{WidthS}(k{-}2, \mathit{drem}) = 1 \qquad \text{for } LQ_{k-2} > \mathit{drem} \ge dJA_{k-1} \tag{392}$$

This $LQ_{k-2} > \mathit{drem}$ is below the RQ_{k-2} condition for $\mathit{WidthS}(\infty)$ at (391), so seeking $\mathit{drem} = \mathit{DOne}(m)$ in the range (392). The possible JA values of DOne which fall in and out of this range are

$$JA_k \ \ge \ LQ_{k-2} \ > \ JA_{k-1},\ JA_{k-1}{-}1 \ \ge \ dJA_{k-1} \ > \ JA_{k-2} \qquad k \ge 7$$

from identities

$$\begin{aligned}
JA_k &= LQ_{k-2} && - [0,0,1] + dJA_{k-5}{+}dJA_{k-8}{+}\cdots \ge LQ_{k-2} && k \ge 5 \\
LQ_{k-2} &= JA_{k-1} && + JA_{k-2} + 1 \qquad\qquad\qquad > JA_{k-1} && k \ge 2 \\
& && && (393) \\
JA_{k-1}{-}1 &= dJA_{k-1} && - [2,1,1] + dJA_{k-3}{+}dJA_{k-6}{+}\cdots \ge dJA_{k-1} && k \ge 7 \\
dJA_{k-1} &= JA_{k-2} && + [1,1,0] + dJA_{k-5}{+}dJA_{k-8}{+}\cdots \ge JA_{k-2} && k \ge 2
\end{aligned}$$

(393) is left boundary as join areas from theorem 29. The others are by induction or some generating function manipulations.

So have $\mathit{drem} = JA_{k-1}$ and/or $JA_{k-1}{-}1$ when they are DOne forms, and in turn d is

$$\begin{aligned} d &= JA_{k-1} + RQ_{k-1} \\ &= JA_{k+2} \qquad\qquad \text{per theorem 26} \end{aligned} \tag{394}$$

So a previous *DOne* which is JA_{k-1} adds 3 to its index for a further *DOne* at JA_{k+2}. Initial *DOne* values are calculated explicitly then for $k \geq 7$ this step is the pattern of every third *JA*.

Generating function *gDOne* is formed by taking *JA* and reducing the index at its 2 mod 3 terms by subtracting the appropriate *dJA*.

$$\mathit{DOne}(m) = JA_{m+3} - [0,0,1].dJA_{m+2} - [0,1,0]$$

Every third term of $gdJA(x)$ is selected for this using third roots of unity in the usual way. Its 1 mod 3 term is shifted by factor x to subtract at 2 mod 3 in *gDOne*.

$$\omega_3 = e^{2\pi i/3} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{3}\,i \qquad \text{3rd root of unity, } +120^\circ$$

$$\begin{aligned} gdJA\,1mod3(x) &= dJA_1\,x + dJA_4\,x^4 + dJA_7\,x^7 + \cdots \\ &= \tfrac{1}{3}\big(\,gdJA(x) + \omega_3^2 gdJA(\omega_3 x) + \omega_3^4 gdJA(\omega_3^2 x)\,\big) && (395) \\ &= \frac{x^4}{1 - 7x^3 + 12x^6 - 8x^9} && (396) \end{aligned}$$

$$gDOne(x) = -1 - \frac{x}{1-x^3} + \frac{1}{x^3}\Big(gJA(x) - x\,gdJA\,1mod3(x)\Big) \qquad \square$$

The denominator in (396) is product of the dragon cubic at x, $\omega_3 x$ and $\omega_3^2 x$ which are the *gdJA* terms at (395).

$$(1-x-2x^3)\,(1-\omega_3 x-2x^3)\,(1-\omega_3^2 x-2x^3) \;=\; 1-7x^3+12x^6-8x^9$$

Depth *JA* is the middle "3" square of figure 28. It is a branch point on the tree spine so a candidate for width 1, but of course it must be established whether earlier branches overlap it.

The number of vertices above a *DOne* depth can be calculated from the 3-index steps the same as above. Stepping to M goes past 2^{k-1} preceding vertices, so whereas the depth has sums of *RQ* from repeated (394) the vertices are sums of corresponding powers of 2. These are the powers in *JND* from theorem 27.

$$\begin{aligned} RQ_{k-1} + RQ_{k-4} + \cdots &= JA_{k+2} \qquad \text{per theorem 26} && (397) \\ 2^{k-1} \;+\; 2^{k-4} \;+ \cdots &= JND_{k+2} \end{aligned}$$

$$\begin{aligned} DOneV(m) &= \sum_{DOne(m) > d \geq 0} WidthS(\infty, d) \\ &= \begin{cases} 0 & \text{if } m{=}0 \\ JND_{m+3} & \text{if } m{\equiv}0 \bmod 3 \text{ and } m > 0 \\ JND_{m+3} - 1 & \text{if } m{\equiv}1 \bmod 3 \\ JND_{m+2} & \text{if } m{\equiv}2 \bmod 3 \end{cases} \\ &= 0,\ 1, 2,\ 9,\ 17, 18,\ 73,\ 145, 146,\ 585,\ 1169, 1170, \ldots \\ &= \text{binary } 0,\ 1, 10,\ 1001,\ 10001, 10010,\ 1001001,\ 10010001, 10010010, \ldots \end{aligned}$$

The n which is each $DOne$ depth follows from the bit pattern necessary to give $TAdepth$ in theorem 97 as sum of RQ at (397).

$$\begin{array}{rl} TAdepth(n) \;=& \begin{array}{|c|c|c|c|c|c|c}\hline RQ_k & \text{gap} & RQ_{k-3} & \text{gap} & RQ_{k-6} & \text{gap} & \cdots \\ \hline \end{array} \\ n \;=& \begin{array}{ccccccc} 1 & 01 & 0 & 10 & 1 & 01 & \text{binary} \end{array} \end{array}$$

The bit pattern is alternating $1, 0$. $m \equiv 2 \bmod 3$ is sum down to $RQ_1 = 2$. Its immediately preceding depth $DOne(m) = JA_{m+3} - 1$ at $m \equiv 1$ is by changing that to $RQ_0 = 1$ by extending its gap above. The n for that is either 1 bigger or 1 smaller depending whether it is an 01 or 10 type gap.

The necessary bit length and ±1 for $m \equiv 1 \bmod 3$ can be written

$$\begin{aligned} DOneN(m) &= \begin{cases} 0 & \text{if } m{=}0 \\ \frac{1}{3}(2^{m+2} - [1,2]) & \text{if } m{\equiv}0 \bmod 3 \text{ and } m > 0 \\ \frac{1}{3}(2^{m+2} + [2,-5]) & \text{if } m{\equiv}1 \bmod 3 \\ \frac{1}{3}(2^{m+1} - [2,1]) & \text{if } m{\equiv}2 \bmod 3 \end{cases} \\ &= 0,\ 1, 2,\ 10,\ 22, 21,\ 85,\ 169, 170,\ 682,\ 1366, 1365, \ldots \\ &= \text{binary } 0,\ 1, 10,\ 1010,\ 10110, 10101,\ 1010101,\ 10101001, 10101010, \ldots \end{aligned}$$

$$TAdepth\big(DOneN(m)\big) = DOne(m)$$

15.3.3 Twindragon Area Tree Independence and Domination

An *independent edge set* in a graph is a set of edges with no end vertices in common, also called a *matching* since it is vertices in pairs with edge between. A *perfect matching* is all vertices in such pairs.

The twindragon area tree has a perfect matching by horizontal pairs of vertices (for $k{\ge}1$). This is the expansion of each $k{-}1$ vertex to 2 adjacent vertices in k, or low bit toggle in the base $i{+}1$ numbering. Or top-down since k is two copies of $k{-}1$ starting from perfect matching of the 2 vertices in $k{=}1$.

An *independent set* in a graph is a set of vertices which have no edges between them. The *independence number* is the maximum number of vertices in any independent set of the graph. The *independence ratio* is the proportion of this to the number of vertices.

Any tree with a perfect matching has independence ratio $\frac{1}{2}$. An independent set can have at most one vertex of each pair, and such a set can be constructed working outwards from any pair by choosing one in the set and one not, then letting that determine the same in neighbouring pairs. So for twindragon area tree k,

$$TAindnum_k = \begin{cases} 1 & \text{if } k{=}0 \\ 2^{k-1} & \text{if } k \ge 1 \end{cases} \qquad TAindRatio_k = \begin{cases} 1 & \text{if } k{=}0 \\ \frac{1}{2} & \text{if } k \ge 1 \end{cases}$$

Taking each neighbour opposite present/absent is unique up to complement, but there are various other sets attaining $TAindnum_k$ too. At each vertex absent from the set its neighbouring pairs can be either way around one present and one absent.

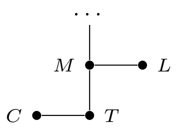
A *dominating set* in a graph is a set of vertices which has every vertex of the graph either in the set or adjacent to one or more of the set. The *domination number* is the size of the smallest set which dominates in a graph.

Theorem 99. *The domination number of twindragon area tree k is*

$$\mathit{TAdomnum}_k = \begin{cases} 1,1,2,4,7 & \textit{if } k=0 \textit{ to } 4 \\ 13.2^{k-5}+1 & \textit{if } k \geq 5 \end{cases}$$

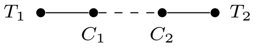
$$= 1,1,2,4,7,14,27,53,105,209,417,\ldots \qquad {\scriptstyle k\geq 5 \text{ A168596}}$$

Proof. The domination number for $k \leq 5$ can be verified explicitly. In k=5, it can be further verified that the start, end and connection vertices are all like C in the following diagram. This is so of $k > 5$ too since further levels are joined copies of the preceding.



In the manner of Cockayne, Goodman, and Hedetniemi[10], the domination number of a tree is obtained by requiring the vertex adjacent to a leaf in the set, and removing it and the leaf from the tree. So leaf L means require M and remove L,M. This gives T dominated, and vertices C--T disconnected from the rest of the tree.

The connection between two $k-1$ trees forming k is C--T pairs



T_1 and T_2 are dominated by their respective M. So just one of C_1 or C_2 suffices, not a vertex from each C--T as when the $k-1$ are separate. The rest of the trees halves are dominated the same as in $k-1$. So

$$\mathit{TAdomnum}_k = 2\,\mathit{TAdomnum}_{k-1} - 1 \qquad k \geq 6 \qquad \square$$

The *domination ratio* is the ratio of domination number to number of vertices in a graph. For the twin alternate area tree this is

$$\mathit{TAdomRatio}_k = \frac{\mathit{TAdomnum}_k}{2^k} = \begin{cases} 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{16} & \text{if } k=0 \text{ to } 4 \\ \frac{13}{32} + \frac{1}{2^k} & \text{if } k \geq 5 \end{cases} \to \frac{13}{32} = 0.40625$$

An *independent dominating set* in a graph is a set of vertices which is both independent and dominating. This is equivalent to being a maximal independent set. A *maximal independent set* is an independent set to which no further vertex can be added and still be an independent set. This means dominating since any undominated vertex would have no neighbour and so could be added and still be independent.

The *independent domination number* of a graph is the size of the smallest independent dominating set. Or equivalently, the size of the smallest maximal independent set and as such also called the *lower independence number*.

Theorem 100. *The independent domination number of twindragon area tree k is the same as the domination number.*

$$TAindomnum_k = TAdomnum_k$$

Proof. Independent dominating sets are a subset of all the dominating sets, so $TAindomnum_k \geq TAdomnum_k$. For $k \leq 5$, it can be verified explicitly there exist independent dominating sets equal to $TAdomnum_k$. The following diagram shows such a set for k=5 (black vertices in the set).

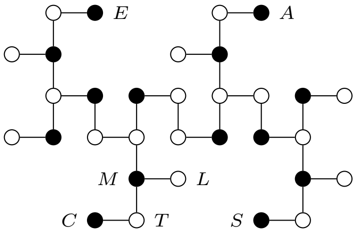


Figure 77:
k=5
twindragon area tree
$TAindomnum_5 = 14$
twice dominated T

This can be used with the construction of theorem 99 to make independent dominating sets $k \geq 6$. On joining at C, vertex T is dominated by M so one of C_1, C_2 can be omitted. S and A which become new E and C have the same M,T pattern. □

A *total dominating set* in a graph is a set of vertices for which all vertices are adjacent to one or more in the set. This differs from an ordinary dominating set in that a vertex in the set does not dominate itself, it must have some neighbour. The *total domination number* is the size of the smallest total dominating set of a graph.

Theorem 101. *The total domination number of twindragon area tree k is*

$$\begin{aligned} TAtotdomnum_k &= \begin{cases} none & \text{if } k{=}0 \\ 2 & \text{if } k{=}1 \\ 2^{k-1} & \text{if } k \geq 2 \end{cases} \\ &= 2, 2, 4, 8, 16, \ldots \qquad k{\geq}1 \end{aligned}$$

The number of sets of this size is

$$\begin{aligned} TAtotdomnumCount_k &= \begin{cases} 0, 1, 1, 1, 1, 9 & \text{if } k = 0 \text{ to } 5 \\ \frac{1}{4}.30^{2^{k-5}} & \text{if } k \geq 6 \end{cases} \\ &= 0, 1, 1, 1, 1, 9, 225, 202500, 164025000000, \ldots \end{aligned}$$

k=0 is a single isolated vertex so it cannot be total dominated, hence no value for $TAtotdomnum_0$ and count $TAtotdomnumCount_0 = 0$.

Proof. The theorem can be verified explicitly for $k \leq 6$. The sets in k=5 can be illustrated

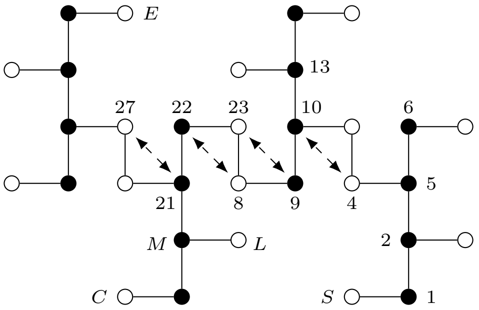


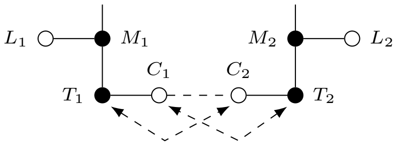
Figure 78:
k=5
twindragon area tree
$TAtotdomnum_5 = 16$

Black vertices are in the set. The configuration shown is two copies of the sole k=4 set attaining $TAtotdomnum_4 = 4$. Any total dominating set includes all vertices with a leaf attached to them, since that is the only way to total dominate those leaves. For example in k=3 this means vertices $1, 2, 5, 6$ must be present. For $k \leq 3$, such vertices with leaves suffice to mutually dominate the rest. For k=4, two copies of those is still smallest.

For k=5, in figure 78, there are further sets. Vertex 9 can move to 23 where it still dominates 8, and 10 is still dominated by 13. Similarly 22 can move to 8, for 4 combinations.

When 22 moves to 8, vertex 9 is dominated by 8 allowing 10 to move to 4. Similarly when $9 \to 23$ can move $21 \to 27$. These are a further 5 combinations for total $TAtotdomnumCount_5 = 9$.

For $k \geq 6$, the connection vertices C meet as



If no cross-domination across the join then the new set size is simply the two halves. M_1 and M_2 must be present to dominate leaves L_1, L_2. They dominate T_1, T_2 too, but all combinations of T_1, C_1, C_2, T_2 present or absent are still at least 2 vertices, and so no reduction in set size through cross-domination.

$$TAtotdomnum_k = 2\, TAtotdomnum_{k-1} \qquad k \geq 6$$

For k=6 number of sets, in the k=5 half sets from figure 78, vertex M_1 is dominated by vertex 21 above it only in some sets, so T_1 can move only in some combinations. Working through the possibilities is $TAtotdomnumCount_6$ = 225.

For $k \geq 7$ number of sets, in figure 78 the S start vertex has its M=2 dominated by 5, and both are present in all $TAtotdomnum_5$ sets. Likewise by symmetry end E. On joining, those ends become S,A,C,E of k=6. So for $k \geq 7$ by joining k=6, always have M dominated from above so 4 combinations C_1, C_2 moves,

$$TAtotdomnumCount_k = 4\, TAtotdomnumCount_{k-1}^2 \qquad k \geq 7 \qquad \square$$

A *semi-total dominating set* in a graph is a set of vertices where each not in the set has a neighbour in the set, and each in the set has a neighbour or a distance 2 away in the set (or both).

Semi-total is similar to total domination, but relaxes to allow set members dominated from up to distance 2 away. It is still a plain dominating set and so falls between the conditions of total and plain dominating.

The *semi-total domination number* of a graph is the size of the smallest semi-total dominating set.

Theorem 102. *The semi-total domination number of twindragon area tree k is the same as the domination number for $k \geq 2$,*

$$TAsemitotdomnum_k = \begin{cases} none & \text{if } k=0 \\ 2 & \text{if } k=1 \\ TAdomnum_k & \text{if } k \geq 2 \end{cases}$$

Proof. The theorem can be verified explicitly for $k \leq 4$. For $k \geq 5$, the independent dominating sets constructed in theorem 100 are also semi-total dominating. In figure 77, vertex M has a vertex in the set 2 above so on joining and removing one of C_1, C_2 still have M semi-total dominated. ☐

The *disjoint domination number* of a graph is the smallest combined size of two disjoint dominating sets. For the twindragon area tree, the two sets can both be the minimum dominating set size *TAdomnum*.

Theorem 103. *The disjoint domination number of twindragon area tree k is*

$$TAdisdomnum_k = \begin{cases} none & \text{if } k{=}0 \\ 2\,TAdomnum_k & \text{if } k \geq 1 \end{cases}$$

$$= 2, 4, 8, 14, 28, 54, 106, \dots \quad k{\geq}1 \qquad k{\geq}5\ \ 2{\times}\text{A168596}$$

The number of pairs of such sets is

$$TAdisdomnumCount_k = \begin{cases} 0, 1, 2, 8, 36 & \text{if } k = 0 \text{ to } 4 \\ 4.5332^{2^{k-5}} & \text{if } k \geq 5 \end{cases}$$

$$= 0, 1, 2, 8, 36, 21328, 113720896, \dots$$

Proof. For $k \leq 5$, the theorem can be verified explicitly. The following diagram shows a disjoint pair of sets in k=5.

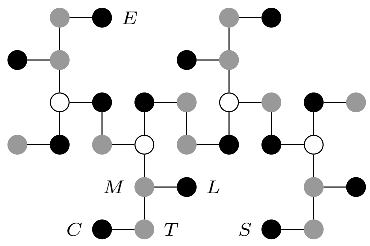
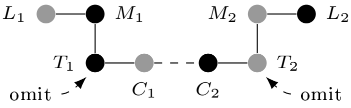


Figure 79:
k=5
twindragon area tree
$TAdisdomnum_5 = 28$
set A black
set B grey

For k=6 comprising two k=5 halves, at the connection take sets A and B the opposite ways around,



k from
two $k-1$ halves

T_1 can be omitted from its set since it and its neighbour C_1 are already dominated by M_1 and C_2 respectively. Likewise by symmetry T_2 can be omitted from its set. So 1 vertex each set dropped across the join as per *TAdomnum* in theorem 99.

The start, end and connection vertices of figure 79 are all of this C,T,M,L form so the same construction and omitted vertices apply for $k > 6$ too.

For the count of set pairs, *TAdisdomnumCount* can be verified explicitly up to k=5. At any leaf, that leaf and its attachment must go one each to the two disjoint sets. For M,L assigned to the sets, C,T can go either way around. But at a connection that is not so. Fixing $M_1 = A$ and $L_1 = B$, must have $C_1 = B$ to dominate T_1, then $C_2 = A$ to dominate C_1, and then $M_2 = B$ to dominate T_2. So no choice for C,T and no independent choice for M_2 second half relative to the first. Thus

$$\mathit{TAdisdomnumCount}_k = \left(\tfrac{1}{2}\,\mathit{TAdisdomnumCount}_{k-1}\right)^2 \qquad k \geq 6 \qquad \square$$

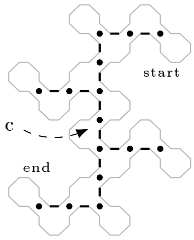
15.4 Twindragon Turn Tree

For any non-crossing closed curve or curve continuing infinitely and not encircling its start on a square grid, the turn at revisited points is the same on each visit. An opposite turn would either enclose either the end or the start.

opposite turns would enclose curve end

opposite turns would enclose curve start

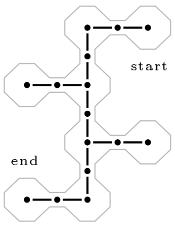
The twindragon is a closed curve of this type. Some of its points are right turns. Those points and the segments between them form a tree.



Twindragon k=4,
right turn points
and segments between

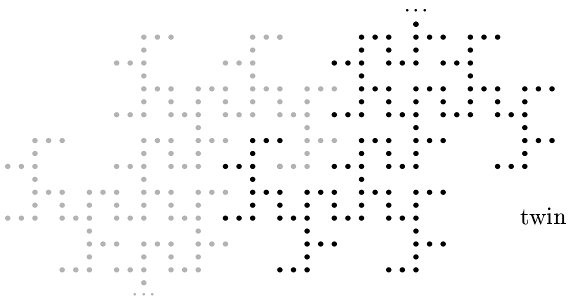
This is twindragon area tree $k-1$ with an extra vertex inserted in each edge. This follows from the same sort of connection arguments used in that tree. Point c is the connection between the two twindragon halves. It is single-visited in those halves so a boundary point and so a left turn. When a connection it swaps to a right turn.

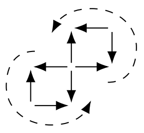
The extra vertices are also the connections between unit squares in the area tree. On expansion each such unit square vertices gives a right turn in its middle, making the turns tree.



Twindragon k=3,
interior unit squares
and right turn points

Two twindragons at the origin form an interlocking pattern



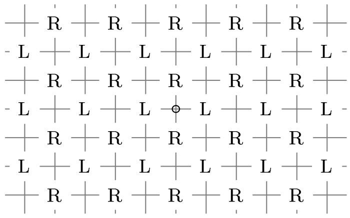


twindragon turn trees

The same pattern holds for 4 dragon curve arms filling the plane if left turns in the north and south arms 1,3 are taken. In the twindragon, the north and south verticals at the origin are traversed in reverse direction, from their end to start. For 4-arm plane filling, those arms are traversed in forward direction and that swaps the turns right ↔ left.

The gaps between the trees are rotated copies of the trees (and the origin not in any tree). The gaps are left turns in the twindragons. Rotating the pattern puts the reverse north, south arms horizontal. Their left turns correspond to right turns of forward direction, and similarly the horizontal arms turning to vertical.

As from (2), the turn at odd n is alternately L and R at $n \equiv 1, 3 \bmod 4$ respectively. Since the curve turns either left or right at every point, this gives the odd turns at every second point in a 2×2 grid.



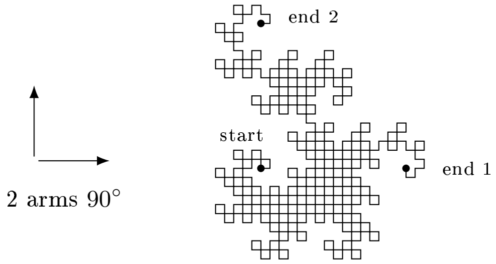
R R R R R
L L L L L L
R R R R R
L L L L L L
R R R R R
L L L L L L
R R R R R

turns
at odd locations

The R turns are at $z \equiv i \bmod b$ in the pattern. The remaining points are filled by the same 2×2 pattern turned 45° and scaled by factor b, and then further powers of b similarly. The base pattern is a simple R in a 2×2 block but repeatedly applying factors of b forms the tree structure.

16 Multiple Arms

16.1 Two Arms



k=8 two arms
$B2_8 = 364$ boundary
$W_8 = 120$
$A2_8 = 165$ area

Theorem 104. *The boundary length between the endpoints of two dragon curves k directed outward at right angles is*

$$W_k = \begin{cases} 2,4 & \text{if } k = 0,1 \\ 2\,L_{k-1} & \text{if } k \geq 2 \end{cases} \qquad \textit{boundary segments}$$

$$= 2, 4, 4, 8, 16, 24, 40, 72, 120, 200, 344, \ldots \qquad k{\geq}1 \quad 4{\times}\text{A203175}$$

$$\textit{Generating function} \quad gW(x) = 2\,\frac{x+1}{1-x-2x^3}$$

And boundary squares,

$$WQ_k = \tfrac{1}{2} W_k = \begin{cases} 1,2 & \text{if } k = 0,1 \\ L_{k-1} & \text{if } k \geq 2 \end{cases}$$

Proof. A W_k curve part expands twice as follows. In k+2 is seen $W_{k+2} = 2\,L_{k+1}$.

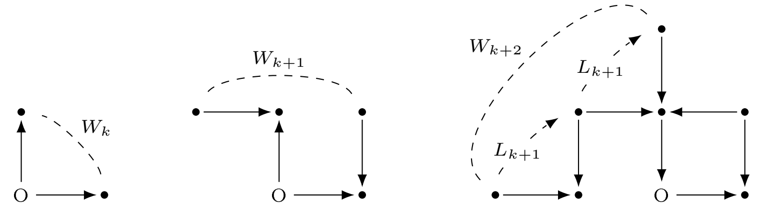
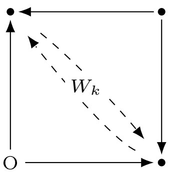


Figure 80: W endpoint expansion

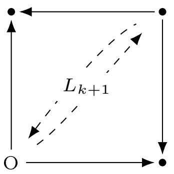
Boundary squares WQ likewise, with $WQ_{k+2} = 2\,LQ_{k+1}$ and $LQ_k = 2L_k$ from (104), for $k \geq 1$. ☐

Notice that W_k and L_{k+1} are not the same configurations. L_{k+1} is two dragons ending at a common point. W_k is two dragons starting from a common point.

In a square of dragons, W_k is boundary across the opposite diagonal and L_{k+1} is boundary across the leading diagonal. Such a square traverses all segments and so ways mesh perfectly.



opposite diagonal
shorter



leading diagonal,
longer

The square is identical in 180° rotation so WQ squares are symmetric in 180° rotation the same as LQ from theorem 14.

L_{k+1} is the longer direction,

$$\begin{aligned} L_{k+1} &= W_k && \text{for } k = 0, 1 \\ L_{k+1} &> W_k && \text{for } k \geq 2 \end{aligned}$$

This can be seen explicitly for $k \leq 6$ and then the L recurrence (100) applied repeatedly gives identities

$$\begin{aligned} L_{k+1} &= 2L_{k-1} + L_{k-2} + 4L_{k-6} && \text{for } k \geq 7 \\ L_{k+1} &= W_k \quad + L_{k-2} + 4L_{k-6} && > W_k \end{aligned}$$

Just numerically, working through the recurrences shows difference of squares is left 2-side squares (theorem 15),

$$LQ_{k+1} - WQ_k = LQ2_{k+2}$$

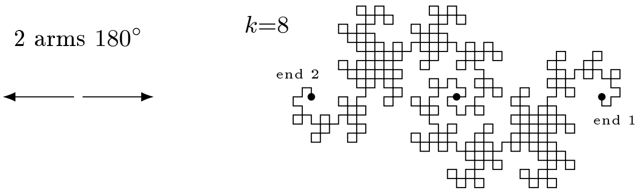
A limit for ratio W_k to L_{k+1} follows from W as L in theorem 104. The result is the same as L over R from (115).

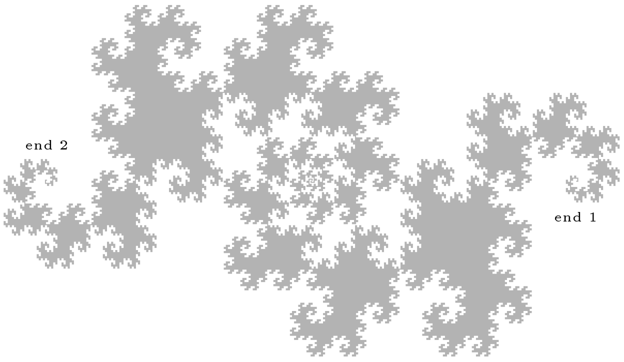
$$\frac{W_k}{L_{k+1}} = \frac{2L_{k-1}}{L_{k+1}} \to \frac{2}{r^2} = r - 1 = 0.695620\ldots$$

Two arms outward boundary length is then a left and right (so B) on the outer side, and W on the inner. For area, the join between two outward curves is JA_{k+1} as shown in figure 27.

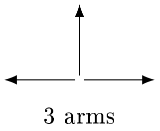
$$\begin{aligned} B2_k &= B_k + W_k \\ &= 4, 8, 12, 24, 44, 72, 124, 216, 364, 616, 1052, \ldots && 4{\times}\text{A003479} \\ A2_k &= 2A_k + JA_{k+1} \\ &= 0, 0, 1, 2, 5, 14, 33, 74, 165, 358, 761, \ldots \end{aligned}$$

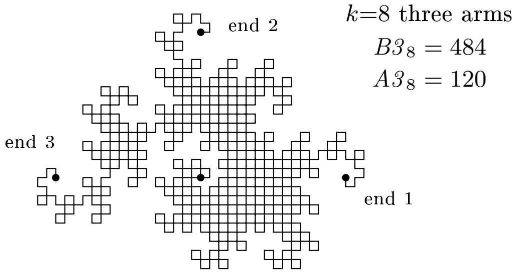
Two arms at 180° do not touch but they do come to within a unit distance of each other. These close approaches are at the bridges of the absent vertical curves in between.





16.2 Three Arms



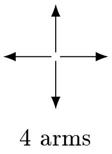


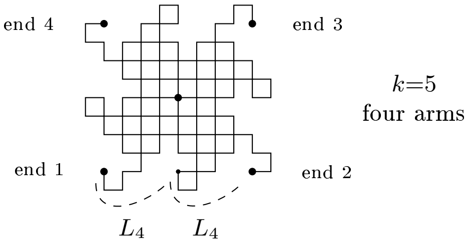
The boundary length is a left and right for B on the straight sides, and two W between. For the area, the two outward joins are JA_{k+1} each, as from figure 27.

$$\begin{aligned} B3_k &= B_k + 2W_k \\ &= 6, 12, 16, 32, 60, 96, 164, 288, 484, 816, 1396, \ldots \\ A3_k &= 3A_k + 2\,JA_{k+1} \\ &= 0, 0, 2, 4, 9, 24, 55, 120, 263, 564, 1187, \ldots \end{aligned}$$

16.3 Four Arms

Four arms has a 4-way rotational symmetry. Dekking[14] calls this a carousel. Each of the four sides are two copies of L_{k-1}, as from the W expansion in figure 80.





The boundary is simply $4W$. The area is the 4 curves plus 4 outward joins which are JA_{k+1} each, as from figure 27.

$$\begin{aligned}B4_k &= 4\,W_k\\ &= 8, 16, 16, 32, 64, 96, 160, 288, 480, 800, \ldots && \scriptstyle k\geq 1\ 16\times\text{A203175}\\ A4_k &= 4A_k + 4JA_{k+1}\\ &= 0, 0, 4, 8, 16, 40, 88, 184, 392, 824, \ldots\end{aligned}$$

In the sample values, the initial $B3_k$ and $B4_k$ are quite close. They are equal at $B3_7 = B4_7 = 288$. But $B3_k$ becomes bigger as k increases. Some recurrence manipulations gives an identity for the difference using *BlobL* so as to see positive.

$$\begin{aligned}B3_k - B4_k &= -2, -4, 0, 0, -4, 0, 4, 0, 4, 16, 20, 32, 68, 112, \ldots\\ &= 8\,BlobL_{k-6} + [20, 24, 20, 8] \qquad \text{for } k \geq 10\end{aligned}$$

A limit for ratio of these lengths follows from their growth as powers of the root r. Limit $L/B \to 2/r^3$ from (126) and $W_k = 2L_{k-1}$ give ratio > 1 so that eventually $B3 > B4$.

$$\begin{aligned}\frac{B3_k}{B4_k} = \frac{B_k + 2W_k}{4W_k} &\to \tfrac{1}{16}r^4 + \tfrac{1}{2}\\ &= 1 + \frac{1}{4r^5 + 4} \;\; > 1 \qquad (398)\\ &= 1.016648\ldots\end{aligned}$$

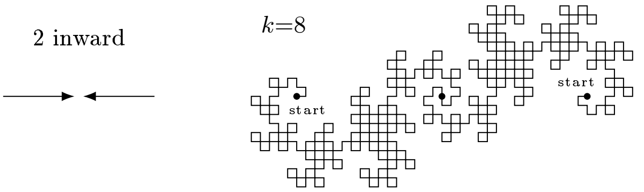
(398) was found by a computer search for an attractive form for the small amount bigger than 1.

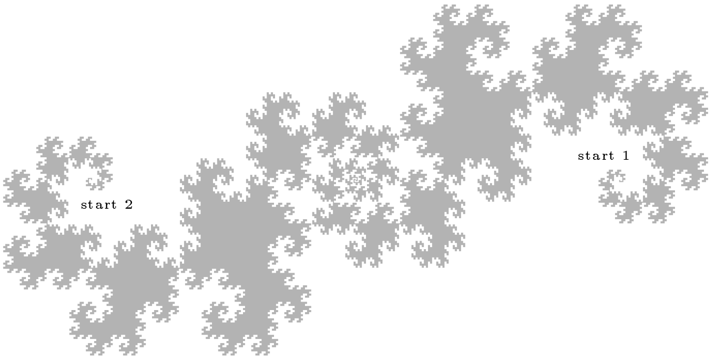
But 4-arm area $A4_k \geq A3_k$ always, so 3-arms is longer boundary but smaller area. 4-arms is in that sense more compact.

16.4 Two Arms Inward

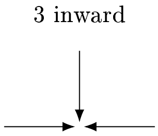
Two arms pointing inward at 90° is simply curve k+1.

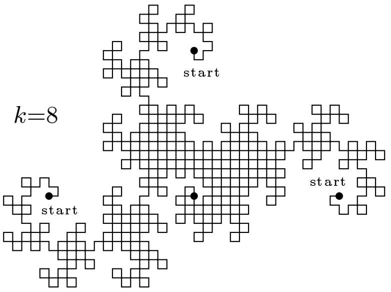
Two arms pointing inward at 180° do not touch but as per the two arms outward (section 16.1) they come to within a unit distance of each other. These close approaches are the bridges of the absent vertical curves in between.





16.5 Three Arms Inward



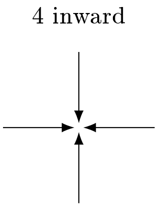


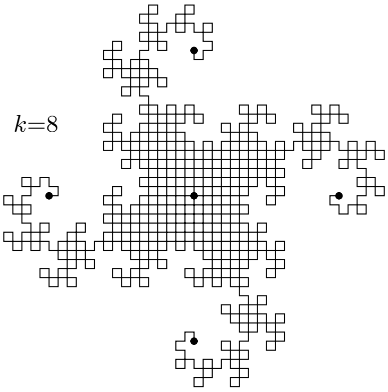
$$\begin{aligned}
B3I_k &= 2L_{k+1} + R_k + L_k \\
&= 6, 12, 24, 40, 68, 120, 204, 344, 588, 1000, 1692, 2872, \ldots \\
BQ3I_k &= 2LQ_{k+1} + RQ_k + LQ_k \quad = BQ_{k+1} + LQ_{k+1} \\
&= 4, 7, 13, 21, 35, 61, 103, 173, 295, 501, 847, 1437, \ldots \\
A3I_k &= 3A_k + 2JA_k \\
&= 0, 0, 0, 2, 7, 18, 45, 106, 237, 518, 1113, 2354, \ldots
\end{aligned}$$

16.6 Four Arms Inward

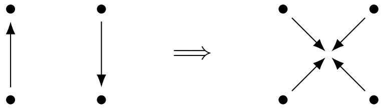
Four arms pointing inward make a symmetric shape.

The centre blob is a little smaller than the corresponding centre in the 4 outward arms of section 16.3, essentially because inward arms meet as join areas JA_k whereas outward arms are the next bigger JA_{k+1} (per figure 27).





Another way to think of this shape is starting from two opposing right sides across a square. Opposing right sides touch and this is that right-side touching.



opposing right sides
of a square

The boundary is 4 left sides of k+1. The area is the 4 component curves with 4 join areas in between, or equivalently 4 left side k+1 areas.

$$
\begin{aligned}
B4I_k &= 4L_{k+1} \\
&= 8, 16, 32, 48, 80, 144, 240, 400, 688, 1168, 1968, 3344, \ldots && \text{8×A203175} \\
BQ4I_k &= 4LQ_{k+1} \\
&= 4, 8, 16, 24, 40, 72, 120, 200, 344, 584, 984, 1672, \ldots && \text{4×A203175} \\
A4I_k &= 4A_k + 4JA_k \quad = 4AL_{k+1} \\
&= 0, 0, 0, 4, 12, 28, 68, 156, 340, 732, 1556, 3260, \ldots && \text{4×A003478}
\end{aligned}
$$

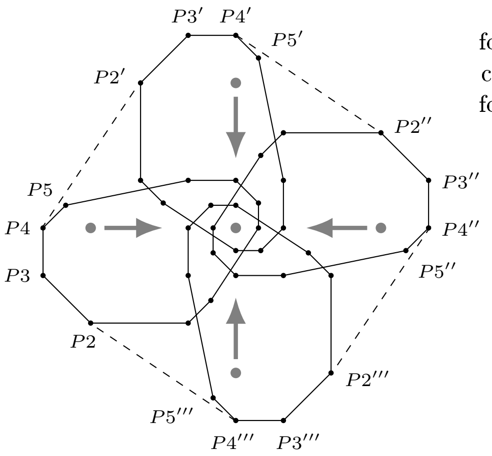
Benedek and Panzone [7] call this shape a mill wheel and draw a 12 vertex convex hull around the fractal using the convex hull of the component dragon fractals. The hull around finite iterations is likewise 12 vertices for $k \geq 4$. There are 3 vertices in each of 4 directions

$$
\begin{aligned}
&\big(b^k - P2(k)\big)\, i^d, && \text{for } d = 0 \text{ to } 3 \\
&\big(b^k - P3(k)\big)\, i^d, \\
&\big(b^k - P4(k)\big)\, i^d
\end{aligned}
$$

For k=0, the hull is just a diamond around the 4 segments. These are just 4 distinct vertices. The three in each direction coincide $b^0 - P2(0) = b^0 - P3(0) = b^0 - P4(0) = 1$.

For $k = 1, 2, 3$, there are just 8 distinct vertices since $P3(k) = P4(k)$ for $k = 1, 3$ and $P2(2) = P3(2)$ for k=2.

For k=4 and k=5, the hull has 12 vertices and can be verified explicitly. For $k \geq 6$, the 10 hull points around the component curves are



four arms inward
convex hull from
four dragon hulls

Adjacent curves are at 90° which is the same as the dragon hull unfolding from figure 38. As from there the hull goes $P4$ to $P2'$ on each side, since P5 is inside that line.

The area of the hull, from the triangular regions delimited by P2, P3, P4 is

$$\begin{aligned} H4IA_k &= \tfrac{1}{3}\Big(14.2^k \; - [10,18,10,14]\,.2^{\lfloor k/2 \rfloor} + [2,2,0,0]\Big) \\ &= 2,4,12,28,62,126,272,560,1142,\ldots \end{aligned}$$

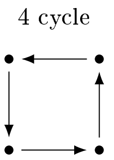
The maximum distance between points in this pattern is opposite P3 vertices by considering vertices pairwise like *Hdiam* intheorem 39. The distance is

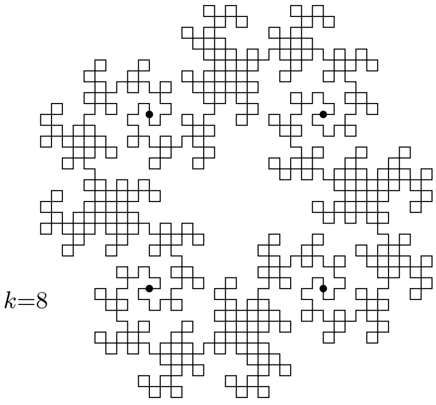
$$\begin{aligned} H4ID_k &= \sqrt[2]{\ }\; \tfrac{1}{9}\Big(68.2^k - [40,80,56,40].2^{\lfloor k/2 \rfloor} + [8,16,20,4]\Big) \\ &= \sqrt[2]{\ }\; 4,8,20,52,104,208,436,932,1864,3728,\ldots \end{aligned}$$

$H4ID_0 = 2$ is straight across the k=0 star. Thereafter $H4ID_k$ is irrational, per the approach of *THdiam* (theorem 57). One leg is a power of 2 when k odd and neither is a multiple of 4 when k even.

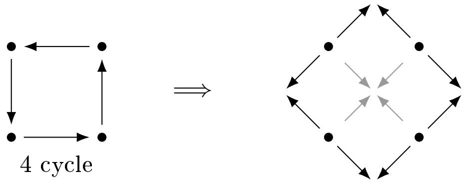
16.7 Four Cycle

Four dragons can be arranged in a square cycle, each head to tail around in the same direction (and therefore not the same as the twindragon).





Expanding one level shows this cycle has sub-part directions per the grid of figure 2, so it is non-crossing etc.



The four cycle sides do not touch each other since in this expansion their ends (the dot locations) are at 180°, per two arms outward 180° in section 16.1.

The segments inside the cycle are the sub-parts shown in grey. They are 4 arms inward (section 16.6).

The boundary length is $4R_k$. The area enclosed can be calculated from the expansion. The 2×2 square divides the plane so the area inside the curve is 4.2^k. The area enclosed by the outside is 4 right sides and 4 left sides so

$$
\begin{aligned}
Acycle_k &= \begin{cases} 1 & \text{if } k{=}0 \\ 2^{k+1} + 4A_{k-1} & \text{if } k \ge 1 \end{cases} \\
&= 1, 4, 8, 16, 32, 68, 144, 300, 624, 1292, \ldots
\end{aligned}
$$

17 Fractional Locations

A fractional point $0 \le f \le 1$ along the dragon curve fractal is located at limit

$$fpoint(f) = \lim_{k\to\infty} \frac{point(\lfloor f.2^k \rfloor)}{b^k} \qquad \text{fractional point}$$

$n = \lfloor f.2^k \rfloor$ is the first k bits below the binary point of f. Stopping there means a point somewhere in a sub-curve of length $1/\sqrt{2}^k$. The extent of that sub-curve is a constant factor of that length so the limit converges to some z.

The location is a change from bits of f to powers $\pm 1/b^j$ with sign change below each 01 bit pair as per (64).

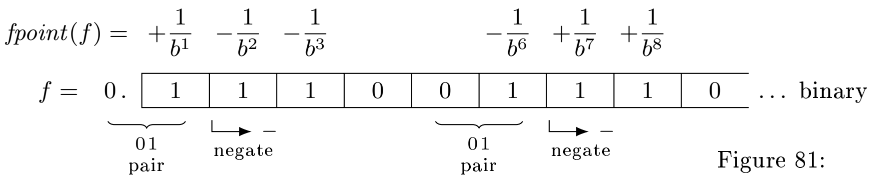
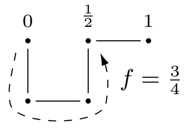


Figure 81: *fpoint* negations

For the 01 pairs rule, the first fractional bit of f is considered to have a high 0 above it (at what would the least significant integer bit), the same as for *point* at (64) above the top of n. For example $\frac{3}{4} = 0.11$ binary has a sign change for the second 1 giving

$$fpoint(\tfrac{3}{4}) = \frac{1}{b^1} - \frac{1}{b^2} = \tfrac{1}{2} \tag{399}$$



When f is rational, its bits are an initial fixed part then a repeating periodic part (of length at most denominator − 1). The b powers and sign changes are then likewise periodic and give a location as some $x{+}iy$ with rational x, y. Davis and Knuth give an example $\frac{1}{3} = .010101\ldots$ which is alternating terms in their general unfolding ζ, and for b is alternating

$$fpoint(\tfrac{1}{3}) = \frac{1}{b^2} - \frac{1}{b^4} + \frac{1}{b^6} - \frac{1}{b^8} + \cdots = \frac{b^{-2} - b^{-4}}{1 - b^{-4}} = \tfrac{1}{5} - \tfrac{2}{5}i \tag{400}$$

If the periodic part of f has an odd number of 01 bit pairs then that is a net negative on the resulting b powers. This can be accounted for in the calculation, or taking the periodic part twice ensures an even number of sign changes.

An exact binary fraction $f = n/2^k$ has two binary representations, one ending $1\,000\ldots$ and the other $0\,111\ldots$. These give the same result in *fpoint* under the 01 pairs since the latter is

$$1/b^{k+1} - 1/b^{k+2} - 1/b^{k+3} - \cdots = 1/b^{k+1} - 1/b^{k+2}/(1 - 1/b) = 1/b^k$$

If f is irrational then it may still have a rational real or imaginary part. The simplest is when all 1-bits of f are at positions $k \equiv 0 \bmod 4$. All $1/b^k$ terms are then imaginary part 0 so entirely real. A finite number of initial other 1-bits are a rational imaginary part $/2^k$ (at most). Conversely if all 1-bits are at $k \equiv 2 \bmod 4$ then the real part is 0 so entirely imaginary.

The Kempner-Mahler number *KM* from (23) is an example of entirely real.

$$\begin{aligned} fpoint(KM) &= \tfrac{1}{2} + \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{2^j}} && \text{entirely real} && (401) \\ &= 0.808609\ldots && && \tfrac{1}{2}{+}\text{A275975} \\ &= 0.110011110\ldots\ \text{binary} && && .1\ \text{then A030300} \end{aligned}$$

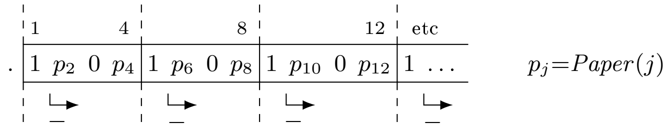
fpoint(*KM*) has initial $1/b^1 - 1/b^2$ whose imaginary parts cancel (as at (399)), then all 1-bits are at $k \equiv 0 \bmod 4$. Each 1-bit like this is an 01 sign change for terms below, giving alternating signs $\pm 1/2^{k/2}$. For *KM*, the result (401) is $\frac{1}{2}$ plus another sum of powers of powers of 2 also of a type Kempner [28] showed is transcendental.

Sum (401) is also considered by Shallit [48] who shows its continued fraction terms arise from an "unfolding" with alternating middle. These terms correspond to turn run lengths of the alternate paperfolding curve (see the author's alternate paperfolding curve write-up) in a similar way to *KM* continued fraction corresponding to dragon curve turn run lengths.

In denser binary fractions, adjacent real or imaginary parts can cancel. An example of this is when bits at positions $k \equiv 1, 3 \bmod 4$ alternate 0,1 giving a rational real part from irrational f. This happens in *PaperConst* (8).

$$\begin{aligned} \mathit{fpoint}(\mathit{PaperConst}) &= \tfrac{2}{3} + \big(\mathit{PaperConst} - \tfrac{2}{3}\big)\, i \\ &= 0.666666\ldots + 0.184069\ldots i \\ &= 0.101010\ldots + 0.001011110\ldots i \quad \text{binary} \end{aligned} \tag{402}$$

The low 2 bits of n determine $\mathit{Paper}(n)$ at odd n, giving a 4-long pattern with alternating $1, 0$ at $n \equiv 1, 3 \bmod 4$



Each $0\,p\,1$ is either 001 or 011 so is a sign change for the b powers below it. Suppose all p=0, then bits 1000 repeat giving $\frac{8}{15}$ and $\mathit{fpoint}(\frac{8}{15}) = \frac{2}{3} - \frac{2}{3}i$.

Each p_j=1 then adds imaginary part $1/2^{j/2}$. At j≡2 mod 4 positions p_2, p_6 etc, the b powers are purely imaginary $1/b^2 = -\frac{1}{2}i$, $1/b^6 = +\frac{1}{8}i$, etc with alternating signs. The negations due to 01 pairs turn them all positive.

Then j≡0 mod 4 positions are $0\,p_j\,1$. If p_j=1 then it is $1/b^j$ and also the b at the 1 changes from positive to negative so net $1/b^j - 2/b^{j+1} = i.1/b^j$. At p_4 this is $-\frac{1}{4}i$, at p_8 it is $+\frac{1}{16}i$, etc, with alternating signs. Again the alternating negations by 01s turn them all positive.

The sequence of values p_2, p_4, p_6 etc are the paperfolding sequence itself again, $\mathit{Paper}(2n) = \mathit{Paper}(n)$, since bit above lowest 1 skips the low 0-bit of $2n$. So these p terms add powers of $\frac{1}{2}$ to the imaginary part at powers according to the paperfolding sequence, and that is *PaperConst*.

In *point* by unfolding (59) of Davis and Knuth, the fraction f might be in the first or second half of the curve. When in the second half the position is measured back from the end. The resulting fraction in the respective half after expansion is then a mapping

$$f \to \begin{cases} 2f & \text{if } f \le \frac{1}{2} \\ 2\,(1-f) & \text{if } f \ge \frac{1}{2} \end{cases} \tag{403}$$

When taking successive bits of f written in binary, this is the fractional position within a sub-fractal. For rational f, the denominator is unchanged (hence eventually repeating).

Integer points in finite iterations of the curve are terminating binary fractions $n/2^k$. These are all still single or double visited in the fractal since at every expansion level their entering and leaving sub-fractals surround the location.

Locations with non-terminating binary fractions can give triple-visited points. Bridge points and each side of them are 15ths triple-visited points (with the bridges themselves being 5ths among the 15ths). The join end is a 7th where three curves meet. These are in fact the only triple-visited points (ahead at theorem 113). Gosper[23] in the OEIS considers the 5ths and 15ths. The 7ths are a little simpler, so take them first.

17.1 7ths Fractional Triples

In figure 29, three sides of a square of sub-curves have a common unit square at the end of the join. As per *JN* and *JNother* limits (160), this is a point $f=3/7$ and $f=5/7$ in the fractal. The locations of all 7ths in the fractal are

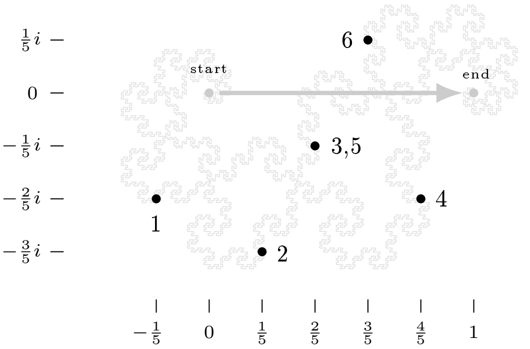


Figure 82: $\mathit{fpoint}(\frac{p}{7})$ fractional locations $\frac{1}{7}$ to $\frac{6}{7}$

The following configurations are all curve arrangements which give triple-visited fractional 7th locations

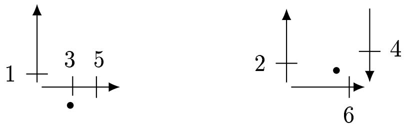


Figure 83: 7ths configurations

$$\begin{aligned} i.\mathit{fpoint}(\tfrac{1}{7}) &= \mathit{fpoint}(\tfrac{3}{7}) &&= \mathit{fpoint}(\tfrac{5}{7}) &&= \tfrac{2}{5}-\tfrac{1}{5}i \\ i.\mathit{fpoint}(\tfrac{2}{7}) &= b - i.\mathit{fpoint}(\tfrac{4}{7}) &&= \mathit{fpoint}(\tfrac{6}{7}) &&= \tfrac{3}{5}+\tfrac{1}{5}i \end{aligned}$$

Location $\frac{2}{5}-\frac{1}{5}i$ of 1,3,5 is shown below its 3,5 segment, but the spiralling down means it is actually on the left side as in figure 82. The right side of 1 is the other visit. For finding triples, it's enough just to know necessary configurations.

Configurations of sub-curves like this are segments in finite iterations. Each such is a triple-visited fractional point of the form

$$\mathit{fpoint}\Big(\frac{s}{7.2^k}\Big) = \mathit{fpoint}\Big(\frac{t}{7.2^k}\Big) = \mathit{fpoint}\Big(\frac{u}{7.2^k}\Big) \qquad \begin{array}{c} 0 \le s,t,u \le 7.2^k \\ \text{integers} \end{array} \tag{404}$$

3 and 5 are in the same segment so are always a difference 2 among two of s,t,u written in least terms (smallest k).

Configuration 1,3,5 expands to configuration 2,4,6. 3 doubles to 6. 5 is 2 back from the end 7 so doubles to 4 from the end, which is forward since that new sub-curve is reverse. This is the f mapping (403). Each fraction expands under that map as

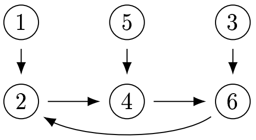
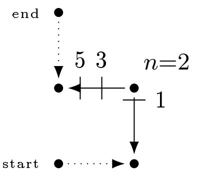


Figure 84:
7ths fractions
expansion

The first triple occurs in k=2. This is a 1,3,5 configuration at n=2 (turned 180° from figure 83).

k=2
7ths 1,3,5
configuration

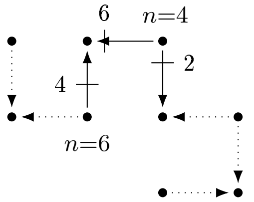


1 is directed back along the curve from n=2 so it is $2.7-1 = 13$. 3 is directed forward so it is $2.7+3 = 17$, and 5 similarly for 19, giving triple $13, 17, 19$.

$$fpoint\left(\tfrac{13}{7.2^2}\right) \ = fpoint\left(\tfrac{17}{7.2^2}\right) \ = fpoint\left(\tfrac{19}{7.2^2}\right) \ = \tfrac{3}{5} - \tfrac{3}{10}i$$

k=2 has a U shape but its segment directions are not configuration 2,4,6. The first 2,4,6 is in k=3

k=3
7ths 2,4,6
configuration



$4.7-2 = 26$
$4.7+6 = 34$
$6.7-4 = 38$
fractions over
$7.2^3 = 56$

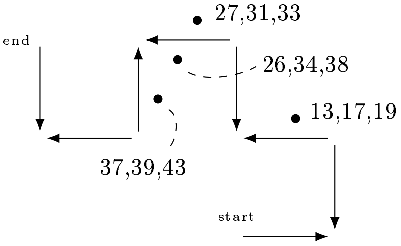
This is simply the first triple 13,17,19 doubled, as for segment expansion of the 1,3,5 configuration. In general, any triple in k doubles to a new triple in k+1. This is numerator doubled corresponding to denominator doubled to 7.2^{k+1}, and so the same *fpoint* location in the fractal.

Level k triples such as 13,17,19 are still present in next level k+1. They are locations in the first half of the curve. Level k triples are also in the second half of k+1 by unfolding. The values in that case measure back from the curve end 7.2^{k+1}.

Additional configurations can be formed across the join of the curve halves. The triples of the first half preceding that join are unchanged and each such join is bigger with bigger k, so triples can be treated as an infinite sequence.

There are total 4 triples in k=3,

k=3
denominator $7.2^3 = 56$
$13, 17, 19$
$26, 34, 38$
$27, 31, 33$
$37, 39, 43$
$SeventhTriples_3 = 4$



13,17,19 is from k=2 unchanged. 37,39,43 is its unfold to measure from the end. 26,34,38 and 27,31,33 are both new across the join. As noted above, 26,34,38 is also a doubling of 13,17,19.

Configuration 2,4,6 has two segments meeting at ends. Its 7th location is the join end *JEQf* relative to segment 6. Segment 2 can be absent when on the curve boundary. An example of 2 absent is the right boundary half-way point *fRQhalf*. As from figure 31, it is a sub-curve join end. The absent segment 2 would be above it.

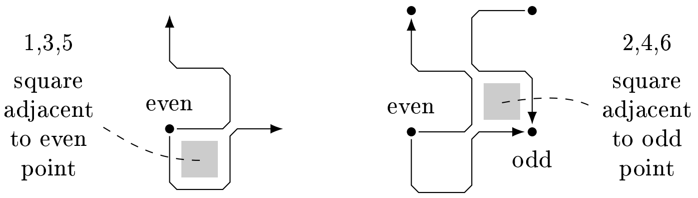
$$\mathit{fpoint}(\tfrac{9}{28}) = \mathit{fpoint}(\tfrac{11}{28}) \ = \mathit{fRQhalf} \ = \tfrac{3}{10} - \tfrac{4}{10}i$$

The two f here are *RQhalf* point number limits $\mathit{RQhalfNS}_k/2^k \to \frac{9}{28}$ and $\mathit{RQhalfNE}_k/2^k \to \frac{11}{28}$ from (132).

Theorem 105. *The number of 7ths fractional triple points* (404) *in curve k is equal to the area of k+2.*

$$\mathit{SeventhTriples}_k = A_{k+2}$$

Proof. The line segments of configurations in figure 83 expanded twice are



The unit squares shown in grey are enclosed. The 1,3,5 configuration encloses a unit square. The 2,4,6 configuration encloses an additional unit square over the 1,3,5. So each configuration is net 1 unit square in k+2.

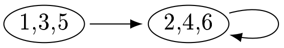
Conversely, k+2 is a 2×2 grid of squares between points from k. Each enclosed unit square in k+2 has a corner at either an odd or even point of k, those being $(x+iy)\,b^2$ with $x+y$ odd or even. An enclosed unit square adjacent to an even must have arisen from segments of a 1,3,5 configuration in k suitably rotated. Likewise when adjacent to an odd point it must have arisen from segments of a 2,4,6 configuration suitably rotated. ☐

Second Proof of Theorem 105. Configurations can also be counted directly.

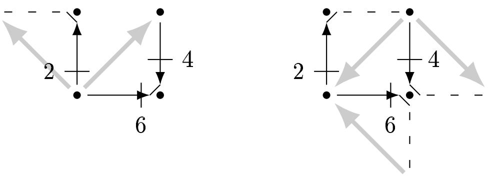
1,3,5 is at an even point so each even $n = 2\,\text{to}\,2^k{-}2$ inclusive gives a 1,3,5. The turn at n determines whether the 1 part or the 3,5 part is in the first or second segment. At a double-visited even point, there are 2 additional cross pairs of segments. Some recurrence manipulation shows the total is left-side area AL_{k+2} from section 4.1.

$$\mathit{Sevenths135}_k = 2\,\mathit{Deven}_k + \begin{cases} 0 & \text{if } k{=}0 \\ 2^{k-1}{-}1 & \text{if } k \geq 1 \end{cases} = AL_{k+2}$$

Configuration 2,4,6 occurs as an expansion of previous level 1,3,5 and 2,4,6.



This is per figure 84 and can also be seen by considering whether segment 2 is first or second in the expansion of the previous level. Determining one segment that way determines the others as firsts or seconds too. When 2 is a first segment the 2,4,6 is an expansion of 1,3,5. When 2 is a second segment the 2,4,6 is an expansion of a previous 2,4,6.



2 as first segment

2 as second segment

Figure 85

So *Sevenths246* is cumulative *Sevenths135* which is AL and that cumulative gives AR as by repeated unfoldings from theorem 24 and (147).

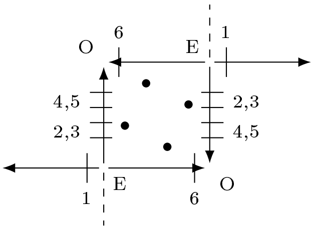
$$\mathit{Sevenths246}_k = \mathit{Sevenths246}_{k-1} + \mathit{Sevenths135}_{k-1} \;= \sum_{j=0}^{k+1} AL_j = AR_{k+2}$$

Or alternatively, 2,4,6 is *BQ3e* as from (127) and a fully enclosed unit square A_k has two such configurations. This is total A_{k+1} as from (142).

$$\mathit{Sevenths246}_k = \mathit{BQ3e}_k + 2A_k \;= A_{k+1} \;= AR_{k+2}$$

Total triples 1,3,5 plus 2,4,6 is then left plus right side areas of k+2. ☐

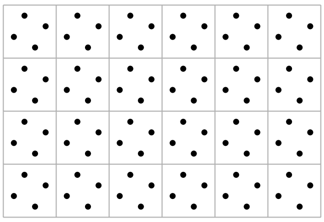
Four triple locations occur in a unit square when there is a 1,3,5 on each vertical and a 2,4,6 upwards and upside-down.



segment arrangement
4 triples in each square,
4 double-visited corners

The two even corners E are double-visited. The two odd corners O might be single or double, though when single there must be some additional segments as otherwise it would be 2-side boundary squares with even middle E, but that does not occur. (2-side boundary is odd middle per figure 20.) Additional segments below and right etc make 3e or 3o squares instead.

The triple locations within the square are symmetric in 90° rotations. When such squares are adjacent they form the following grid. The locations are in groups of 4 along 2:1 slopes. The integer points of the grid are skipped, being only double-visited, not triple-visited.



grid of 7ths
fractional locations

Taking the union of all triples gives numerators which are part of some triple (similar to what Gosper does for 15ths in OEIS A260747),

13, 17, 19, 26, 27, 31, 33, 34, 37, 38, 39, 43, 52, 54, … numerators
4 2 7 1 4 2 1 3 1 1 4 9 2 … differences

Theorem 106. *The differences between successive numerators of 7th fractional triples are 1, 2, 3, 4, 7, 9. These differences all occur infinitely often.*

Proof. Every segment contains some numerator since configuration 1,3,5 puts 1 and 3,5 respectively in the segments each side of every even point (which way around according as left or right turn). So it suffices to consider differences within a segment and across a point.

Taking each segment of figure 83 as middle m gives the following set of segments. Arrows are shown in direction of expansion, so all expand on the right. Knowing these present or absent suffices to determine which 7ths triples are in m.

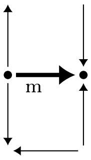


Figure 86:
7th triples
configurations union,
m as each segment

For a difference going across a point, it's necessary also to know 7ths in the next segment after m, in curve start to end direction.

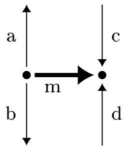
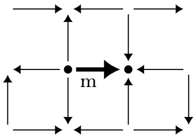


Figure 87:
next segment

If m is forward along the curve (an even segment) then the next is c or d. If m is backward along the curve (an odd segment) then the next is a or b. If m is last in the curve then it has no next.

Take a further union with m of figure 86 also at each a, b, c, d position. This gives a set of segments determining which 7ths are in the next segment too.



7th triples
union and next

On expansion, the first and second segment of m has a corresponding set of surrounding segments. Beginning from a single segment, and making successive expansions, including noting which of a, b, c, d is next in curve order, gives total 105 configurations. Triples occurring within m, and the first triple in next segment, show the only differences are per the theorem.

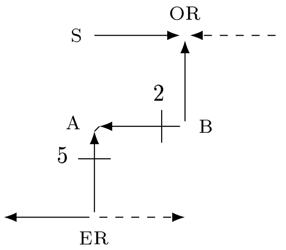
Mutual recurrences on the configurations, and the differences they contain, gives how many of each difference occur in given k. All grow without bound. □

It's possible instead to explicitly work through cases of a segment and the triples it can contain due to its surrounds, and next triple across an odd or even point. Doing so is quite tedious. An attraction of the mechanical approach is that it gives a state machine which can count how many of each difference, and which n has how many triples.

From the state machine, the number of times difference 7 occurs is

$$\begin{aligned} \mathit{SeventhsDiff7}_k &= \begin{cases} 0 & \text{if } k \le 2 \\ k-2 & \text{if } k \ge 2 \end{cases} \\ &= 0, 0, 0, 1, 2, 3, 4, 5, 6, \ldots \end{aligned} \tag{405}$$

It is 2 to 5 across an odd point ($7{-}2 + 7{-}5 = 7$) of the form



7ths
2 to 5
difference 7

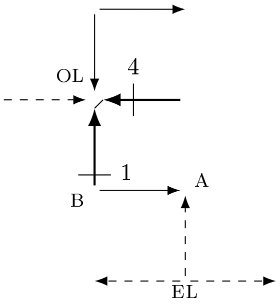
2 is from the U above and 5 is from the L below. Single-visited A means no 6 after 5, nor 4 after 2. Single-visited B means no 3,5,6 after 2.

Such a configuration is at curve start for $k \ge 3$ (point S is the curve start), and corresponding unfold at curve end for $k \ge 4$, and then each ER--OR bridge. Bridges alternate so there is 1 new such in each expansion, giving count (405).

From the state machine, the number of times difference 9 occurs is

$$\begin{aligned} \mathit{SeventhsDiff9}_k &= \begin{cases} 0 & \text{if } k \le 3 \\ k-3 & \text{if } k \ge 3 \end{cases} \\ &= 0, 0, 0, 0, 1, 2, 3, 4, 5, \ldots \end{aligned} \tag{406}$$

It is 1 to 4 across an odd point ($7{-}1 + 7{-}4 = 9$) of the form

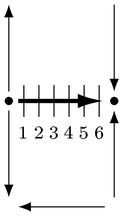


7ths
1 to 4
difference 9

4 is from the U above and 1 is the L at B. Single-visited A means no 2,4 after 1, nor 5,6 after 4. Single-visited B means no 3,5,6 after 1.

Such a configuration is in k=4 with OL as the sole double-visited point there (curve start being towards EL), and each EL--OL bridge. Again, bridges alternate so there is 1 new such in each expansion, for count (406).

A full set of triples 1 to 6 in a segment occurs when it has all 5 neighbours from figure 86, so configurations 1,3,5 and 2,4,6 all put their fractions into m.



segments for 7ths
full set of triples 1 to 6

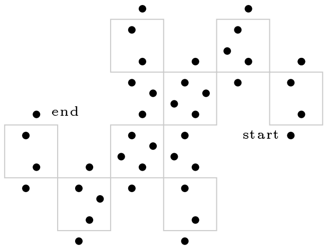
This is an enclosed unit square below, and above it either *BQ3o* boundary square or another enclosed unit square (further segment at the top).

Boundary segments grow only as r^k so fully enclosed segments are eventually the majority and occur in ever longer consecutive runs. In the numerator differences, this means runs of $1,1,1,1,1,2$.

Working through the state machine gives counts of segments with all, or not all, of 1…6 as triples. $\mathit{SeventhsNotAll}_k$ is the cubic recurrence, plus linear term.

$$\begin{aligned} \mathit{SeventhsAll}_k &= 2^k - \mathit{SeventhsNotAll}_k \\ &= 0,0,0,0,0,1,4,17,56,157,\ldots \\ \mathit{SeventhsNotAll}_k &= \mathit{SeventhsNotAll}_{k-1} + 2\,\mathit{SeventhsNotAll}_{k-3} \qquad k \ge 7 \quad (407) \\ &\qquad + 8k - 37 \\ &= 1,2,4,8,16,31,60,111,200,355,\ldots \end{aligned}$$

For the twindragon, the same argument as theorem 105 gives the number of triples $\mathit{TSeventhTriples}_k = \mathit{TA}_{k+2}$ twindragon area.



k=3
twindragon 7ths
triple points
$TSeventhTriples_3$
$= TA_{3+2} = 40$

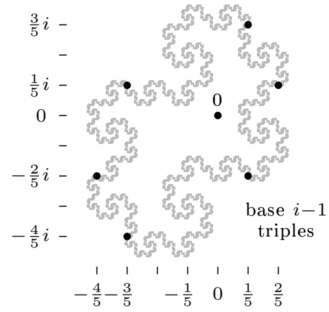
A union of all 7ths numerators occurring eventually in some triple in the twindragon can be taken.

$$\begin{array}{l}3,\ 5,\ 6,\ 10, 12, 13, 17, 19, 20, 24, 26, 27, 31, 33, 34, 37, \dots \text{ numerators} \\ \ \ 2\ \ 1\ \ 4\ \ 2\ \ 1\ \ 4\ \ 2\ \ 1\ \ 4\ \ 2\ \ 1\ \ 4\ \ 2\ \ 1\ \ 3\ \ \dots \text{ differences}\end{array}$$

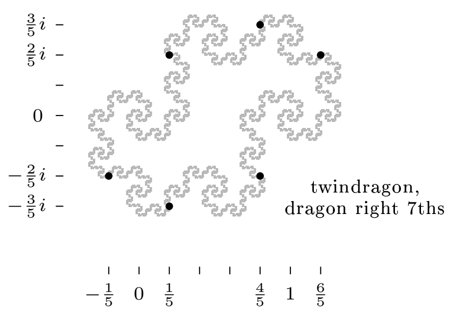
Twindragon numerators are a superset of the dragon numerators. In the twindragon, the second copy of the curve completes some triples on the dragon left side. For example 3 and 5 are numerators since the twindragon has a vertical segment at the start to complete a 1,3,5 configuration there. The 1 part is in the second copy of the curve.

Numerator differences 7 and 9 from theorem 106 do not occur in the twindragon since it has no bridges, but the others do, for differences $1, 2, 3, 4$ all occurring infinitely.

Gilbert [20] considers triple-visited points in complex base $i{-}1$ (and other $i{-}n$). The triple-visited fractional points in $i{-}1$ correspond to dragon curve 7ths on the right boundary. This is where three $i{-}1$ sub-tiles meet.



base $i{-}1$
triples



twindragon,
dragon right 7ths

The base $i{-}1$ triples are 7ths $1, 2, 4$, or $3, 5, 6$ in its f fractionals.

Theorem 107. *A 7th location is visited only by the 7ths of figure 83.*

Proof. A 7th location can be taken as a 2,4,6 triple by increasing k as necessary so fractions $/2^k$ are not in lowest terms. The location is $\frac{3}{5}+\frac{1}{5}i$ relative to the middle 6 segment of the U. Convex hulls around other (non-U) surrounding segments are

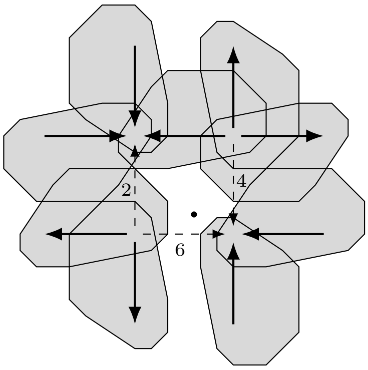


Figure 88:
hulls of other
segments surrounding
7th location
$\frac{3}{5}+\frac{1}{5}i$

These other segments are all whose convex hulls extend into the U square. None touch or contain the 7th location.

So any additional visits to this location could only be from within one of the segments 2,4,6. But in the manner of figure 85, a 2,4,6 expands in $k+1$ to a corresponding 3 smaller segments around the location. They are then the same as figure 88. Anything except the new 3 smaller segments do not touch or contain the location. ☐

17.2 15ths Fractional Triples

Gosper[23] in the OEIS considers triple points in the dragon curve fractal with denominators 15.2^k, so a same location

$$fpoint\Big(\frac{s}{15.2^k}\Big) = fpoint\Big(\frac{t}{15.2^k}\Big) = fpoint\Big(\frac{u}{15.2^k}\Big) \qquad \begin{matrix} 0 \le s,t,u \le 15.2^k \\ \text{integers} \end{matrix} \tag{408}$$

Such 15ths are the curve bridges, and the boundary points of left and right adjacent sub-curves which touch those bridges. This can be seen in the limits for the bridge point numbers from theorem 63, scaled to middle curve length 1,

$$\frac{BridgeLother_k}{2^k} \to \frac{2}{15} \qquad \frac{BlobN_k}{2^k} \to \frac{2}{5} = \frac{6}{15} \qquad \frac{BridgeRother_k}{2^k} \to \frac{22}{15} = 2 - \frac{8}{15} \tag{409}$$

The locations of all 15ths in the fractal are

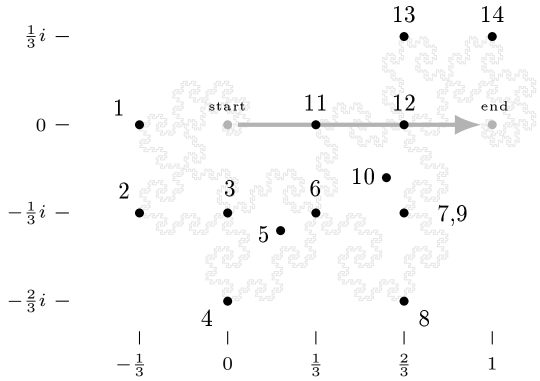


Figure 89:
$fpoint(\frac{p}{15})$
fractional
locations
$\frac{1}{15}$ to $\frac{14}{15}$

6 is before the middle biggest blob per *BlobN* limit (409), which is location c_0 of Ngai and Nguyen[38]. 12 is after this middle biggest blob. 3 is before the

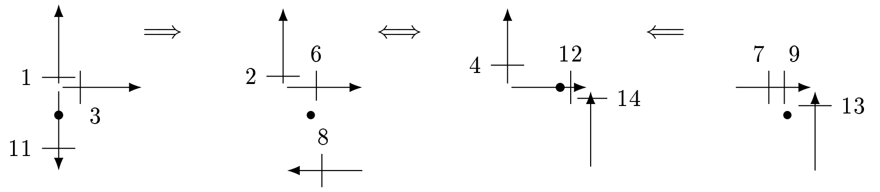
second biggest blob. 7 and 9 are at the same location and are on the boundary of the middle biggest blob. In the second half of the curve, 9 is the bridge after biggest blob in that half since $2(1-\frac{9}{15})=\frac{12}{15}$.

$3,6,9,12$ are bridge points (including 9 as bridge in the second sub-curve). They reduce to fifths 1/5 through 4/5 (ahead in section 17.2.1).

$1,2,4,8,14$ are in fact vertices of the convex hull $P4, P3, P2, P1, P7$ respectively, per locations given by Benedek and Panzone[7], and limits $\mathit{P1N}(k)/2^k \to 8/15$ etc from section 7.

Similar to the 7ths above, factor 2^k in the denominator (408) can be taken as an expansion level in finite iterations of the curve. The fractions 1/15 to 14/15 are then offsets into a segment of curve k along the direction of expansion for that segment.

Locations are multiples of $\frac{1}{3}$ and $\frac{1}{3}i$, except for 5 and 10. The following configurations are all arrangements of segments which give 15th triple locations.



1,3,11 at $z{=}{-}\frac{1}{3}i$
before second
biggest blob

2,6,8 at $z{=}\frac{1}{3}{-}\frac{1}{3}i$
before
biggest blob

4,12,14 at $z{=}\frac{2}{3}$
after
biggest blob

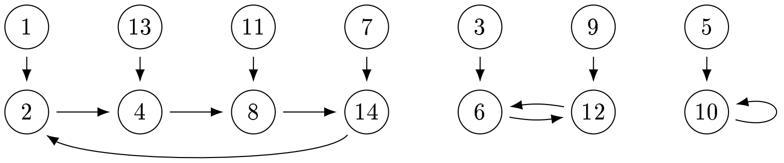
7,9,13 at $z{=}\frac{2}{3}{-}\frac{1}{3}i$
end half,
after biggest

Figure 90: 15ths sub-curve configurations

The middle segment $3,6,9,12$ in each case is the one with the bridge point (for 9, a bridge in the second half). The curling and spiralling of the curves means this is not on the straight line from its start to end, except for 12.

Configuration 2,6,8 is the curve arms of theorem 63, with the first sub-curve of the right side arm $-90°$ omitted (that segment doesn't need to be present to form a triple). Configurations expand to each other per the ⇒ arrows shown.

Under the f mapping (403) each fraction expands



Fractions $f{=}\frac{5}{15}{=}\frac{1}{3}$ and $f{=}\frac{10}{15}{=}\frac{2}{3}$ are at

$$\mathit{fpoint}(\tfrac{1}{3}) = \tfrac{1}{5}{-}\tfrac{2}{5}i \qquad \mathit{fpoint}(\tfrac{2}{3}) = \tfrac{3}{5}{-}\tfrac{1}{5}i \tag{410}$$

and each expands to 10 again. There are no adjacent sub-curves which repeat those points. This is essentially since they are not on the fractal boundary. That is clear enough in figure 89 or can be seen by considering locations in big enough k that the points have sub-curve crossing lines or hulls entirely surrounding. $k{=}5$ suffices for $f{=}\frac{2}{3}$. It is at $\mathit{fpoint}(\frac{2}{3}).b^5 = -3{-}\frac{1}{5} + (-1{-}\frac{3}{5})i$

which is in an enclosed unit square, and the blob crossing lines of the sides surround the point. $k{=}6$ has $f{=}\frac{1}{3}$ at the same location. Dragon curves touch only at their boundaries so 5, 10 are not repeated in 15ths.

Similar to the 7ths, solutions for numerators s, t, u in (408) are configurations from figure 90 which occur in a curve level k.

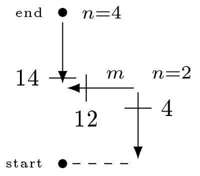
The first triple occurs in the two segments of $k{=}1$ as configuration 7,9,13 (rotated $-90°$). Fractions 7 and 9 measure back from the end of the second segment at $n{=}2$ so become $2.15 - 7 = 23$ and $2.15 - 9 = 21$. 13 measures from the start of the first segment which is $n{=}0$. So triple $13, 21, 23$.

$$\mathit{fpoint}\big(\tfrac{13}{30}\big) \ = \mathit{fpoint}\big(\tfrac{21}{30}\big) \ = \mathit{fpoint}\big(\tfrac{23}{30}\big) \ = \tfrac{1}{2} - \tfrac{1}{6} i$$

Every odd n is a configuration 7,9,13 like this. The turn left or right at n determines whether the 7,9 part or the 13 part is the first in the curve. $\mathit{turn}(n)$ is bit above lowest 1 (section 1.2) so with n odd it is left or right according as $n = 4m{+}1$ or $n = 4m{+}3$. These become sets of triples for $m \geq 0$

$$\begin{aligned} 15\,(4m{+}1 - 1) + 13 \text{ and } 15\,(4m{+}1 + 1) - 7, 9 \quad &= 60m \ + 13, 21, 23 \\ 15\,(4m{+}3 - 1) + 7, 9 \text{ and } 15\,(4m{+}3 + 1) - 13 \quad &= 60m \ + 37, 39, 47 \end{aligned}$$

Configuration 4,12,14 is the last three segments of $k{=}2$, rotated 180°.



$k{=}2$ sub-curves
left and right curves
adjacent to middle m

The offsets from those segment numbers $n{=}2$ and $n{=}4$ gives triple

$$2.15{-}4 = 26 \qquad 2.15{+}12 = 42 \qquad 4.15{-}14 = 46$$

This is simply 13,21,23 doubled, splitting 7,9 into separate sub-curves. This triple is the midpoint of the biggest blob as illustrated in figure 67 and limits

$$\frac{\mathit{MidSN}_k}{2^k} \to \frac{13}{30} \qquad \frac{\mathit{MidMNs}_k}{2^k} \to \frac{\mathit{MidMNe}_k}{2^k} \to \frac{21}{30} \qquad \frac{\mathit{MidEN}_k}{2^k} \to \frac{23}{30}$$

Like the 7ths, any triple can be doubled to make another at the same location in $k{+}1$ expanded curve. Original triples remain in the first half of the curve and unfolded in the second half, plus possible further configurations crossing the join between those halves.

Gosper takes A260748 as the smallest member of each 15th triple and sorts triples by that smallest member, so A260748 is each s which will make a triple in combination with some subsequent segments, possibly across a join.

Configurations can be made by any segments, consecutive or not. Configuration 1,3,11 is three segments around one point so two of them must be consecutive for the curve arriving at and then leaving that point.

Theorem 108. *The number of 15th fractional triple points in dragon curve k is*

$$\begin{aligned}
\mathit{FifteenthTriples}_k &= 2^k - 1 + 2D_k + 4\sum_{j=0}^{k-1} D_j \qquad (411)\\
&= 4.2^k - \bigl(2B_k - 2dJA_{k-1} - 2k + 1\bigr) \qquad k \ge 1 \quad (412)\\
&= 0, 1, 3, 7, 17, 43, 105, 247, 565, 1259, 2745, \ldots\\
\mathit{gFifteenthTriples}(x) &= \frac{1}{1{-}x} + \frac{2}{(1{-}x)^2} + \frac{4}{1{-}2x} - \frac{7{+}5x{+}8x^2}{1{-}x{-}2x^3}
\end{aligned}$$

Proof. Configuration 2,6,8 is an expansion of 1,3,11 or 4,12,14. It arises only as such an expansion, as seen by considering the middle segment as either first or second of a segment from the previous level. The two cases determine the other segments of that level and are 1,3,11 or 4,12,14.

Similarly 4,12,14 arises as an expansion of 7,9,13 or 2,6,8 and only from them.

So the number of 2,6,8 and 4,12,14 in k are the previous $\mathit{FifteenthTriples}_{k-1}$.

The number of 1,3,11 configurations in k is 4 at each even double-visited point, by taking all 4 rotations of 3 segments there.

The number of 7,9,13 configurations is all the odd points, and 2 extras at each double-visited odd point.

$$\begin{aligned}
\mathit{FifteenthTriples}_k &= \mathit{FifteenthTriples}_{k-1} + 4\mathit{Deven}_k + 2\mathit{Dodd}_k + 2^{k-1}\\
&= \mathit{FifteenthTriples}_{k-1} + 2D_k + 2D_{k-1} + 2^{k-1} \qquad (413)
\end{aligned}$$

For (413), two of the odds and evens combine to two D_k, then $\mathit{Deven}_k = D_{k-1}$ (second proof of theorem 32), for sum (411).

Each D contains $\frac{1}{2}2^k$ and splitting out those powers gives 4.2^k per (412). Roughly speaking, surrounded segments have all 12 possible 15ths going into some triple, for $12/3 = 4$ triples. Surrounded segments grow as 2^k, then less some near the boundary which lack necessary nearby segments to complete. □

Configurations 1,3,11 and 7,9,13 at double-visited even and odd points respectively could (in principle) have their bit patterns determined by n and $\mathit{other}(n)$ per section 1.5.

Gosper notes in A260747 (union of all numerators) differences between successive numerators $1, 2, 3, 4, 5, 8, 11, 20, 21$.

13, 21, 23, 26, 37, 39, 42, 46, 47, 52, 73, 74, 78, … numerators A260747
4 2 3 11 2 3 4 1 5 21 1 4 … differences

Difference 20 first occurs from 113 to 133.

Theorem 109. *The differences between successive numerators which are in some 15th fractional triple are 1, 2, 3, 4, 5, 8, 11, 20, 21. Difference 21 occurs once, the others occur infinitely often.*

Proof. Proceed in a similar manner to 7ths differences theorem 106. Every segment contains some numerator since configuration 7,9,13 puts 7,9 and 13 respectively in segments each side of every odd point. So it suffices to consider differences within a segment and across a point.

Take each segment of the figure 90 configurations as a middle m for union

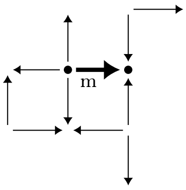
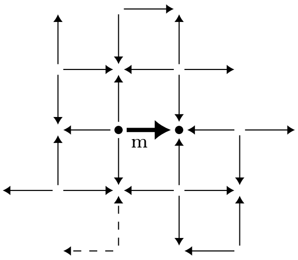


Figure 91:
15th triples
configurations union,
m as each segment

Then this union also at the next segment a, b, c, d as in figure 87 is



15th triples
union and nexts

With the two additional segments shown dashed, these segments on expansion determine a corresponding set for the first or second half of m.

Beginning from a single segment, and making successive expansions, including noting which of a, b, c, d (figure 87) is next, gives a total 335 configurations. Triples occurring within m, and next triple in next segment, show the only differences are per the theorem.

Mutual recurrences on the configurations, and the differences they contain, gives how many of each difference occur in given k. All grow without bound except 21 which is only

$$\begin{aligned} \mathit{FifteenthsDiff21}_k &= \begin{cases} 0 & \text{if } k \le 2 \\ 1 & \text{if } k = 3 \,\text{or}\, 4 \\ 2 & \text{if } k \ge 5 \end{cases} \\ &= 0, 0, 0, 1, 1, 2, 2, 2, 2, \dots \end{aligned}$$

This is at curve start, and an unfolded copy at curve end. For triples continued infinitely (k unbounded) there is only the 1 occurrence at curve start. □

Like the 7ths, it's possible instead to explicitly work through cases of a segment and its surrounds. Doing so is a little tedious, though taking possible differences in increasing order helps. Differences 1 to 4 occur variously and then there are relatively few combinations able to keep a clear gap to make bigger differences. The mechanical approach gives counts of how many of each difference occur.

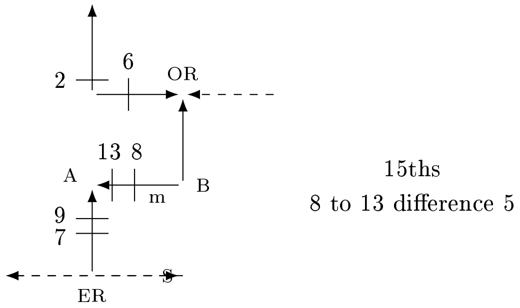
The number of times difference 2 occurs is related to *TAwS* (380) from twindragon area tree Wiener index, but just numerically, without obvious geometric etc relation,

$$\begin{aligned} \mathit{FifteenthsDiff2}_k &= 3.2^k + 1 - \frac{\mathit{TAwS}_{k+1}}{2^{k-1}} \\ &= 0, 1, 2, 5, 12, 33, 84, 197, 448, \dots \end{aligned}$$

The number of times difference 5 occurs is

$$\mathit{FifteenthsDiff5}_k = \begin{cases} 0 & \text{if } k \le 2 \\ k-2 & \text{if } k \ge 2 \end{cases} \tag{414}$$
$$= 0, 0, 0, 1, 2, 3, 4, 5, 6, \ldots$$

It is 8 to 13 within m in configurations of the form



Single-visited A means no 9 or 12 in m. Single-visited B means no 11 in m. Two single-visited points turning different ways means either a bridge or curve start or end. There is one such configuration at curve start, being m as $k{=}3, n{=}3$ directed towards curve start (point S is the curve start). Corresponding unfold at curve end is $k{=}4, n{=}12$. Thereafter it is ER--OR bridges, with the further segments shown dashed.

Bridge types alternate so each k expansion gains 1 new ER--OR bridge and hence count (414). For triples in k continued infinitely, those before the join are unchanged by expansion.

A segment with all of figure 91 has all 15ths triples except the 0, 5, 10 which are never triples. Differences in and between such segments are a repeating sequence $1, 1, 1, 2$. These are the majority since the number of segments within distance 2 of the boundary are at most a multiple of the boundary length B and so grow only as r^k whereas total segments grow as 2^k. So (like the 7ths) these full differences $1, 1, 1, 2$ become the majority and form ever longer consecutive runs.

Theorem 110. *A 15th location is visited only by the 15ths of figure 90.*

Proof. A 15th location can be taken as 2,6,8 by expanding (increasing k) as necessary. The arrows $\Longrightarrow$ in figure 90 show how one expands to another. Expanding either 1 or 2 times reaches 2,6,8. The location of 2,6,8 is $\frac{1}{3} - \frac{1}{3}i$ relative to its 6 segment. Convex hulls of surrounding non-15th segments are shown in figure 92 below.

These hulls are those which extend into the square below 6. None of them touch or contain the 15th location. The hull to the left of the point is close, but does not touch. Its vertex below is P6 rotated to $\frac{1}{3} - \frac{2}{3}i$ and the hull side goes up and leftward to P5 at $\frac{1}{6} + \frac{1}{6}i$.

So other visits to the location can only be from within one of the 2,6,8 segments.

But 2 expansions returns to 2,6,8 again and a corresponding 3 smaller segments around the location. Other halves of the originals are some of the surrounding segments of figure 92 and so are too far away. ☐

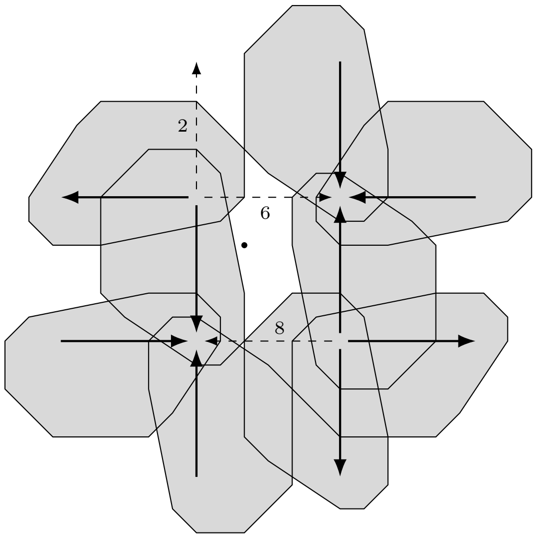


Figure 92:
hulls of other
segments surrounding
15th location
$\frac{1}{3} - \frac{1}{3}i$

17.2.1 5ths Differences

As noted above, 5ths fractions are the bridge points in 15th triples. They are the multiples of 3 among the 15ths, and that 3 can be divided out to consider them in 5ths.

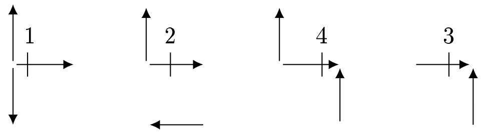


Figure 93:
5ths
configurations

Gosper notes in A260482 (union of all 5ths numerators) differences between successive numerators $1, 2, 3, 4, 5, 6, 9, 10$ (and there is an initial 12).

5ths numerators A260482

7, 13, 14, 26, 27, 28, 33, 37, 47, 52, 53, 54, 56, 57, 66, 67, 69, 71, 73, 74, 77, …

6 1 12 1 1 5 4 10 5 1 1 2 1 9 1 2 2 2 1 3 …

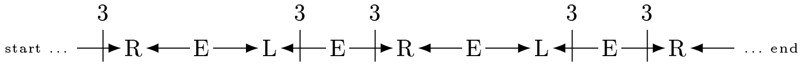
differences

Theorem 111. *The differences between successive numerators which are a 5th in some fractional triple are* $1, 2, 3, 4, 5, 6, 9, 10, 12$*. Difference 12 occurs once, the others occur infinitely often.*

Proof. In a similar manner to theorem 106 and theorem 109, take the union of the 5ths configurations figure 93. These segments present or absent determines all 5ths in m.

Figure 94:
5ths union

At the last triple in a segment, difference is to the first triple of the next segment which has a triple. The 3 configuration in figure 93 is a 3 before an odd point right turn and after an odd point left turn. As from section 1.2, the turns at odd points alternate L,R so



Each E is an even point. It can turn left or right so R to L in two forms

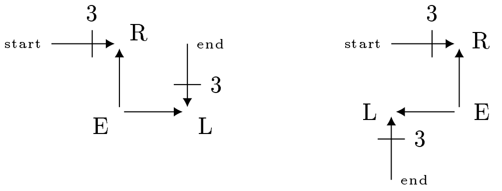
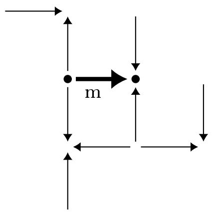
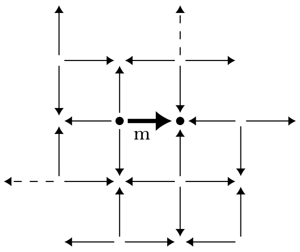


Figure 95:
5ths R to L

At any segment, it suffices to go forward by 1, 2 or 3 segments to reach a 3. The worst case is segment preceding an R which must go forward 3 segments to after an L. If m is each segment in figure 95 then the union of segments for steps forward to after L is



Then to know all 5ths in each such segment is figure 94 at each position



With the two additional segments shown dashed, these segments on expansion determine a corresponding set. Successive expansions, and tracking 3 following segments, gives total 356 configurations. Triples within m and to the first of a following segment show the only differences occurring are per the theorem.

Mutual recurrences on the configurations, and the differences they contain, gives how many of each occur in given k. All grow without bound except 12 which is only

$$\mathit{FifthsDiff12}_k = \begin{cases} 0 & \text{if } k \le 2 \\ 1 & \text{if } k = 3 \,\text{or}\, 4 \\ 2 & \text{if } k \ge 5 \end{cases}$$

$$= 0, 0, 0, 1, 1, 2, 2, 2, 2, \ldots$$

This is one 12 at curve start and another at curve end (like 21 in 15ths). For k unbounded there is just the one at curve start. $\square$

Like the 7ths and 15ths, it's possible instead to explicitly work through cases of a segment and its surrounds but doing so is a little tedious. There are only 4 possible 5ths in each segment, but it's necessary to consider next segments with no 5ths to find differences 10 and 12, and to show no other big differences.

In the configurations for the proof, the location and 7ths in 3 segments following is more than necessary when there is a 7th in the 1st or 2nd following. There might be opportunity to reduce distance or unify some configurations, but 356 is already manageable for computer calculation.

Or going bigger instead, configurations could be formed ignoring the possible turns of figure 95, just know a triple 3 occurs at most distance 3 ahead so put the 5ths union (figure 94) at all positions up to 3 away. The resulting diamond (and tracking 3 following segments) has 1034 configurations occurring and gives the same results.

17.3 Fractional Boundary

An f can be identified as boundary or non-boundary from its bits high to low using the *Rpred* state machine from theorem 21 figure 23 plus additional boundary cases at certain 15ths.

Theorem 112. *Fractional f on the boundary of the dragon fractal are*

$$\begin{aligned}
\mathit{fRpred}(f) = 1 \text{ iff } \; & \mathit{Rpred}(\lfloor f.2^k \rfloor) = 1 \text{ for all } k \\
& \text{or } \mathit{SpredLeft}_k(2n) \quad \text{if } f = (2n{-}\tfrac{1}{15} \text{ or } 2n{+}\tfrac{11}{15})/2^k && (415)\\
& \text{or } \mathit{SpredLeft}_k(2n{+}1) \quad \text{if } f = (2n{+}\tfrac{7}{15})/2^k && (416)\\
& \text{or } f = \tfrac{7}{15} && (417)\\
\mathit{fLpred}(f) = 1 \text{ iff } \; & \mathit{Lpred}(\lfloor f.2^k \rfloor) = 1 \text{ for all } k \\
& \text{or } \mathit{SpredRight}_k(2n) \quad \text{if } f = (2n{+}\tfrac{1}{15} \text{ or } 2n{-}\tfrac{11}{15})/2^k && (418)\\
& \text{or } \mathit{SpredRight}_k(2n{-}1) \quad \text{if } f = (2n{-}\tfrac{7}{15})/2^k \\
\mathit{fBpred}(f) = \; & \mathit{fRpred}(f) \text{ or } \mathit{fLpred}(f)
\end{aligned}$$

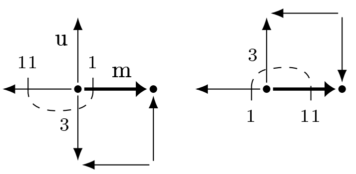
where single visited n with left or right turn are

$$\begin{aligned}
\mathit{SpredLeft}_k(n) &= 0{<}n{<}2^k \text{ and } \mathit{Spred}_k(n) \text{ and } \mathit{turn}(n) = 1 \quad \text{(left)}\\
\mathit{SpredRight}_k(n) &= 0{<}n{<}2^k \text{ and } \mathit{Spred}_k(n) \text{ and } \mathit{turn}(n) = -1 \quad \text{(right)}
\end{aligned}$$

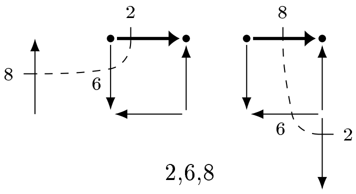
SpredLeft and *SpredRight* are for a turn within level k. There is no turn at curve start $n{=}0$ nor at end $n{=}2^k$ so false there.

Case (417) $f{=}\frac{7}{15}$ would be in (416) as $n{=}0, k{=}0$ if $\mathit{SpredLeft}_0(1)$ was reckoned true, being $2n{+}1 = 2^k$ end of its level $k{=}0$. The curve continued infinitely does turn left there, but $k{=}0$ is the only level where 2^k end is odd so that it matters.

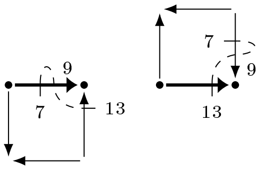
Proof. *Rpred* and *Lpred* are 3 enclosing segments for non-boundary on their respective sides. But in the fractal, those enclosing segments have bridge points per Ngai and Nguyen [38]. Points of a segment m which is otherwise enclosed can be at those bridges and so on the boundary beyond. These points and bridges are the 15th triples from section 17.2,



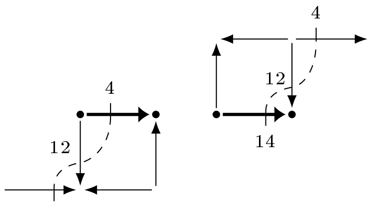
1,3,11



2,6,8



7,9,13



4,12,14

In 1,3,11 with 1 in m, that $1/15$ is on the right of m and at the location of the bridge 3. If segment 3 is present then the other side of it is the left of the curve. If 11 is present too then it encloses the point. So $1/15$ is left boundary when 3 present but 11 absent. The start of m is an even point $2n$. To have 11 absent requires single-visited. To then have 3 present requires a right turn, hence *SpredRight* in (418).

11 in m similarly, it being the left of the sub-curve and the other side of the bridge 3 is the right of the curve, hence (415).

In 7,9,13, an absent 13 ensures point 7 is on the right boundary. Start of m is an even n and its end is $2n+1$. So at $2n+1$ single-visited and left turn away from 13, hence (416). 7,9 are in the same segment so always occur together, so there is no 13 across bridge 9 to absent 7.

2,6,8 and 4,12,14 arise only as expansions of a previous 1,3,11 or 7,9,13 per theorem 108, and with the bridge part remaining a bridge and the two sides remaining separate. So relevant presence or absence in the 1,3,11 and 7,9,13 forms determines that in 2,6,8 or 4,12,14.

It can be noted for 7,9,13 that *Rpred* is already true when 13 is absent, since that 13 is among the right side segments considered by *Rpred*. Also the 1,3,11 case has *Lpred* true since single-visited and 3 present means top segment u is absent. This is *Rpred* and *Lpred* giving the right answer for f at that many bits. But further bits of f expand to 2,6,8 and 4,12,14 and for them the absent segment is further away and its absence does not imply *Lpred* or *Rpred* true. It can, and does, happen that those become false so that the 15ths cases are necessary.

m is shown in direction of expansion. When this is back along the curve, the sides right and left swap, hence the opposite $-1, -11, -7$ cases. For 7,9,13 reverse, $2n$ is the start of m and its arrow end is $2n-1$ which is to be single-visited right turn to avoid 13. ☐

Second Proof of Theorem 112. A sub-curve m has its convex hull touched or overlapped by the hulls of the following surrounding segments

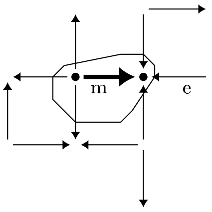


Figure 96:
surrounding segments
whose hulls touch
or overlap,
hull of m

It can be noted this is 15ths union figure 91 with additional end segment e. The 15ths are a subset since of course another sub-curve making a 15th triple with m is a touch of m.

If m has all segments of figure 96 surrounding it then it is non-boundary since, by construction, it is does not touch or overlap the hull of any absent outside. Conversely, if m has one or more of the segments of figure 96 absent, then that is some part of the hull of m which is outside the curve and therefore some of m possibly on the boundary.

Hulls beyond figure 96, so not touching m, can be illustrated

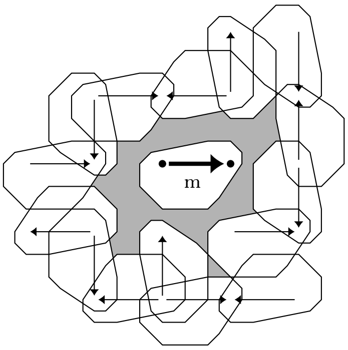


Figure 97:
non-touching
hulls

When m is surrounded by all segments of figure 96, the grey area here is a minimum amount of filled region surrounding m. The closest approach of an absent outside is $\frac{1}{6}$ at the right, and the same downwards from m start.

The non-touching hulls in figure 97 are just those close enough to delimit the grey filled region. Actually that region will be bigger than shown, since the two 3-segment points in the lower row of figure 96 will have their 4th segment present too so the curve can both enter and leave, but knowing that is not necessary.

As a remark, the triangle at the bottom touches the rest at a point, and has another much smaller triangle at its bottom right also touching at a point. Another small triangle is at top right from corresponding relative hulls.

Similar to theorem 106, a given m has some of the segments of figure 96 here surrounding it. Initially at k=0 there are none. On expansion there are new segments around the first and second half sub-curves. The segments of figure 96 suffice to determine a corresponding set around both new halves. The result is a finite set of configurations for a state machine traversed by bits of f.

m and the other segments have arrows shown in the direction of expansion. A configuration can be m forward or reverse along the direction of the curve.

On expansion in a forward segment, the first half is for bit=0 and the second for bit=1. In a reverse segment, the other way around. For both cases, the direction in the new configuration is forward if bit=0 or reverse if bit=1.

A fully surrounded configuration expands to fully surrounded for next bit 0 or 1. So if the bits of f ever reach fully surrounded then it remains so always. If f never reaches fully surrounded then that is an absent sub-curve at distance $\leq 1/\sqrt{2}^k$ for ever increasing k, so m an arbitrarily small distance from the outside, and so a boundary point.

$$fBpred(f) = \begin{cases} 0 & \text{if ever reach fully surrounded} \\ 1 & \text{if never fully surrounded} \end{cases}$$

To distinguish right and left boundary, segments of the curve always turn left or right and so divide the plane into alternating left or right side squares (eg. as for area in figure 24).

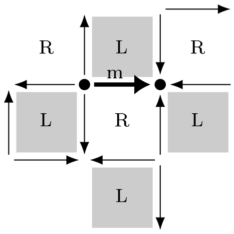


Figure 98:
possible boundary squares,
left and right sides
(swap when m reverse
along the curve)

The actual sub-curves are curling spiralling shapes, but they divide the plane into logical squares. If going from m to some absent sub-curve requires crossing another sub-curve then that goes to the other side of the curve.

In figure 98, squares are shown with just the touching hull segments of figure 96. If a square has at least 1 of its sides segments shown, but not all of them, then this is some of its R or L as boundary for m.

Figure 98 is shown with R and L on the side of expansion. If m is reverse along the curve then the sides R and L along the curve swap. If a configuration has no R in the direction of the curve then on expansion it has no R again, for next bit either 0 or 1. Similarly L.

$$fRpred(f) = \begin{cases} 1 & \text{if always an part-sided R square} \\ 0 & \text{if ever reach all complete sides R squares} \end{cases}$$

$$fLpred(f) = \begin{cases} 1 & \text{if always a part-sided L square} \\ 0 & \text{if ever reach all complete sides L squares} \end{cases}$$

Total 92 configurations arise. There are 26 with R fully enclosed, and 26 with L fully enclosed. 2 are common to them, being the full set of segments with m forward or reverse along the curve. The state machine reduces to a slightly complicated 30 states for R, or 32 states for L. Their union for B is 91 states (forward and reverse fully enclosed can merge).

The conditions of the theorem can be expressed as state machines too. $Rpred_k$ and $Lpred_k$ high to low each have state 3eB which is "eventually enclosed" in the sense that more bits from f, no matter what $0,1$ values, will reach enclosed. This state is reckoned enclosed since f always has further bits (low

0s or low 1s if an exact fraction $/2^k$). This is necessary for comparing to the hulls state machine. The hulls state machine itself does not have any eventually enclosed states (any state not enclosed is able, by some bits of f, to remain not enclosed).

The 15ths cases are extras over the Rpred_k or Lpred_k conditions. An f of the *Spred* and given 15th form should be reckoned as boundary. $\mathit{SpredRight}_k$ and $\mathit{SpredLeft}_k$ are intersection of Spred_k with *turn* bit above lowest 1 forms.

Then for *fRpred*, intersect $\mathit{SpredLeft}_k$ with low bit 0 so even $2n$, and to that concatenate 1011 repeating for $+\frac{11}{15}$. Similarly *fLpred* with $\mathit{SpredRight}_k$ and 0001 for $+\frac{1}{15}$.

The subtraction cases are conveniently reckoned as addition from preceding integer. *fRpred* case $-\frac{1}{15}$ is $2n{-}1+\frac{14}{15}$ so concatenate 1110 repeating to a $2n{-}1$ which has $2n$ as an $\mathit{SpredRight}_k$. Such subtract from all strings of a DFA is formed in usual ways, reckoning a borrow propagating up as far as necessary (as noted for consecutive enclosures in section 6). Here a borrow propagates only as far as the lowest 1-bit, above which is the turn bit (another 1-bit for right). Similarly *fLpred* case $-\frac{11}{15}$.

fRpred case $+\frac{7}{15}$ is also such a DFA decrement. It is $2n$ concatenate 0111 repeat, where $2n{+}1$ is $\mathit{SpredLeft}_k$, so decrement all odd strings in $\mathit{SpredLeft}_k$.

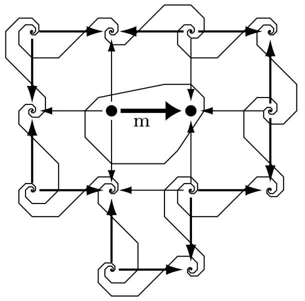
fLpred case $-\frac{7}{15}$ is not this way. Its f is $2n{-}1{+}\frac{8}{15}$ where $2n{-}1$ is $\mathit{SpredRight}_k$, so just concatenate 1000 repeating onto an odd $\mathit{SpredRight}_k$.

In all cases any bit-prefix of these patterns is accepted too as still potentially these forms.

State machine manipulations show that *Rpred* eventually-enclosed and union the 15ths cases gives *fRpred* by hulls touch, and that none of the 15ths cases stated can be omitted. Similarly *fLpred.* ☐

For *fRpred*, the 15ths bit patterns are all 0111 repeating, but with $-1, +11$ or $+7$ starting at a different position in the pattern (and different *SpredLeft* of even or odd above the repeats). Similarly *fLpred* is 1000 repeating for its $+1$, -11 or -7, which is bit flip opposite.

The surrounding segments at figure 97 can also be chosen further out, such as an upper bound for the convex hull, or using blob crossing lines to delimit the closest any outside can be. In the following diagram, if the segments with crossing lines shown are present then any outside is a non-zero distance away from m.



blob crossing lines
surrounding hull of m

The result of a bigger set of surrounding segments is more configurations, and some possibly "eventually enclosed" in the sense (from the proof) that more bits of f, no matter what 0,1, will lead to enclosed. The f bit patterns matched are the same.

Any f is an infinite bit string. After some bits, a state machine does not know whether the rest of f is good, but it does know when the bits so far are not good, ie. f is certainly non-boundary, no matter what might follow. This is an enclosed (or eventually enclosed) sub-curve m.

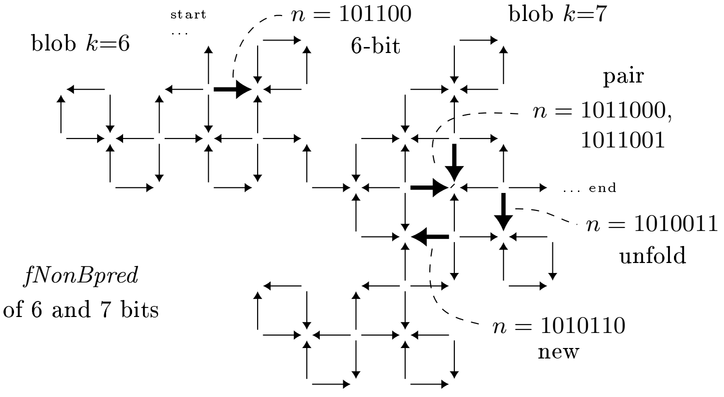
The first prefixes of f recognised as non-boundary by *fNonBpred* are

$$\begin{array}{lll} .101100 & & \text{6 bit prefix of } \textit{fNonBpred} \qquad (419) \\ .0101100 \quad .1010011 \quad .1010110 & & \text{7 bit prefixes} \\ .1011000 \quad .1011001 & & \qquad\qquad\qquad\qquad (420) \end{array}$$

The 7-bit .0101100 is an initial 0 then the 6-bit .101100. This is since if f is enclosed then so is $\frac{1}{2}f$ in the first half of the curve. .1010011 is the same but 0↔1 bit flipped. It is the 6-bit in the second half back from the curve end $1-\frac{1}{2}f$ (subtracting from 1 represented as .111...). A 0 or 1 can be appended to the 6-bit too which is (420). The remaining 7-bit is new (the third shown).

Their locations are illustrated in the following diagram. The segments are all and only those with all the surrounding segments of figure 96. There are only a few in the first few curve levels but soon they become the majority.

6-bit $n = 101100$ is on the boundary but is a 3eB type square which, as noted in the proof, is "eventually enclosed" so not *fBpred*. Likewise unfold of the 6-bit.



The number of non-boundary f recognised in initial k bits can be found by counting states which occur. The state machine transitions are mutual recurrences on these counts. Some recurrence or generating function manipulation gives totals

$$\begin{aligned} \textit{fNonR}_k &= 2^k - \textit{fR}_k \qquad \text{non-right-boundary in } k \text{ bits} \\ &= 0, 0, 0, 0, 1, 4, 12, 34, 90, 222, 522, 1186, \ldots \\ \textit{fR}_k &= \begin{cases} 1, 2, 4, 8, 15, 28 & \text{if } k \le 5 \\ R_k + \textit{SixB}_{k-3} - 2k + 8 & \text{if } k \ge 6 \end{cases} \qquad (421) \\ &= 1, 2, 4, 8, 15, 28, 52, 94, 166, 290, 502, 862, \ldots \end{aligned}$$

$$\begin{aligned} \textit{fNonL}_k &= 2^k - \textit{fL}_k \qquad \text{non-left-boundary in } k \text{ bits} \\ &= 0, 0, 0, 1, 3, 9, 25, 61, 139, 309, 675, 1449, \ldots \end{aligned}$$

$$
\begin{aligned}
\mathit{fL}_k &= \begin{cases} 1,2,4,7,13,23,39,67 & \text{if } k \le 7 \\ L_k + 2\mathit{SixB}_{k-5} - 2k + 17 & \text{if } k \ge 8 \end{cases} \\
&= 2\mathit{fR}_{k-2} + 2k - 3 \qquad \text{for } k \ge 3 \qquad\qquad (422) \\
&= 1,2,4,7,13,23,39,67,117,203,349,599,\ldots
\end{aligned}
$$

$$
\begin{aligned}
\mathit{fNonB}_k &= 2^k - \mathit{fB}_k \qquad \text{non-boundary in } k \text{ bits} \\
&= 0,0,0,0,0,0,1,5,23,81,247,673,\ldots \\
\mathit{fB}_k &= \begin{cases} 1,2,4,8,16,32,63 & \text{if } k \le 6 \\ B_k + \mathit{SixB}_{k-2} - 16k + 61 & \text{if } k \ge 7 \end{cases} \\
&= 1,2,4,8,16,32,63,123,233,431,777,1375,\ldots
\end{aligned}
$$

fR is how many strings of k bits are still candidates to be boundary f, ie. at least one f with those initial bits is on the boundary. fR_k is written using finite right boundary segments R_k since such segments, other than some of 3eB, are in *fR*. The additional segments in *fR* are counted by *SixB* from (272). But *SixB* is used only as a convenient cubic. (It is complex base $i{+}1$ close to the boundary and so twindragon and so two dragon right sides.)

fL and *fB* are written similarly. *fB* is the union of *fR* and *fL*, ie. segments which still have some point on either R or L boundary. Some segments near bridges and curve start and end are both *fR* and *fL*. They are counted just once by *fB*, so *fB* is smaller than $\mathit{fR} + \mathit{fL}$. The number of boths is difference

$$
\mathit{fR}_k + \mathit{fL}_k - \mathit{fB}_k = \begin{cases} 1,2,4,7,12,19,28 & \text{if } k \le 6 \\ 12k - 46 & \text{if } k \ge 7 \end{cases} \qquad (423)
$$

In the *SixB* forms (421) etc, $\mathit{fR} + \mathit{fL}$ is $\mathit{SixB}_{k-3} + 2\mathit{SixB}_{k-5}$ and since *SixB* is the cubic recurrence this is equal to SixB_{k-2} in *fB*, leaving (423) just linear in k after initial exceptions.

fL form (422) is essentially left boundary as two right boundaries size -2 as in theorem 18.

R_k is in fR_k since an R_k even segment has its R square below with <3 sides so *fRpred*. An R_k odd segment is reverse so the L square above is right boundary. It is <2 sides except for 3eB (see figure 22). R_k counts 3eB as boundary, but for *fR* it is not, unless one of the other L squares is boundary.

There are $\mathit{RQ3e}_k$ many 3eB segments. Some recurrence manipulations gives an identity with all 3eB subtracted. The extras (other L squares being boundary) are then, just numerically, a delayed *SeventhsNotAll* from (407).

$$
\mathit{fR}_k = R_k - \mathit{RQ3e}_k + \mathit{SeventhsNotAll}_{k-3} + 2k - 14
$$

$\mathit{fNonB}_{6,7} = 1,5$ are the prefixes shown at (419). As noted there, a 0 or 1 can be appended to each preceding bit string since all of that string is non-boundary. Counts of new non-boundary, meaning those not extending a previous, can be found by subtracting twice the $k{-}1$ counts. These counts of new grow as the cubic r^k.

$$
\mathit{fNonRnew}_k = \begin{cases} 0 & \text{if } k = 0 \\ \mathit{fNonR}_k - 2\mathit{fNonR}_{k-1} & \text{if } k \ge 1 \end{cases}
$$

$$= 0, 0, 0, 0, 1, 2, 4, 10, 22, 42, 78, 142, 250, \dots$$

$$\mathit{fNonLnew}_k = \begin{cases} 0 & \text{if } k = 0 \\ \mathit{fNonL}_k - 2\mathit{fNonL}_{k-1} & \text{if } k \geq 1 \end{cases}$$

$$= 0, 0, 0, 1, 1, 3, 7, 11, 17, 31, 57, 99, 173, \dots$$

$$\mathit{fNonBnew}_k = \begin{cases} 0 & \text{if } k = 0 \\ \mathit{fNonB}_k - 2\mathit{fNonB}_{k-1} & \text{if } k \geq 1 \end{cases}$$

$$= 0, 0, 0, 0, 0, 0, 1, \ \ 3, 13, 35, 85, 179, 349, \dots$$

The number of f which are *fNonRpred* is uncountably infinite, since once reaching "non", further bits of f can be an arbitrary real.

The number of f which are boundary *fRpred* is uncountably infinite too. That can be seen in the *Rpred* state machine (figure 23) where there are various different ways bits of f can loop back so as to always stay away from "non". For example 100 is 1e,1o,2eA returning to 1e, and 1101000 is 1e,1o,2eB,... returning to 1e. The bits of an arbitrary real can be coded as 0 → 100, 1 → 1101000 so there are at least as many *fRpred* as reals. The same holds for *fLpred*, and then their union is *fBpred*.

Per the topology shown by Ngai and Nguyen [38], the only points on both left and right boundary are curve start and end and the bridges, which is

$$\begin{aligned} &\mathit{fRpred}(f){=}1 \text{ and } \mathit{fLpred}(f){=}1 \\ &\text{iff } f = 0,\ 1,\ \frac{2}{5.2^k},\ 1{-}\frac{2}{5.2^k} \qquad \text{integer } k \\ &\qquad = 0, \dots \tfrac{1}{20}, \tfrac{1}{10}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{9}{10}, \tfrac{19}{20}, \dots, 1 \\ &\qquad = \text{binary } .00\dots00 \quad 0110\ 0110\dots \qquad k \text{ leading 0s} && (424) \\ &\qquad\qquad\qquad\ .11\dots11 \quad 1001\ 1001\dots \qquad k \text{ leading 1s} && (425) \end{aligned}$$

In the state machines for *fRpred* and *fLpred*, the intersection of their bit patterns is these 5ths (424),(425). Roughly speaking, in the boundary squares figure 98, the only m able to be near both left and right boundary squares are those on or beside a bridge. Then expansion of bridges takes anything except the relevant 5th too far away eventually.

When considering whether a given f is on the boundary, it can often be known or proved f is not a 15th and so not subject to the extra cases of theorem 112. For example any irrational is not a 15th. A boundary point is then an f which never reaches enclosed in the *Rpred* or *Lpred* state machines.

The initial states of *Rpred* accept binary fractions where each 1-bit has 2 or more 0-bits below so 100...1000... etc. If this pattern of runs is periodic then it is rational, or if not then irrational. Various sparse binary irrational constants have this form. The Kempner-Mahler number *KM* from (23) is an example. Its initial bits take a deeper excursion, then it is sparse so on the right boundary,

$$\mathit{fRpred}(KM) = 1 \tag{426}$$

Theorem 113. *The number of visits to the location of a given f in the dragon fractal is*

$$fVisits(f) = \begin{cases} Visits_k(n) & \textit{if } f = \frac{n}{2^k} \textit{ for integer } n, k \\ 1, 2, 3 \textit{ of 7ths} & \textit{if } f = \frac{n}{7.2^k} \textit{ integer } n, k \textit{ and } n \not\equiv 0 \bmod 7 \\ 1, 2, 3 \textit{ of 15ths} & \textit{if } f = \frac{n}{15.2^k} \textit{ integer } n, k \textit{ and } n \not\equiv 0 \bmod 3 \\ & \textit{and otherwise} \\ \quad 2 & \textit{if fNonBpred but sub-bits fBpred} \\ \quad 1 & \textit{otherwise} \end{cases} \tag{427}$$

Sub-bits *fBpred* means the bits of f at and below some position are *fBpred*, so f is on the boundary of the sub-curve there.

Proof. An exact fraction $f = n/2^k$ is a vertex of curve k and is visited there the same as $Visits_k$ from (174). By plane filling, those visits enclose the point so no other sub-curves touch it. There are no additional visits from within those k sub-curves either since on repeated expansion there is an ever smaller 4 surrounding sub-curves.

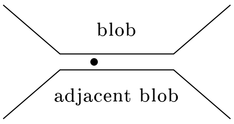
The claimed cases of whole curve *fBpred* boundary or not, and sub-curve eventually or never *fBpred*, are

	whole curve	
	fBpred	*fNonBpred*
sub-curve eventually *fBpred*	1	2
sub-curve never *fBpred*	no such	1

Per Ngai and Nguyen [38], each blob is a topological disc and the blobs of the whole curve touch only at the bridge points.

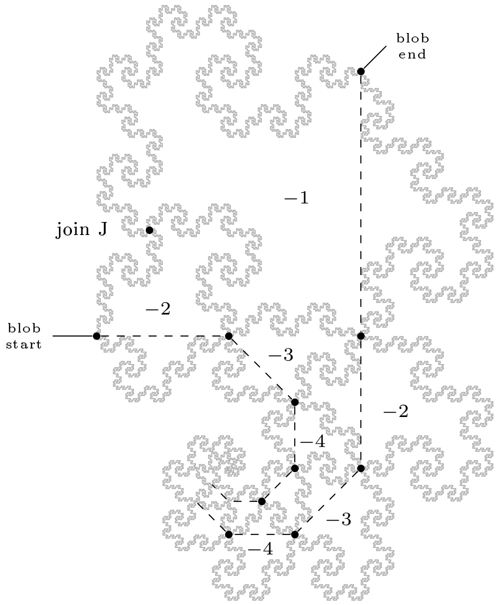
An f which is on the boundary of some sub-curve is on the boundary of a blob. There are countably infinite blobs in a sub-curve and which one has f is determined by where f falls between the bridge points.

f may have an adjacent blob from another sub-curve too, or perhaps more then one such.



f on blob boundary
and adjacent other blob

Boundaries between blobs can be seen by considering how a blob is comprised of smaller blobs. On expanding down to level $k{+}1$ sub-curves, the biggest blob of each k sub-curve breaks into smaller blobs in the manner of section 12.4.1 new biggest blob from smaller blobs,



middle biggest blob 0
comprised of smaller
blobs $-1, -2$ etc

boundaries are between
15ths bridges, 7th join

Within the $k{+}1$ half sub-curves, the now smaller blobs of the first touch each other at bridge points, the same as the whole. Likewise the second. Between the halves, the bridge points of each do not coincide, so the boundaries are between bridge points. Blobs -2 and -1 touch at the join point J too. So common boundary between blobs always splits at the bridge points and at the join J.

An f which is not a 7th (join) or 15th (bridge) is now on the boundary of a smaller blob, and possible adjacent smaller blob. But only one adjacent, since such an adjacent splits only at 7ths and 15ths too.

By repeated expansions, an f not a 7th or 15th remains on the boundary of an ever smaller blob, so only 1 visit in its own sub-curve. And when the location is on the boundary of an adjacent blob, there likewise ever smaller and so only 1 visit. When f is on the whole-curve boundary there is no such adjacent, so no 2nd visit.

By 7ths visits theorem 107 and 15ths visits theorem 110, an f which is a 7th or 15th has 1,2,3 visits according to the presence or absence of relevant surrounding sub-curves. ☐

(427) has 15ths cases excluding $n \equiv 0 \bmod 3$, so not 0,5,10 mod 15. 0 mod 15 is the exact $n/2^k$. $5, 10$ are non-boundary as noted with the 15ths expansion, or by putting them through *fBpred*, and consequently are single-visited the same as other points never on a sub-curve boundary.

Roughly speaking, the curve can be thought of as a patchwork of blobs of various sizes. Common boundary between any two is only ever between 7th or 15th locations, but a countably infinite grid of those locations.

Second Proof of Theorem 113. A mechanical approach can be made based on ways another f' might be at the same location as a given f. The two f and f' differ at some bit position. The bits of f down to there are f in sub-curve m, and f' in a different but nearby sub-curve.

$f = .\ \boxed{\quad\text{some bits}\quad}$ m subcurve

$f' = .\ \boxed{\ \text{different bits}\ }$ s subcurve

f' must be in one of the segments s of figure 96 since they are all which are close enough for their convex hull to overlap the hull of m and therefore could have a point in common with m.

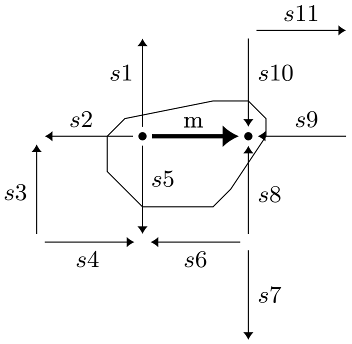
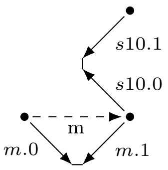


Figure 99:
surrounding segments
whose hulls touch
or overlap,
hull of m

An f visited ≥ 2 times must have some part of an adjacent s remaining close enough to m on all subsequent expansions. For example, suppose f' is in segment s10. On expansion s10 becomes new segments A and B,



Bit 0 of f is the first expanded segment $m.0$. $s10.0$ and $s10.1$ are both too far away, ie. they are not among the segments of figure 99.

Bit 1 of f is the second expanded segment $m.1$. Relative to it, $s10.0$ and $s10.1$ are respectively s5 and s4. So a 1-bit (in f) goes to require either s4 or s5 close enough. It doesn't matter which of s4 or s5, since both have some part of the f' which is different from f.

A requirement for a given s to remain close enough to m is reckoned as a state. On expansion, a bit (0 or 1) of f goes to none, one or either of two new required s which are the destination states for that bit.

If there are no new states then that is non-accepting. For these bits of f, the original s is too far away to have a point in common with f.

If there are two new states then either satisfies the s requirement. This can be expressed by an NFA where any among multiple transitions can be taken by a given bit of f. The equivalent in a DFA, and the usual way an NFA is converted to a DFA, is to reckon a state as a set of s, at least one of which must remain close enough.

When m is forward along the curve, bit 1 goes to m reverse along the curve, the same as in the *fBpred* state machine. In figure 100, the s states are m forward along the curve and the r states are the same relative segments but m reverse along the curve.

When reverse, the expanded segments of m are bits $1, 0$ instead of $0, 1$. In both cases bit 0 goes to new state forward (s) and bit 1 goes to new state reverse (r). The starting state is likewise s when the initial "some bits" of f end with a 0, or r where they end with a 1.

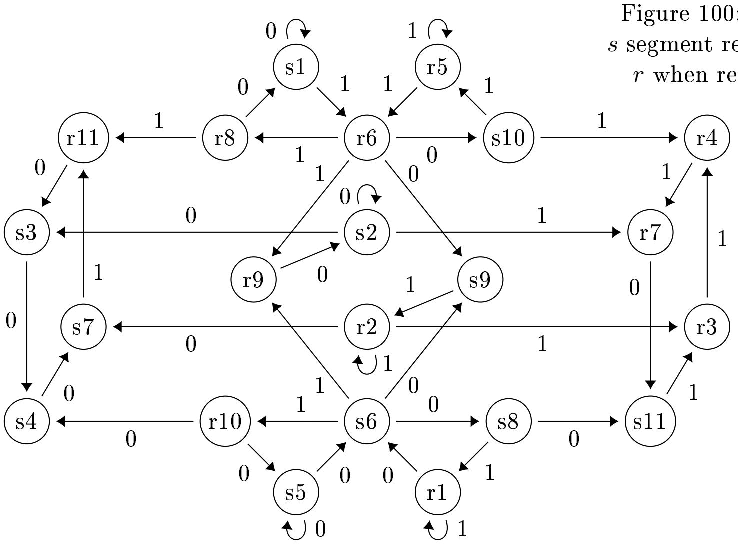


Figure 100: NFA
s segment required,
r when reverse

2 other f', f'' at the same location as f is constructed by considering enough bits that all three are in different sub-curves, so f in m and f', f'' in two s. Both of those 2 must remain close enough to m. This is state machine intersection of starting at each of those two s. Or two r when m is reverse along the curve. Any pair of two others suit, so union all combinations.

3 or more f', f'', f''' etc others at the same location as f is a similar state machine intersection (any 3 of s, or any 3 of r). The conditions of the theorem can be compared to various intersections.

The theorem is no $fVisits(f) \geq 5$, which is no 4 other s continuing infinitely. Working through the state machine shows the intersection of any 4 starting s, or 4 starting r, give only finite bit strings. Some are up to 5 bits long, but none continue infinitely (which bits of f must do).

The theorem is exact $f = n/2^k$ at double-visited points n. Such an exact f has two representations in binary, one ending infinite zeros 000… and the other ending infinite ones 111…. So the theorem is for 3 other s continuing infinitely to be only s1,s2,s5 by 0s or s8,s9,s10 by 1s (and vice-versa r). Working through the state machine shows this is so, and therefore no $fVisits(f) \geq 4$.

.	$Dpred(n)$	000…
.	$Dpred(n{+}1)$	111…

The theorem for $fVisits(f) = 3$, which is 2 other s, is the 7ths and 15ths fractions of respective segment arrangements from figure 83 and figure 90. One of their segments is m and the other 2 segments are other s. 15th triple 7,9,13 has only one other and so does not apply. The fraction in m is a repeating bit pattern.

In the state machine, 2 other s combinations will include the 3 other s combinations too, so these bit patterns for 2 other are in addition to the exact 0s and 1s of 3 other. Working through the state machine shows all and only 2 other s combinations and their bits are 7ths, 15ths, and exacts.

The states are conceived as representing an s required to be present, but they apply also to an s required to be absent. So the "2 other" combinations are also those with one s present and one s absent, or 2 both absent. These are the theorem 7ths and 15ths cases $fVisits = 1, 2$.

An f is boundary if and only if m as the initial segment has at least 1 other s remaining absent. This can be any s, so s1 to s11 as multiple starts in the NFA. Working through the state machine shows the result is the same as *fBpred*.

The remaining theorem $fVisits = 2$ is when sub-curve boundary but whole curve non-boundary. This is sub-curve 1 other s required present, but not a 2nd required absent. A 2nd requirement means the 2 other 7ths and 15ths cases as above, hence "otherwise" in the theorem.

The remaining theorem $fVisits = 1$ is in three ways. Firstly 7ths and 15ths per above. Secondly by whole curve boundary, which is 1 other s required absent, after the "otherwise" of the 7ths and 15ths. And finally 0 other s ever remaining, which means never sub-curve boundary. □

As noted in the proof, the DFA equivalent of the NFA is to form states as a set of s where at least one is required to remain present. On expansion, a 0-bit or 1-bit goes to a new set. These sets are precisely the segment configurations occurring in the *fBpred* state machine.

In the NFA diagram (figure 100), the cycles s3,s4,s7,r1 and r3,r4,r7,s11 at the left and right are the 15ths. Notice there is no way out from them. Once reached, no other bit patterns are possible. Geometrically, this is since the hulls of s3, s4, s7, s11 touch m at only one point each (a hull vertex). The single sequence of bits in the state machine shows there is only one f in m at each of these locations, which is per theorem 110.

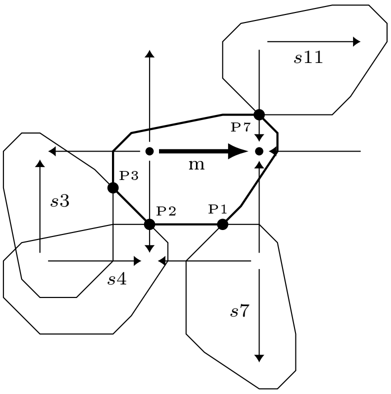


Figure 101:
s3, s4, s7, s11
hulls touch m at a
single point each,
being m hull vertices
P1,P2,P3,P7

7ths bit patterns 001 and 110 are in the 3-state loops drawn at top and bottom in the NFA. The most direct is starting s10 where 110 repeating is $\frac{6}{7}$ limit of $JN_k/2^k$ from (156).

These loops allow other bit patterns too. The self-loops such as 0 at s1 can be used to make a 15th. Roughly speaking, these are nearby segments which can be part of a 15th if desired, but not forced like the far away ones are.

Initial bits of f do not determine *fVisits*. Such bits are a sub-curve, and within any sub-curve there are all possible $fVisits = 1, 2, 3$.

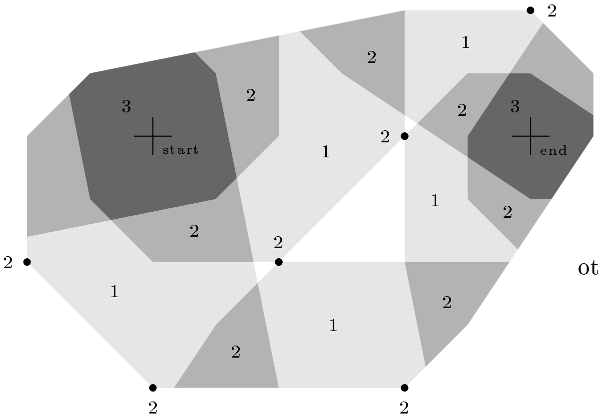
$fVisits = 1$ is at rational $\frac{1}{3}$, or irrational by any non-periodic sequence of two *fNonBpred* bit strings.

$fVisits = 2$ is at 7ths and 15ths cases when whole curve boundary, and also occurs when not. Any exact double-visited finite point $/2^k$ is a rational 2 visits. Or construct an irrational by *fNonBpred* prefix so not whole curve boundary, then stay in *fBpred* states by a non-repeating pattern (of two types of loop say).

$fVisits = 3$ are never on the whole curve boundary, and within the curve they are always solely the rational 7ths and 15ths. Consequently they are countably infinite, whereas both $fVisits = 1, 2$ are uncountable.

In the proof, the state machine shows no 4 other s are possible to make an $fVisits \geq 5$. Geometrically, this is as simple as no set of 4 other surrounding hulls having an area or point which is common to all and to m. The state machine finds a gap between any combination after a few bits expansion.

Regions of m with 0 to 3 other hulls touching or overlapping can be illustrated

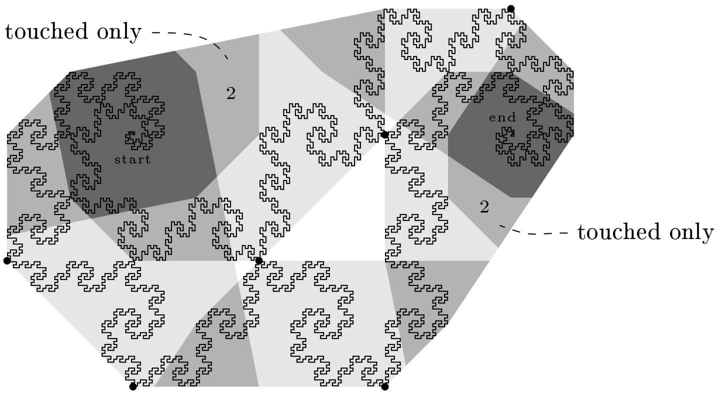


other hull overlaps

The blank (white) triangle regions in the middle are where no other hulls overlap m. Hull boundaries are inclusive, so the lines between regions are the bigger number of overlaps. The dots shown are isolated points with more overlap than their region. They are hull vertices P1,P2,P3,P7 (figure 101) and two similar type meetings inside. The insides points are since the regions each side which touch are inclusive of their boundary.

The regions of 3 other at curve start and end would be candidates for $fVisits = 4$, but per the proof they are not, essentially since they are always consecutive sub-curves arriving at then leaving the $/2^k$ start and end points.

The curve extends into all these regions, though for the top left and right-most 2s regions the curve only touches.



17.4 Fixed Point

Fractional f on the centreline (line passing through curve start and end) are characterized by the segment state machine from section 5.5. Any sub-curve without at least one end on the centreline is, per its convex hull, too far away to put any point on the centreline.

fCentrelinePred$(f) = f$ bits high to low
never reach "non" in *CentrelineSegmentPred* state machine figure 32

The *CentrelineSegmentPred* state machine can be simplified by taking 4 bits at a time. For finite iterations, this would be k a multiple of 4. For fractional f this is just a convenient grouping, since f has bits continuing infinitely. (2-bits at a time is also a simplification, to 6 states.)

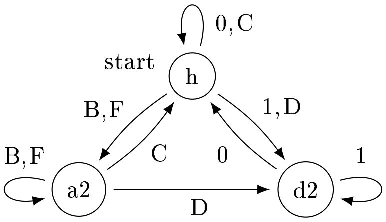
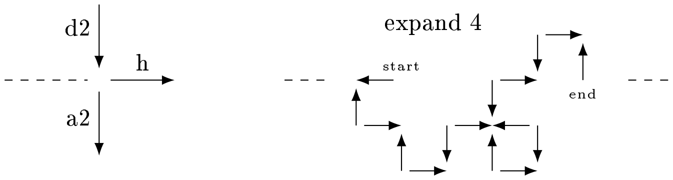


Figure 102:
f hexadecimal
fractional digits,
high to low

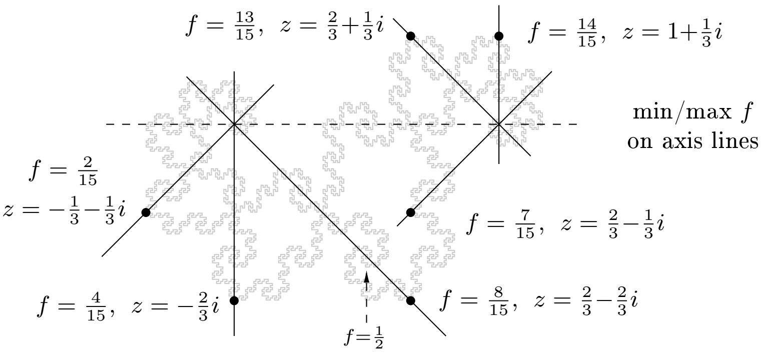
These states are the orientations shown in figure 33. On expanding 4 times, from the respective orientation, these 3 suffice for the resulting new states (turned 180° if necessary as that does not change the centreline).



Fractional f on other lines relative to curve start or end are by starting the full binary *CentrelineSegmentsPred* state machine in the various states shown in figure 34, the same as finite iterations.

The biggest f on lines at 45° angles through curve start follow by taking a 1-bit if possible at each state. From state s3 this is s3,s2,s1,d2 and back to s3 for repeating bit pattern $0,0,1,0$ which is $f=\frac{2}{15}$. From states s2 or s1 similarly for $\frac{4}{15}$ and $\frac{8}{15}$. These are convex hull vertices too.

The smallest f on lines through curve end is by taking bit 0 if possible if possible at each state. From state e3 this is e3,e2,e1,a2 and back to e3 for repeating bit pattern $0,1,1,1$ which is $f=\frac{7}{15}$. From states e2 or e1 similarly for $\frac{14}{15}$ and $\frac{13}{15}$. The latter two are convex hull vertices.



$f=\frac{7}{15}$ is the smallest on the line South-West from the curve end. This is in the first half of the curve and in that half the South-West line is perpendicular to the centreline so that $\frac{7}{15}$ is half of $f=\frac{14}{15}$ from the whole curve.

Visually, the South-West line passes through more curve below $f=\frac{7}{15}$ and it might be wondered that some of this could be smaller f, but it's not. All that's there is either second half curling back up from $f{=}\frac{1}{2}$, or bigger parts of the first half working their way to $f{=}\frac{1}{2}$. Location $\frac{2}{3}-\frac{1}{3}i$ is a bridge point in the second half of the curve (a 7,9,13 type 15th, with absent 13), so that $\frac{7}{15}$ is on the right boundary.

Points on the centreline are $\mathit{fpoint}(f)$ entirely real, which allows the possibility of "fixed points" $\mathit{fpoint}(f)=f$. $f{=}0$ and $f{=}1$ are simple such f, and there is another too.

Theorem 114. *The dragon fractal has a fixed point* $\mathit{fpoint}(\mathit{ffixed})=\mathit{ffixed}$ *at*

$$\begin{aligned}
\mathit{ffixed} &= \sum_{j=0}^{\infty} \mathit{run}_j \qquad \textit{where } \mathit{run}_j = \frac{1}{2^{s_j}} - \frac{3}{2^{e_j}} \\
&= 1 - \frac{3}{2^4} + \frac{1}{2^8} - \frac{3}{2^{16}} + \frac{1}{2^{28}} - \frac{3}{2^{32}} + \frac{1}{2^{56}} - \frac{3}{2^{60}} + \cdots \\
&= 0.11010000111111010000\ldots \ \textit{binary} \\
&= 0.816360476\ldots
\end{aligned}$$

where the run starts s and ends e are given by "3 from 2" mutual recurrences

$$\begin{array}{lll}
s_{3h} = 2s_{2h} & s_{3h+1} = 2e_{2h} & s_{3h+2} = 2e_{2h+1} - 4 \\
e_{3h} = 2e_{2h} - 4 & e_{3h+1} = 2s_{2h+1} & e_{3h+2} = 2e_{2h+1} \\
\quad \textit{starting } s_0, e_0 = 0, 4 & &
\end{array} \tag{428}$$

$$\begin{aligned} s_j &= 0,\ \ 8, 28, 56,\ \ 64, 116, 128, 224, 236, 256, 440, \ldots \\ e_j &= 4, 16, 32, 60, 112, 120, 220, 232, 240, 436, 448, \ldots \end{aligned}$$

Proof. Each run is bits of the following form (hexadecimal "FF...FFD")

$$\begin{array}{rc|c|c} \mathit{run} = & \ldots 0\,0 & 1111\ldots1111\ \ 1101 & 0\,0\ldots \\ \hline & s \lrcorner\!\!\uparrow & \qquad\qquad\quad e \lrcorner\!\!\uparrow & \end{array}$$

Per *fpoint* figure 81, the b powers and negate below 01 bit pairs for such a run are

$$\begin{aligned} \mathit{fpoint}(\mathit{run}) &= \frac{1}{b^{s+1}} \ - \frac{1}{b^{s+2}} - \frac{1}{b^{s+3}} - \cdots - \frac{1}{e^{e-2}} \ - \frac{1}{b^{e}} \\ &= \frac{1}{b^{s+1}} \ - \Big(\frac{1}{b^{s+1}} - \frac{1}{b^{e-2}}\Big)/(b{-}1) \ - \frac{1}{b^{e}} \\ &= \frac{1}{b^{s}} + \frac{1}{b^{e}} \end{aligned} \tag{429}$$

s, e are multiples of 4 since the initial values are multiples of 4 and the recurrences maintain that. Then with $b^4 = -1$ have

$$\mathit{fpoint}(\mathit{run}) \ = \frac{(-1)^{s/4}}{2^{s/2}} + \frac{(-1)^{e/4}}{2^{e/2}} \tag{430}$$

The s numerator is $+1$ when $s \equiv 0 \bmod 8$ or -1 when $s \equiv 4 \bmod 8$. Likewise e. So forms,

$$\mathit{fpoint}(\mathit{run}) = \left\{ \begin{array}{ll} \pm & 10\ldots00\ 01 \\ \pm & 01\ldots11\ 11 \end{array} \right. \\ \qquad\qquad s/2 \lrcorner\!\!\uparrow \qquad\quad \uparrow\!\!\llcorner\ e/2$$

The bits of *run* have two 01 bit pairs, including high 0s reckoned above the first run in f, so there's no net sign change to consider when taking combinations of them in *fpoint*.

2 runs in *ffixed* are built using *fpoint* of 3 runs.

$$\begin{array}{rlr} & \overbrace{1111\ldots11\,0\,1}^{\mathit{run}_{2h}} & 0\,\overbrace{1111\ldots11\,0\,1}^{\mathit{run}_{2h+1}} \\ = & \underbrace{1111\ldots11}_{\substack{\mathit{fpoint} \\ (\mathit{run}_{3h})}} \ \ \underbrace{1\,00\ldots00\,1}_{\substack{\mathit{fpoint} \\ (\mathit{run}_{3h+1})}} & \underbrace{-\,11}_{\substack{\mathit{fpoint} \\ (\mathit{run}_{3h+2})}} \end{array}$$

$$\mathit{run}_{2h} + \mathit{run}_{2h+1} \ = \mathit{fpoint}(\mathit{run}_{3h}) + \mathit{fpoint}(\mathit{run}_{3h+1}) + \mathit{fpoint}(\mathit{run}_{3h+2}) \tag{431}$$

Recurrences (428) for s_{3h} etc give these bit combinations. $s_{3h} = 2s_{2h}$ so it is a multiple of 8 and so $+1/2^{s_{2h}}$ in (430). $e_{3h} = 2e_{2h} - 4$ which is 4 mod 8 so $-1/2^{e_{2h}-2}$. Together they are the 1s shown $\mathit{fpoint}(\mathit{run}_{3h})$ and which are the initial 1s of run_{2h}. s_{3h+1} is $+1$ at e_{2h} which is the final bit of run_{2h}.

e_{3h+1} is $+1$ at s_{2h+1}. Then $s_{3h+2} {=} 2e_{2h+1} - 4$ gives $-1/2^{e_{2h+1}-2}$. It borrows down from s_{2h+1} to produce the initial 1s of run_{2h+1}. Then $e_{3h+2} = 2e_{2h+1}$ is $+1/2^{e_{2h+1}}$ for the final bit of run_{2h+1}.

$fpoint(run_{3h+2}) = -4+1 = -3 =$ binary -11 at that position. This is the $-3/2^e$ part of run_{2h+1}. The same in run_{2h} is the combination end of $3h$ and start of $3h{+}1$.

Runs do not touch or overlap since the initial $j=0,1$ are runs $e_j - s_j \geq 4$ and gap $e_{j+1} - s_j \geq 4$. Given such lengths and gaps ≥ 4, the doubling and possible -4 in the recurrences are likewise runs and gaps ≥ 4. This and no net negations for *fpoint* between runs allow them to be combined

$$run_{2h} + run_{2h+1} = fpoint(\, run_{3h} + run_{3h+1} + run_{3h+2}\,) \qquad (432)$$

and similarly any number of pairs of runs formed by *fpoint* of triplets. ☐

The first two runs $j=0,1$ are made by *fpoint* of themselves and run $j{=}2$,

$$\begin{aligned}
&fpoint(\,.1101) &&= \;.11\\
&fpoint(\,.0000\,0000\,1111\,1101) &&= \;.0001\,0001\\
&fpoint(\,\underbrace{.0000\ldots0000}_{\text{28 zero bits}}1101) &&= -.0000\,0000\,0000\,0011\\
& && \overline{\;.1101 \qquad 1111\,1101\;}
\end{aligned}$$

Folded representation (60) can be used for a run too. The 2 blocks of 1s become 4 folded terms

$$\begin{aligned}
run &= \frac{1}{2^s} - \frac{1}{2^{e-2}} + \frac{1}{2^{e-1}} - \frac{1}{2^e} && \text{folded}\\
fpoint(run) &= \frac{1}{b^s} + (-i)\,\frac{1}{b^{e-2}} + (-i)^2\,\frac{1}{b^{e-1}} + (-i)^3\,\frac{1}{b^e} && \text{revolving}\\
&= \frac{1}{b^s} + \Big(+(-i)b^2 + (-i)^2 b + (-i)^3\Big)\,\frac{1}{b^e}\\
&= \frac{1}{b^s} + \frac{1}{b^e} \qquad \text{per (429)}
\end{aligned}$$

A pair of runs $2h, 2h{+}1$ can be thought of as "expanding" to a triplet $3h, 3h{+}1, 3h{+}2$. The s,e recurrences are

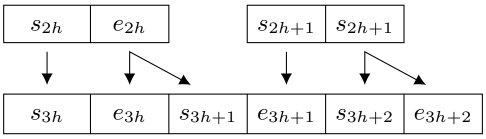


Figure 103

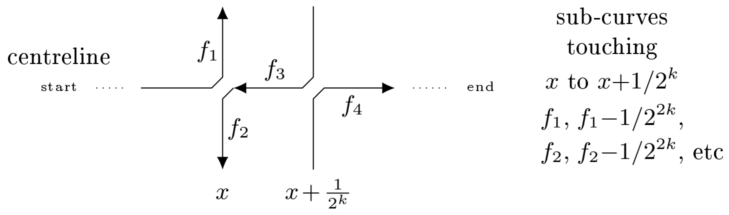
For constructing a list of runs, it suffices to take successive pairs s,e like this and append to the list their respective resulting expansion, after the initial runs which are their own expansions. See u ahead at page 355 for similar with s,e taken together.

ffixed satisfies *fCentrelinePred* by its construction of *fpoint*(*ffixed*) entirely real. *ffixed* in hexadecimal is runs *FF…FFD* which satisfy the hexadecimal state machine figure 102 (and no matter what run lengths in fact).

ffixed satisfies *fRpred* too, since in *Rpred* state machine figure 23, each run and following gap ≥ 4 zero bits returns to state 1e, so never reaching "non".

Theorem 115. *The only fixed points fpoint*$(f) = f$ *in the dragon fractal are* $f = 0, 1,$ *ffixed.*

Proof. 0 and 1 are clearly fixed points, and *ffixed* is by theorem 114. To show these are the only fixed points, consider an x of k many fractional bits which is a segment of the centreline from x to $x{+}1/2^k$,



Visits to x or $x{+}1/2^k$ are f with $2k$ many bits. The aim is to have high parts equal

$$x = \ .\boxed{aaaa}$$
$$f = \ .\boxed{aaaa}\boxed{\ \dots\ }$$

$f_{1,2}$ are sub-curves leaving location x. $f_{3,4}$ are sub-curves leaving location $x{+}1/2^k$. The turns shown are examples and can be either left or right. Each $f - 1/2^{2k}$ is a sub-curve preceding f. Altogether there are up to 7 sub-curves traversing or surrounding the x segment. These are all the sub-curves close enough that their convex hull touches the x segment.

If at least one touching segment has f high bits matching x then x might contain a fixed point. If none of the f has high bits matching x then there is no fixed point within segment x, since irrespective what lower bits might follow in such an f, the high bits are still $x \neq f$.

The x "segment" might not be a sub-curve (ie. not traversed). It may be that x extends from inside to outside the curve. Or $x{+}1/2^k$ inside and x outside. If neither end is visited then there are no surrounding sub-curves at all and no *fpoint* in the segment.

These bit match tests can be done in integers with an x location of curve level $2k$, and x directed i^k so towards curve end. *unpoint*(x) and *unpoint*$(x{+}1)$ are then n in $2k$ bits. Ignore any n bigger than 2^{2k} since that is not in level $2k$, and ignore any in an adjacent curve arm too.

A search for candidate x segments can begin at $x{=}0, k{=}0$ for the whole fractal. At k bits, extend to $k{+}1$ by trying new low 0 or 1 bit on each x candidate. Doing so down to $k{=}4$ shows 3 segments are x candidates,

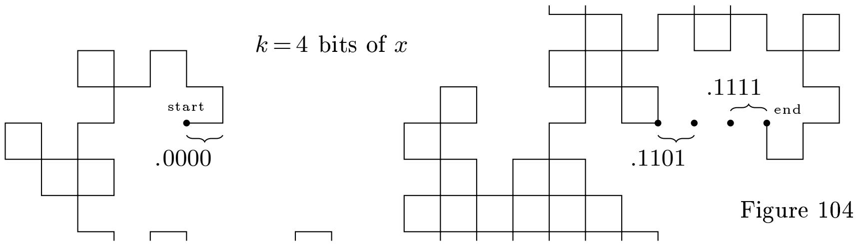


Figure 104

For curve start, segments of each possible orientation which intersect an x=0 to $1/2^k$ are

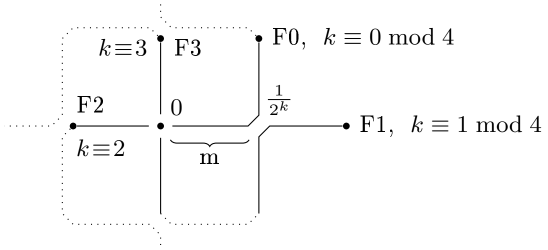


Figure 105:
sub-curves
around x=0 to $1/2^k$

The segment leaving the origin is in one of the 4 directions and the others are absent (where other arms would go). Case k=4 shown in figure 104 is $k \equiv 0 \bmod 4$. Its intersection with m extends to point F0 which is 2 sub-curves so $f \leq 2/2^{2k}$ which for k=4 is $f \leq 1/2^7$. This constrains $x \leq 1/2^7$ too so as to be equal.

The greatest f intersecting m is case $k \equiv 1$ which is 4 segments ending at F1 so $f \leq 4/2^{2k} = 1/2^{2k-2}$. Starting from k=7, repeated $k \to 2k-2$ grows without bound so that the x range able to contain a fixed point becomes arbitrarily small and so the only fixed point there is f=0.

At curve end, the same applies. The second half of the curve is the same as the first half but directed towards the start so $x \geq 1 - 1/2^k$ is $f \geq 1 - 1/2^{2k-2}$ at least. Starting from $x \geq 1 - 1/2^4$ as in figure 104 and repeatedly applying the $k \to 2k-2$ constraint shows f arbitrarily close to curve end 1 and so the only fixed point there is f=1.

Segment x = .1101 contains *ffixed*. Continue working down through x candidates and surrounding f until k=112 bits. There is just one x candidate there. This x is *ffixed* runs to j=4 inclusive. It and its f = *funpoint*(x) are

	j=0		j=1		j=2		j=3		j=4
	4	8	16	28	32	56	60	64	112
x = .	1101	0000	11...1101	00...00	1101	0...0	1101	0...0	11...1101
f = .	1101	0000	11...1101	00...00	1101	0...0	1101	0...0	11...1101
	0000	1101	0...0	11...1101	0001				
	116	120	128	220	224				
		j=5		j=6	extra				

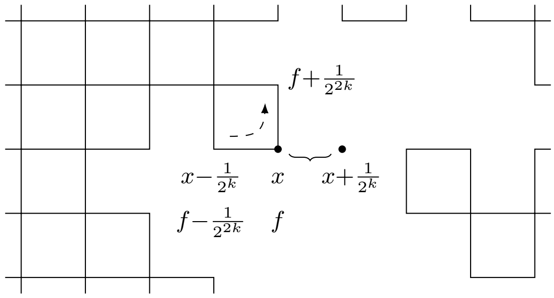


Figure 106:
k = 112 bits
candidate x

Any x comprising runs to an even j has its f in corresponding form of runs and extra 1 bit. That follows from the *ffixed* construction through to the $11\ldots11$ of the j in x, and extra 1 in f for the lowest 1 in x. This extra 1 in f is immediately above the $3h{+}1$ second new run. That second new makes this 1 and also a further lower 1, but in x here do not have the further lower.

x is single-visited in its level $2k = 224$ curve, by putting f through *Spred*, or rather through the *other* bit pattern of figure 13 which defines *Spred*. For any even j, the runs are a non-terminating *other* so only single-visited.

$$f = \underbrace{\boxed{0.11\ldots11}\,\overset{t}{\boxed{0}}\,\boxed{100\ldots00}}_{run}\,\overset{t}{\boxed{0}}\,\underbrace{\boxed{011\ldots11}\,\overset{t}{\boxed{0}}\,\boxed{100}}_{run}\,\overset{t}{\boxed{0}}\,\underbrace{\boxed{1}}_{\text{extra}}$$

other pattern
non-terminating
so single-visited

unpoint on the $x{+}1/2^k$ bits, which ends $1101 + 1 = 1110$, shows $x{+}1/2^k$ is not visited at all. This also follows from *Spred* of $f{+}1/2^{2k}$ which has the low 1-bit of f moved up one place then the same pattern of *other* runs so it too is single-visited. The bits of f show direction North and turn left at x, as illustrated in figure 106, for any x of even runs. If $x{+}1/2^k$ was visited then it would be a 1-wide gap where an adjacent curve could not both enter and leave for non-crossing plane filling of the absent segments at $f{+}1/2^{2k}$. Hence $x{+}1/2^k$ is not visited.

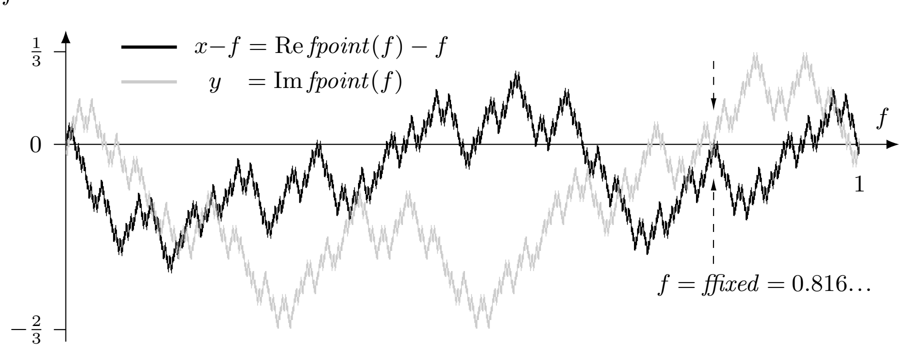
The only f's around segment x to consider are thus its f and $f{-}1/2^{2k}$. In the latter, $-1/2^{2k}$ subtracts away the extra low 1 on f. Both have the same bits down to $j{=}6$ inclusive, and so constrain x to equal those bits.

For even $j \geq 4$, there are always at least 2 more runs in f than in x like this, because the *ffixed* construction of even j (in x) is a run at $3j/2$ which is $\geq j{+}2$ (in f) when $j \geq 4$.

This constraint on x takes it to a bigger even j where the same argument applies and so, by repeating, the sole fixed point is *ffixed*. ☐

Runs down to even j are convenient in the proof since their x is single-visited, has nothing at $x{+}1/2^k$, and its f form is easily reduced for $f{-}1/2^{2k}$. Other k, after some initial small values, seem to have just one candidate x too, but sometimes both x and $x{+}1/2^k$ are visited, and sometimes double-visited. This is more fs, and only bits common to all would constrain (and thus progress) the x bits.

The fixed point solutions can be illustrated by plotting $x{-}f$ and y against f.



$x{-}f = 0$ axis touches are where $\operatorname{Re}\mathit{fpoint}(f) = f$. $y{=}0$ axis touches (the grey line) are *fCentrelinePred* points and are $\operatorname{Im}\mathit{fpoint}(f) = 0$. Both occur in various places, but simultaneously only at $f = 0, 1, \mathit{ffixed}$.

The minimum and maximum y shown are the hull extents $-\frac{2}{3}$ and $+\frac{1}{3}$.

The minimum and maximum $x{-}f$ are $-\frac{7}{15}$ and $+\frac{4}{15}$. These follow by considering sub-curves in each direction, and forward or reverse, and how they expand. Once the f increments become small enough, there is a unique square of sub-curves with maximum $x{-}f$, giving a 15th. Similarly the minimum. The minimum occurs at hull vertex P3 (which happens to be the minimum x too). The maximum occurs at hull vertex P10.

s, e at (428) are mutual recurrences. They could be expressed as individual recurrences by noting that say s goes to e but comes back later to s when certain ternary digits. Or e goes to s and then stays there only for the $3h$ case which is low ternary 0s.

But a more convenient single recurrence is to unite s, e in a single u sequence. *ffixed* is then alternating coefficients $+1$ or -3.

$$\begin{aligned} u_m &= \begin{cases} s_{m/2} & \text{if } m \text{ even} \\ e_{(m-1)/2} & \text{if } m \text{ odd} \end{cases} \qquad \text{starting } m{=}0 \\ &= s_0, e_0, s_1, e_2, s_3, e_3, \ldots \\ &= 0, 4, 8, 16, 28, 32, 56, 60, 64, 112, \ldots \\ \mathit{ffixed} &= \sum_{m=0}^{\infty} \frac{[1, -3]_m}{2^{u_m}} \end{aligned}$$

s, e cases $j \bmod 3$ at (428) would be $m \bmod 6$ for u. But $0, 1, 2$ are the same as $3, 4, 5$ so $m \bmod 3$ suffices,

$$u_{3h} = 2u_{2h} \qquad u_{3h+1} = 2u_{2h+1} - 4 \qquad u_{3h+2} = 2u_{2h+1} \qquad \text{starting } u_0 = 0,\ u_1 = 4 \tag{433}$$

$$u_m = \begin{cases} 0, 4 & \text{if } m = 0, 1 \\ 2u_{\left\lfloor \frac{2m+1}{3} \right\rfloor} - (4 \text{ if } m \equiv 1 \bmod 3) & \text{if } m \ge 2 \end{cases} \tag{434}$$

Recurrences (433) show the generating function for u satisfies

$$gu(x^2) \;=\; \big(1{+}\frac{1}{x}{+}x\big)\, gu(x^3) \;+\; \big(1{-}\frac{1}{x}{-}x\big)\, gu(-x^3) \;-\; \frac{4x^2}{1 - x^6} \tag{435}$$

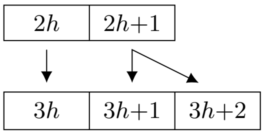
This is a usual $\frac{1}{2}(gu(x) + gu(-x))$ to select even terms $2h$, then substitute x^3 to spread them to 0 mod 6, then $2\times$ per the recurrence cancels $\frac{1}{2}$. Similarly odds $2h{+}1$ but to both $2, 4$ mod 6, and fixed -4 at 2 mod 6. The result is the whole sequence spread by x^2.

The equivalent of run from *fpoint* at (431) using u is

$$\mathit{run}_j = \frac{1}{2^{u_{2j}}} - \frac{3}{2^{u_{2j+1}}} = \mathit{fpoint}\left(\frac{1}{2^{u_{3j}}} - \frac{3}{2^{u_{3j+1}}} + \frac{1}{2^{u_{3j+2}}} \right) \tag{436}$$

Two runs like (431) would be two of (436) here, but the combined *fpoint* at (432) is more subtle in u since 3 terms of u are a net negate in *fpoint*, which causes the next triplet of u to be coefficients $-3, 1, -3$.

The recurrence cases are an "expansion" of even m to one new term and odd m to two new terms. This is each of the two expansions in figure 103.



Successive expansions of a list of values can be illustrated in a tree, or rather a path and a tree,

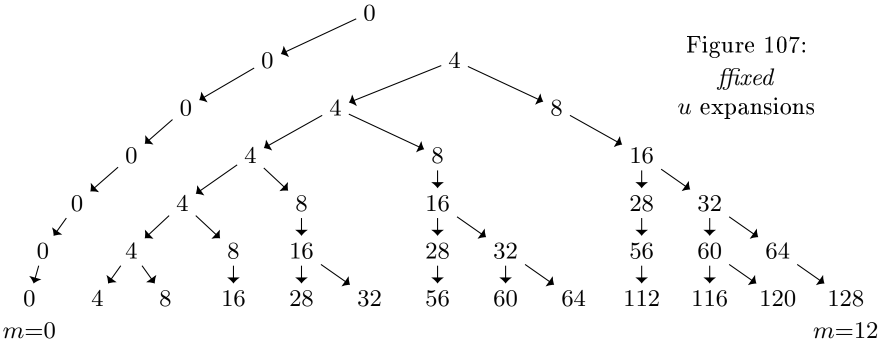


Figure 107: *ffixed* u expansions

The terms in each row repeat the previous row, then some more. For parent u, a single child is $2u$. Two children are left child $2u-4$, and right child $2u$. Whether 1 or 2 children goes according to even or odd position across the row. $m=0$ is left-most and is single child 0 each time, hence the path of 0s down.

The last m in each row is repeated $\lceil 3m/2 \rceil$, since it is $2h \to 3h$ when m there even, or $2h+1 \to 3h+2$ when m odd,

$$\begin{aligned} w_k &= \left\lceil \tfrac{3}{2} w_{k-1} \right\rceil \qquad \text{starting } w_0 = 1 \qquad\qquad \text{last } m \text{ in row} & (437) \\ &= 1, 2, 3, 5, 8, 12, 18, 27, 41, 62, 93, \ldots & \text{A061419} \end{aligned}$$

The two cases $2h$ and second of $2h+1$ are both $2u$ so the last u_m in each row is a power of 2 starting 4 at the top of the tree ($k=0$).

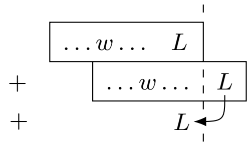
$$u_{w_k} = 2^{k+2} \tag{438}$$

u_{w_k} is start s or end e of a run according to parity $w_k \bmod 2$. Wolfram [50] notes the apparent randomness in this sequence (and various similar).

$$\begin{aligned} \mathit{wparity}_k &= 0, 1 \ \equiv w_k \bmod 2 & (439) \\ &= 1, 0, 1, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 1, 0, \ldots & \text{A205083} \end{aligned}$$

Theorem 116. *$\mathit{wparity}_k$ is not periodic nor eventually periodic.*

Proof. The effect of $\lceil \frac{3}{2} w \rceil$ is to add a copy of w shifted down one bit and its low bit L brought up to apply the ceiling,

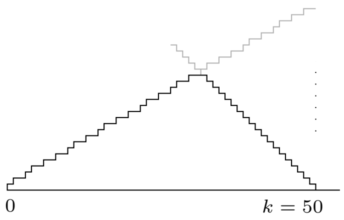


$$= \lceil \tfrac{3}{2} w \rceil \qquad (440)$$

So successive steps move bits of w down to influence the least significant bit. If two w, w' have lowest bit difference at bit position p then after p steps they differ mod 2.

If *wparity* has period $t \geq 1$ from some k onwards then $\textit{wparity}_k$ onwards and $\textit{wparity}_{k+t}$ onwards are the same sequences. But $w_k \neq w_{k+t}$, since w is strictly increasing and different values are eventually different subsequent mod 2 sequences. ☐

At (440), to know the lowest bit of $\lceil \frac{3}{2} w \rceil$ requires knowing the low 2 bits of w. In general, knowing the low t bits requires low $t{+}1$ bits of the previous w. This allows $\textit{wparity}_k$ to be calculated from successive w_j truncated to low $k-j+1$ bits. When j is small, w_j is already smaller than this many bits, but nearing the target k doesn't need full w_j.



w_j terms truncated sufficing for w_k mod 2

w grows by about $\log_2 \frac{3}{2} = 0.584962\ldots$ bits each step. Truncate begins where $j \log_2 \frac{3}{2} \sim k{-}j{+}1$ which is proportion $j/k \sim \log_3 2 = 0.630929\ldots$ (A102525). The peak size in bits there, as a proportion of what the full w would be, is the same $(\log_2 w_j)/(\log_2 w_k) \sim \log_3 2$, so peak space reduced to about $\frac{2}{3}$ the full value.

The tree of figure 107 can be restricted to just new u values in each row, as illustrated in figure 108.

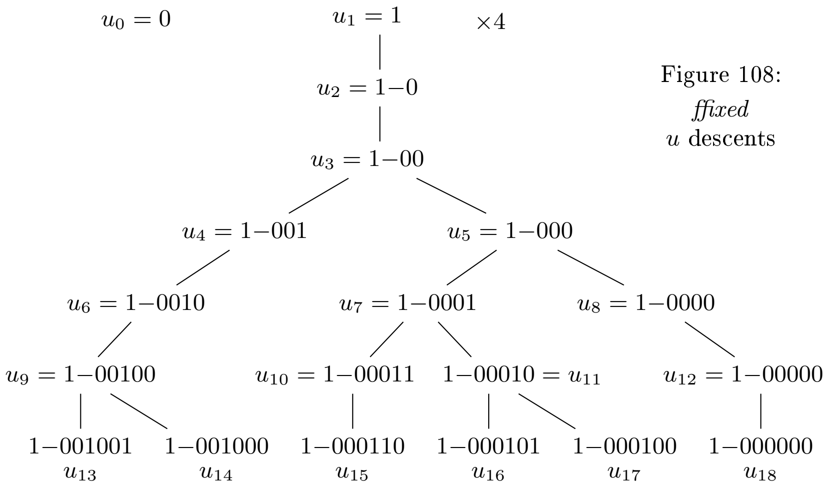


Figure 108: *ffixed* u descents

When calculating u for given m, each application of recurrence (433) is a step up to the parent u. Recurrence (433) is 2× and possible -4 until reaching $u_1 = 4$. Taking the factor 4 out means a high 1-bit and subtractions -1 at various bit positions below it. The vertex labels in figure 108 are a high 1 then bits subtracted.

For example,

$$u_{13} = 1{-}001001 \;= 4.\Big(\text{binary}(1\,000000) - \text{binary}(001001)\Big) \;= 220$$

The tree pattern is u_m of even m has a single child with additional low 0 bit, and u_m of odd m has two children, with additional low 1 bit or 0 bit. The 1 bit is subtracted so is the smaller value. Or equivalently, $m \equiv 0, 2 \bmod 3$ has low 0 on whatever value its parent, and $m \equiv 1 \bmod 3$ has low -1 (becomes -4) on whatever value its parent.

The depth of m, being how many steps of (433) up to reach $u_1 = 4$, follows from the common index reduction in (434). Depth is how many $\lfloor (2m{+}1)/3 \rfloor$, which is round$(\frac{2}{3}m)$, until reaching $m{=}1$.

$$\begin{aligned} \mathit{uDepth}(m) &= \begin{cases} 0 & \text{if } m{=}1 \\ \mathit{uDepth}\Big(\lfloor (2m{+}1)/3 \rfloor\Big) + 1 & \text{if } m \ge 2 \end{cases} \\ &= 0, 1, 2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, \ldots \qquad \text{A061420} \end{aligned}$$

The last u_m in each row is all 0 bits, being successive 0 as the only child of even m parent or second child of odd m parent. This is $m = w_k$ for u power-of-2 at (438). The row widths in figure 108 are w increments, and are how many u of given bit length. The last of each row is the first of each bit length. Reckoning the row containing u_2 as $k{=}0$, row width is

$$\begin{aligned} dw_k &= w_{k+1} - w_k \;= \lceil w_k/2 \rceil \\ &= 1, 1, 2, 3, 4, 6, 9, 14, 21, 31, \ldots \qquad \text{A005428} \end{aligned}$$

The first u of a row is $m = w_k{+}1$. It gains a low 0 when it is the only child of an even m parent, or it gains a low 1 when it is the first child of an odd m parent. The subtracted part is therefore k many successive *wparity*, from high to low and with 0↔1 flips

$$\begin{aligned} u_{w_k+1} &= 2^{k+3} \;- 4\,.\,\text{binary}\big(1{-}\mathit{wparity}_0,\ \ldots,\ 1{-}\mathit{wparity}_{k-1}\big) && (441) \\ &= 2^{k+2} + 4\; + 4.\mathit{wbin}_k && (442) \\ &= 8, 16, 28, 56, 112, 220, 436, 868, 1736, \ldots \\ w_k{+}1 &= 2,\ \ 3,\ \ 4,\ \ 6,\ \ \ 9,\ \ 13,\ \ 19,\ \ 28,\ \ \ 42, \ldots && \text{A061418} \\ \mathit{wbin}_k &= \text{binary}\ \mathit{wparity}_0,\ \ldots\ \mathit{wparity}_{k-1} \qquad \text{high to low} \\ &= \sum_{j=0}^{k-1} \mathit{wparity}_j\,.\,2^{k-1-j} \\ &= \text{binary}\ 0, 1, 10, 101, 1011, 10110, 101100, \ldots \\ &= 0, 1, 2, 5, 11, 22, 44, 88, 177, \ldots \end{aligned}$$

(442) is since (441) is a subtract of a ones-complement negative. The low

+4 changes some low bits of *wbin*. For a limit fraction, that +4 becomes ever smaller so integer 1 then *wparity* fractional bits,

$$\begin{aligned}\frac{u_{w_{k+1}}}{2^{k+2}} &\to 1 + \sum_{j=0}^{\infty} \frac{wparity_j}{2^{j+1}} \qquad (443)\\ &= 1.1011000110\ldots \quad \text{binary}\\ &= 1.693949\ldots\end{aligned}$$

Theorem 117. *The proportion of* 1*-bits to* 0*-bits up to bit position* n *in ffixed oscillates (does not converge).*

Proof. Let

$$\begin{aligned}ones(n) &= \text{number of 1s in the first } n \text{ many bits of } \mathit{ffixed}\\ &= 0,1,2,2,3,3,3,3,3,4,5,6,7,8,9,\ldots \qquad n \geq 0\\ ones_k &= ones(2^{k+2}) \qquad \text{number of 1s in the first } 2^{k+2} \text{ bits}\\ &= \sum_{j=0}^{\lceil w_k/2\rceil - 1} u_{2j+1} - u_{2j} - 1\\ &= 3,3,10,13,16,66,167,390,443,533,\ldots\end{aligned}$$

At (442), the second highest bit of u_{w_k+1} is $wparity_0 = 1$ so that

$$u_{w_k+1} \geq \tfrac{3}{2}2^{k+2} + 4 \qquad \text{for } k \geq 1 \qquad (444)$$

This means the step from $u_{w_k} = 2^{k+2}$ to u_{w_k+1} is at least $\frac{1}{2}$ of its bit position 2^{k+2}. Suppose limit $ones(n)/n \to Q$, so that for any ϵ there exists a k with

$$Q - \epsilon < \frac{ones(n)}{n} < Q + \epsilon \qquad \text{for all } n \geq 2^{k+2}$$

If w_k odd, so that u_{w_k} to u_{w_k+1} is 0-bits gap, then the number of 1s at $n = \frac{3}{2}2^{k+2}$ is the same as at $n = 2^{k+2}$. But this pushes the ratio $ones(n)/n$ down below $Q - \epsilon$ when ϵ is small enough relative to the supposed limit Q,

$$\frac{ones_k}{\frac{3}{2}2^{k+2}} = \frac{2}{3}\frac{ones_k}{2^{k+2}} < \tfrac{2}{3}(Q+\epsilon) < Q - \epsilon \qquad \text{when } \epsilon < \tfrac{1}{5}Q \qquad (445)$$

If w_k even, so that u_{w_k} to u_{w_k+1} is a run of 1-bits, then they increase the ratio $ones(n)/n$ at $\frac{3}{2}2^{k+2}$ up above ϵ when ϵ small enough relative to the supposed limit Q. The run ends at least +4 after half way (444), for $k \geq 1$, so it is all 1s to half way.

$$\frac{ones_k + \frac{1}{2}2^{k+2}}{\frac{3}{2}2^{k+2}} = \tfrac{1}{3} + \tfrac{2}{3}\frac{ones_k}{2^{k+2}} > \tfrac{1}{3} + \tfrac{2}{3}(Q-\epsilon) > Q + \epsilon \quad \text{when } \epsilon < \tfrac{1}{5}(1-Q) \qquad (446)$$

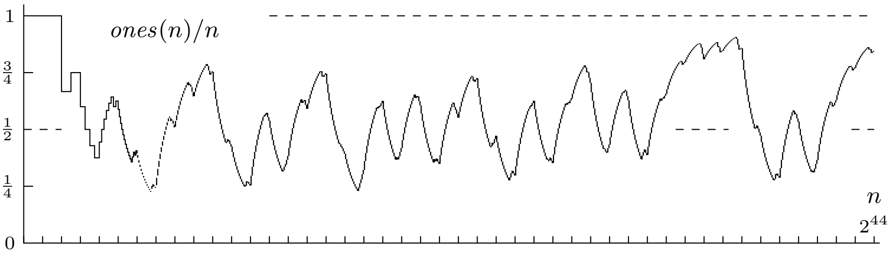
If limit Q=0 then more 0s at (445) do not push the proportion out of bounds, but (446) still does. Or conversely if limit Q=1 then (446) does not but (445) does.

From theorem 116, $w \bmod 2$ is not periodic, and in particular therefore not eventually always even or always odd. So there is always both another block of 1s and another block of 0s. ☐

That w is not eventually always even is noted by Washburn[42] and follows simply since w is non-zero and $\frac{3}{2}w$ successively strips its factors of 2 until odd.

ones$(n)/n$ convergence fails because u_{w_k} to u_{w_k+1} is a block of 1s or 0s with length which is a constant factor of its position. Factor $\frac{1}{2}$ is used in the proof, or the actual factor approaches 0.693... as at (443). A smaller factor would cause oscillation similarly. For example $\frac{1}{4}$ would be $1+\frac{1}{4} = \frac{5}{4}$ instead of $\frac{3}{2}$ at (445), etc.

Oscillation of *ones*$(n)/n$ can be illustrated in a plot. Each block of $2^k \le n < 2^{k+1}$ is scaled to the same width, and linear scale within those blocks.



Each gap of 0s in *ffixed* is a falling hyperbola where *ones*(n) is unchanged as n increases. Each run of 1s is a rising hyperbola where *ones*(n) increases with n and so increases the mean. The run or gap u_{w_k} to u_{w_k+1} from the proof is the hyperbola part at the start of block 2^k.

w_k grows only as $(\frac{3}{2})^k$, since each ceiling is

$$w_{k+1} \;\le\; \tfrac{3}{2}w_k + \tfrac{1}{2}$$

which applied repeatedly from $w_0 = 1$ is an upper bound

$$w_k \;\le\; 2.(\tfrac{3}{2})^k - 1$$

2^{k+2} bits in w_k many runs or gaps is therefore either mean run length or mean gap length, or both, diverging as $(\frac{4}{3})^k$. In fact both diverge since the step at u_{w_k} alone, ignoring all other steps which would be part of total runs or total gaps, is bigger than $(\frac{3}{2})^k$. Then as in theorem 117, the step at w_k is infinitely either a run or a gap.

$$\text{mean step} \ge \frac{u_{w_k+1} - u_{w_k}}{1 + \lfloor w_k/2 \rfloor} \ge \frac{\frac{1}{2}2^{k+2}}{2\,(\frac{3}{2})^k} \;=\; (\tfrac{4}{3})^k$$

A simple consequence of unbounded runs and gaps is that the bit pattern in *ffixed* is not eventually periodic and therefore *ffixed* is irrational.

17.4.1 Fractional Base $\frac{3}{2}$

The recurrence for u_m can be expressed by writing index m in a type of fractional base $\frac{3}{2}$. A fractional base has digits $a_k \dots a_0$ and they are coefficients of powers of the base. In this case,

$$m \;=\; a_k(\tfrac{3}{2})^k + \cdots + a_2(\tfrac{3}{2})^2 + a_1(\tfrac{3}{2}) + a_0$$

high digit $a_k = 1$ always, other digits $0, -\frac{1}{2},$ or $+\frac{1}{2}$

An integer $m \geq 1$ has a unique representation in this digits form if "integer prefixes" are demanded. Integer prefixes mean that if one or more low digits are discarded then the rest (with powers decreased accordingly) is still an integer. This occurs when the low digit is chosen so the rest is a multiple of the base $\frac{3}{2}$,

$$a_0 = \begin{cases} 0 & \text{if } m \equiv 0 \bmod 3 \\ -\frac{1}{2} & \text{if } m \equiv 1 \bmod 3 \\ \frac{1}{2} & \text{if } m \equiv 2 \bmod 3 \end{cases} \tag{447}$$

Such a low digit is removed by subtracting then dividing out the base. Then repeat for further digits, ending at high 1. That high $m{=}1$ would be digit $-\frac{1}{2}$ and then 1 in the place above, which repeats endlessly, hence the rule to stop at high digit 1.

$$\text{above } = (m - a_0)/(\tfrac{3}{2}) = (2m - 2a_0)/3 \quad = \text{integer} \tag{448}$$

$$\begin{array}{rrrrrrrl}
1 = & & & & & & 1 & \\
2 = & & & & & 1, & \frac{1}{2} & \quad \text{base } \frac{3}{2} \\
3 = & & & & 1, & \frac{1}{2}, & 0 & \quad \text{digits } 0, -\frac{1}{2}, +\frac{1}{2} \\
4 = & & & 1, & \frac{1}{2}, & 0, & -\frac{1}{2} & \quad \text{and high } 1 \\
5 = & & & 1, & \frac{1}{2}, & 0, & \frac{1}{2} & \\
6 = & & 1, & \frac{1}{2}, & 0, & -\frac{1}{2}, & 0 & \\
7 = & & 1, & \frac{1}{2}, & 0, & \frac{1}{2}, & -\frac{1}{2} & \\
8 = & & 1, & \frac{1}{2}, & 0, & \frac{1}{2}, & \frac{1}{2} & \\
9 = & 1, & \frac{1}{2}, & 0, & -\frac{1}{2}, & 0, & 0 & \\
10 = & 1, & \frac{1}{2}, & 0, & \frac{1}{2}, & -\frac{1}{2}, & -\frac{1}{2} &
\end{array}$$

Digits $0, -\frac{1}{2}, +\frac{1}{2}$ are chosen because they make (448) the same as the new index in u recurrence (433). u is then given by m in base $\frac{3}{2}$ converted to binary with -4 where digit $-\frac{1}{2}$. Or rather, with -1 as a bit then multiply throughout by 4.

$$u_m = 4 . \begin{cases} \text{digits of } m \text{ in base } \frac{3}{2} \text{ as above} \\ \text{change digits } -\frac{1}{2} \to -1 \text{ and } \frac{1}{2} \to 0 \\ \text{interpret as binary ("bits" } 0, 1, -1) \end{cases} \tag{449}$$

The converse, finding m from u, follows by considering bits of u high to low. Taking $-u$ makes the -1 "bits" from the conversion into ordinary 1 bits. A 1-bit is digit $-\frac{1}{2}$. A 0-bit is either 0 or $+\frac{1}{2}$ and the two are distinguished by the parity $m \bmod 2$, since even m descends to 0 and odd m descends to $\pm\frac{1}{2}$.

Similar works to determine the bit of *ffixed* at a particular position n. Find the smallest m with $n \leq u_m$ by working down taking the 1-bit subtraction whenever doing so preserves $n \leq u_m$, at each relevant bit position in n. Then m odd means n in a run, or m even means n in a gap,

$$\textit{ffixed} \text{ bit } n = \begin{cases} 1 & \text{if } m \text{ odd and } n \neq u_m{-}1 \\ 0 & \text{otherwise} \end{cases}$$
$$\text{where } m \text{ smallest } m \geq 1 \text{ with } n \leq u_m$$

At (438), it can be noted that u at the end of each row is all zero bits. They are from $m{=}w_k$ in base $\frac{3}{2}$ having all digits 0 or $\frac{1}{2}$ (and high 1). These w_k are all and only m which can be made without digit $-\frac{1}{2}$, since integer prefixing means only one choice of 0 or $\frac{1}{2}$ for an additional low digit.

Whether w_k digits are 0 or $\frac{1}{2}$ goes as the parity (439) of successive previous w. If w_{k-1} is even then new low digit 0. If w_{k-1} is odd then new low digit $\frac{1}{2}$ to go up for the ceiling in (437). So w_k in base $\frac{3}{2}$ is its own preceding w_0 through to w_{k-1} mod 2, and w_0 at the high end.

$$\begin{aligned} w_k &= 1,\ \tfrac{1}{2} \mathit{wparity}_0,\ \tfrac{1}{2} \mathit{wparity}_1,\ \ldots,\ \tfrac{1}{2} \mathit{wparity}_{k-1} \quad \text{base } \tfrac{3}{2} \text{ digits} \\ &= (\tfrac{3}{2})^k + \sum_{j=0}^{k-1} \tfrac{1}{2}\, \mathit{wparity}_j \,.\, 2^{k-1-j} \end{aligned} \tag{450}$$

AMM problem E2604 [42] posed by Wang and answered by Washburn forms w_k using a special constant c

$$w_k = \big\lfloor c \,.\, (\tfrac{3}{2})^k \big\rfloor \tag{451}$$

This c is the digits from (450) with radix point immediately below the high 1. Then $(\frac{3}{2})^k$ is a base $\frac{3}{2}$ shift up, and floor chops digits below the radix point.

$$\begin{aligned} c &= 1.\tfrac{1}{2}\,0\,\tfrac{1}{2}\,\tfrac{1}{2}\,000\,\tfrac{1}{2}\,\tfrac{1}{2}\,0\,\tfrac{1}{2}\,0 \ldots \quad \text{base } \tfrac{3}{2} \text{ digits} \\ &= 1.622270 \ldots \qquad\qquad \text{A083286} \end{aligned}$$

Odlyzko and Wilf [39] reach the same c in connection with the Josephus filter problem. Finch [17, section 2.30.1] notes that computationally (451) is not useful unless c is already known to sufficient precision. Effectively c simply codes all digits and (451) takes k many from it. Various sequences growing by factor and small *digit* quantities can be treated similarly. Those with periodic repeating digits give a rational constant.

m in base $\frac{3}{2}$ can be incremented by propagating a carry up through digits. u_m can be iterated by keeping m in that form

$$\begin{array}{rrcl} \text{increment digit} & 0 & \to & -\tfrac{1}{2},\ \text{carry to above} \\ & -\tfrac{1}{2} & \to & \tfrac{1}{2},\ \text{stop} \\ & \tfrac{1}{2} & \to & 0,\ \text{carry to above} \\ & 1 & \to & 1, \tfrac{1}{2},\ \text{at top, stop} \end{array}$$

Each increment sends a digit to the low of $m{+}1$ as in (447) and wrap-around $\frac{1}{2}$ back to 0. Carry propagates as $+1$ to the next higher digit position, according as the digit change. Digit $-\frac{1}{2} \to \frac{1}{2}$ is the desired increment $+1$ so no further carry.

$0 {\to} -\frac{1}{2}$ is low $-\frac{1}{2} + 1 = +\frac{3}{2}$, and so $+1$ in the next higher position. Similarly $\frac{1}{2} {\to} 0$. In an integer base, a carry $+1$ propagates only through the biggest digit, but here carry $+1$ propagates both from biggest $\frac{1}{2}$ and from 0.

Digits could be represented by any 3 distinct values. Putting a factor 2 through everything makes integers, being effectively $2m$ in base $\frac{3}{2}$ with digits $0, -1, +1$ and high 2.

Carry stops immediately when $m{\equiv}1$ mod 3. This is since it is the first child of an odd parent and increment steps to second child and nothing changed above. In general, the number of digits changed when incrementing m to $m{+}1$ is

$$\begin{aligned} \mathit{uProp}(m) &= \begin{cases} 1 & \text{if } m \equiv 1 \bmod 3 \\ 1 + \mathit{uProp}\Big(\big\lfloor (2m{+}1)/3 \big\rfloor\Big) & \text{if } m \equiv 0, 2 \bmod 3 \end{cases} \\ &= 1, 2, 3, 1, 4, 2, 1, 5, 3, 1, 2, 6, 1, 4, 2, 1, 3, 7, 1, \ldots \quad m \ge 1 \qquad \text{A087088} \end{aligned}$$

New highs in uProp are at $m = w_k$ since that is the last of a given length and so carry propagates through all digits to increase the length.

A rough calculation for mean carry propagation distance can be made as follows. $\frac{1}{3}$ of all m are 1 mod 3 which change just 1 digit. Remaining $\frac{2}{3}$ of all m change the low digit plus the mean of what's above, and above is all integers. So $P = \frac{1}{3}.1 + \frac{2}{3}(P{+}1)$ which has solution $P = 3$ mean digits changed. Incrementing in base $\frac{3}{2}$ is thus "constant amortized time" on average, like ordinary integer bases, and likewise so is incrementing u_m to next by keeping base $\frac{3}{2}$ digits.

Theorem 118. *For computer calculation, an iteration u_m to u_{m+1} can be implemented by bit-twiddling with an auxiliary quantity uB_m,*

$$u_{m+1} = u_m + 4 + (uB_m \text{ AND } \mathit{mask}) \tag{452}$$

$$uB_{m+1} = \mathit{high} + \big(uB_m \text{ AND } \text{NOT}(\mathit{mask})\big) \tag{453}$$

$$\mathit{mask} = u_m \text{ XOR } u_m{-}1 \tag{454}$$

$$\mathit{high} = u_m \text{ AND } -u_m \tag{455}$$

where

$$\begin{aligned} uB_m &= 4.\begin{cases} \textit{digits of } m \textit{ in base } \frac{3}{2} \textit{ as described above} \\ \quad \textit{binary bit } 1 \textit{ for digit } \frac{1}{2} \\ \quad \textit{binary bit } 0 \textit{ otherwise} \end{cases} \\ &= 0, 0, 4, 8, 16, 20, 32, 40, 44, 64, 80, 84, 88, 128, \ldots \end{aligned}$$

$$uB_{3h} = 2uB_{2h} \qquad uB_{3h+1} = 2uB_{2h+1} \qquad uB_{3h+2} = 2uB_{2h+1} + 4 \tag{456}$$

$$\textit{starting } uB_0 = 0,\ uB_1 = 4$$

Proof. Per (449), u is bit 0 for base $\frac{3}{2}$ digits $0, +\frac{1}{2}$. Those both propagate carry, but to different new digit. The auxiliary quantity uB is maintained to distinguish those which are $+\frac{1}{2}$. uB has the same factor 4 as u so its bits are aligned to u.

mask at (454) is 1-bits at and below the lowest 1-bit of u. These bits change in u, uB.

high at (455) is the highest 1-bit of *mask*, which is the lowest 1-bit of u. If $u{=}0$ (which is the initial $u_0 = 0$) then $\mathit{high} = 0$.

u at (452) gains the *mask* low bits from uB, and $+4$. This follows from what a base $\frac{3}{2}$ increment does to the conversion in (449). Carry extends up to the lowest digit $-\frac{1}{2}$, which is lowest -1 in bits (449). This is the lowest 1-bit in u (the -1 negative doesn't change that). It is located for *mask* by $u{-}1$ borrowing up to there (similar to what *MaskAboveLowestOne* at (7) does).

Digit $-\frac{1}{2}$ at *high* becomes $+\frac{1}{2}$, so changes from bit -1 to bit 0, so add *high* to u. Low 0 bits of u which were digit $+\frac{1}{2}$ remain 0 bits, but those which were digit 0 become bit -1. The latter are where uB has 0 bits, so subtract NOT, and subtract 3 more to leave the lowest 2 bits as zeros (multiple of 4). Thus,

$$u_{m+1} = u_m + \mathit{high} - \Big(\big(\mathrm{NOT}(uB_m) - 3\big) \text{ AND } \mathrm{RSHIFT}(\mathit{mask}, 1) \Big)$$

But *high* subtract the ones-complement NOT of bits below it is the same as add and $+1$, which with $+3$ becomes $+4$, hence (452).

uB at (453) has its low *mask* bits cleared (they moved to u), because digits $+\frac{1}{2}$ they indicated have incremented to 0, and what had been digit 0s increment to $-\frac{1}{2}$, both of which are not $+\frac{1}{2}$. The $-\frac{1}{2}$ at *high* is now $+\frac{1}{2}$, so set it alone in the low *mask* bits. ☐

When $m = w_k$, the increment increases the length of u and all bits of uB are put to u so uB becomes the *high* alone,

$$uB_{w_k+1} = 2^{k+2}$$

For addition at (452), some CPUs have "effective address" instructions which can, in one instruction, do such an add two quantities and extra constant (+4). These instructions are usually designed for array indexing but can be used for any purpose. u is 0 bits where the masked uB adds, so can instead BITOR then +4 if desired. The +4 must add, not OR, since uB can have a 1-bit at that position.

In general, the low k many digits of base $\frac{3}{2}$ can take all 3^k combinations of digit values, though this imposes some modulo restrictions on the possible value above. Any low bit pattern in uB can therefore arise by low base $\frac{3}{2}$ digits corresponding to the uB bits in *mask*.

uB is, strictly speaking, redundant in that it can be derived from u in similar way to the u to m inverse described after (449). But that works through all bits of u whereas maintaining uB is a fixed number of operations acting just on the low ends.

The recurrences for u at (433) and uB at (456) can be used for a starting pair u, uB from which to iterate. The iteration does not use the index m as such. The recurrences are the same $\mathit{round}(\frac{2}{3}m)$ in their cases, but they accumulate different -4 or $+4$.

Base $\frac{3}{2}$ representations can be formed with various digit and rule variations. Tanton[49] uses digits $0, 1, 2$ and integer prefixing expressed as "exploding dots" for teaching bases and arithmetic in bases. There, n many dots start in the units column and 3 dots "explode" to become 2 dots in the place above, and repeat until everywhere reduced to $0, 1, 2$.

The corresponding explode here for digits $0, -\frac{1}{2}, +\frac{1}{2}$ would be $\frac{3}{2}$ becoming 1 in the place above and repeat until reaching high 1 and everywhere else $0, -\frac{1}{2}, +\frac{1}{2}$.

17.4.2 Fixed Point Lengths and Gaps

Successive increments of u are

$$\begin{aligned} du_m &= u_{m+1} - u_m \\ &= 4, 4, 8, 12, 4, 24, 4, 4, 48, 4, \ldots \end{aligned}$$

The bit-twiddling (452) has du in the low bits of uB. But uB has further higher bits too, in general, so only there

$$uB_m \text{ AND } mask \;=\; du_m - 4$$

$mask$ is all bits of uB when incrementing from u_{w_k} so equality there

$$uB_{w_k} = du_{w_k} - 4$$

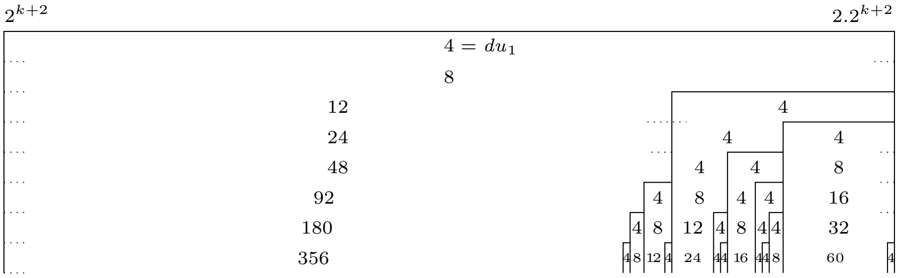
For du, the u recurrence (433) becomes

$$\begin{aligned} &du_{3h} = 2\,du_{2h} - 4 \qquad du_{3h+1} = 4 \qquad du_{3h+2} = 2\,du_{2h+1} \\ &\text{starting } du_0 = 4,\ du_1 = 4 \end{aligned} \tag{457}$$

Recurrences (433) give the generating function of du satisfying, in a similar way to gu at (435),

$$gdu(x^2) \;=\; \big(1{+}x\big)\,gdu(x^3) \;+\; \big(1{-}x\big)\,gdu(-x^3) \;+\; \frac{-4+4x^2}{1-x^6}$$

du increments which make up a block $u = 2^{k+2}$ to 2.2^{k+2}, starting $k{=}0$, can be illustrated as follows. $du_0 = 4$ is omitted. It would be from 0 to 4.



The total across a row is 2^{k+2}. A given du descends to the next row as either $2\,du - 4$ or $2\,du$ per (457). When -4, a new chain 4 starts immediately after. Those starts are at each $m \equiv 1 \bmod 3$.

The left-most chain down is, starting $k{=}0$,

$$\begin{aligned} du_{w_k} &= u_{w_k+1} - u_{w_k} \\ &= u_{w_k+1} - 2^{k+2} \\ &= 4.wbin_k + 4 \\ &= 4, 8, 12, 24, 48, 92, 180, 356, 712, 1424, \ldots \end{aligned} \tag{458}$$

The du pattern of $2\,du - 4$ or $2\,du$ goes according to $wparity_k$ at (439), with -4 where even and no -4 when odd. This is (442) without the high 2^{k+2} and hence (458).

$du_{3h+1} = 4$ is seen in (457). Other 4s also occur from $du_{2h} = 4$.

Theorem 119. *Increments* $du_m = 4$ *are characterized from index* m *by*

$$du4pred(m) = \begin{cases} 1 & \textit{if } m{=}0 \\ \begin{array}{l} TernaryLowestNon0(m) \\ \quad + TernaryCountLow0s(m) \end{array} \bmod 2 & \textit{if } m \ge 1 \end{cases}$$

$$= 1, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0, 0, 1, 0, \dots$$

$$=1 \textit{ at } m = 0, 1, 4, 6, 7, 9, 10, 13, 15, 16, 19, 22, 24, 25, \dots \qquad m{\ge}1 \text{ A189715}$$

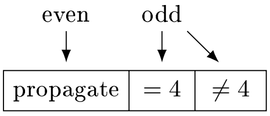
$$=0 \textit{ at } m = 2, 3, 5, 8, 11, 12, 14, 17, 18, 20, 21, 23, 26, \dots \qquad \text{A189716}$$

$$TernaryLowestNon0(n) = 1, 2, 1, 1, 2, 2, 1, 2, 1, 1, 2, \dots \quad n{\ge}1 \qquad \text{A060236}$$

$$TernaryCountLow0s(n) = 0, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, \dots \quad n{\ge}1 \qquad \text{A007949}$$

Proof. Per (457), $du_{3h+1} = 4$ always, and from theorem 114 runs and gaps are ≥ 4 so $du_{3h+2} = 2\,du_{2h+1} \ne 4$ always. These are the theorem with no low ternary 0s on m.

$du_{3h} = 2\,du_{2h} - 4$ is 4 if and only if $du_{2h}{=}4$, so it propagates the predicate.



$$du4pred(3h) = du4pred(2h)$$

If $2h \equiv 0 \bmod 3$ then the same again, until (starting from $h{\ne}0$) after some number t steps reach an $s \equiv 1, 2 \bmod 3$,

$$du4pred(3h) = du4pred(2^t.s)$$

The low ternary digit of s is 1 or 2 and each of the t many factors of 2 flips it $1 \leftrightarrow 2$, thereby flipping which of those cases, hence *TernaryLowestNon0* and *TernaryCountLow0s* in *du4pred*. ☐

Factor 2 leaving low ternary 0s unchanged and flipping the lowest 1, 2 is

$$du4pred(2h) = 1 - du4pred(h) \qquad h \ge 1 \tag{459}$$

This and $m \equiv 1, 2 \bmod 3$ cases could be a defining recurrence for *du4pred*, but expressing by ternary digits shows a "morphism" of flip first, fixed second and third,

$$\begin{array}{ll} 0 \to 1, 1, 0 & \text{apply even number of times,} \\ 1 \to 0, 1, 0 & \qquad \text{starting from single 1} \end{array}$$

Ternary also shows *du4pred* is the left turn predicate of the alternate terdragon curve (Davis and Knuth [12, end of section 5]), except at $m{=}0$ which is curve start and the curve has nothing preceding for it to turn relative to.

A consequence of this is that the number of $du{=}4$s and number of $du{\ne}4$s up to m differ by the alternate terdragon net direction *AltDir*, which (my terdragon write-up) is at most $\pm\lceil \log_9 m \rceil$. So mean proportion of $du{=}4$s converges

$$\lim_{m \to \infty} \frac{\text{num } du{=}4s}{m} \to \frac{1}{2} \tag{460}$$

The length of each *ffixed* bit run is the even du terms

$$\begin{aligned} l_j = e_j - s_j \;&= du_{2j} \qquad\qquad \text{run length} && (461)\\ &= 4, 8, 4, 4, 48, 4, 92, 8, 4, 180, 8, 4, 4, 32, 4, \dots \end{aligned}$$

(459) shows that lengths $l_j = 4$ occur opposite to $du_j = 4$, though the index doubling (461) means they do not refer to the same u bit location.

$$\begin{aligned} \mathit{l4pred}(j) &= \begin{cases} 1 & \text{if } j = 0 \\ 1 - \mathit{du4pred}(j) & \text{if } j \geq 1 \end{cases} \\ &= 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, 1, 1, 0, 1, \dots && \scriptstyle j\geq 1 \text{ A156595} \end{aligned}$$

The gap of 0s after each *ffixed* run is the odd terms of du

$$\begin{aligned} g_j = s_{j+1} - e_j \;&= du_{2j+1} \qquad\qquad \text{gap length} \\ &= 4, 12, 24, 4, 4, 8, 4, 4, 16, 4, 12, 8, 4, 356, 8, \dots \end{aligned}$$

Gaps $g_j = 4$ are characterized from index j by lowest 0 or 2 and its position.

$$\begin{aligned} \mathit{g4pred}(j) &= \mathit{du4pred}(2j{+}1) \\ &= \left\{ \begin{array}{l} \frac{1}{2}\, \mathit{TernaryLowestNon1}(j) \\ +\, 1 + \mathit{TernaryCountLow1s}(j) \end{array} \right. \mod 2 && (462) \\ &= 1, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 0, 1, 0, 0, \dots && \scriptstyle 0\leftrightarrow 1 \text{ flip A189706} \\ =1 \text{ at } j &= 0, 3, 4, 6,\ \ 7,\ \ 9, 12, 15, 16, 18, 21, 24, 25, 27, \dots \\ =0 \text{ at } j &= 1, 2, 5, 8, 10, 11, 13, 14, 17, 19, 20, 22, 23, 26, \dots \end{aligned}$$

$$\begin{aligned} \mathit{TernaryLowestNon1}(n) &= 0, 0, 2, 0, 0, 2, 0, 2, 2, 0, 0, \dots && \scriptstyle 2\times\text{A116178} \\ \mathit{TernaryCountLow1s}(n) &= 0, 1, 0, 0, 2, 0, 0, 1, 0, 0, 1, \dots && \scriptstyle \text{A253786} \end{aligned}$$

The ternary digits form (462) follow from what $2j{+}1$ becomes in *du4pred*. Low 1s ×2 become low 2s, then +1 is low 0s.

The low ternary digit pattern of *g4pred* is "always, propagate, never" so morphism

$$\begin{aligned} \mathit{g4pred} = \qquad & 0 \to 1, 1, 0 \qquad\qquad \text{starting from 1} \\ & 1 \to 1, 0, 0 \end{aligned}$$

The proportion of gaps $g{=}4$ and $g{\neq}4$ has limit $\frac{1}{2}$ each, the same as the whole du proportions (460). This is clear in blocks of 3^k since *g4pred* is just a digit flip $0 \leftrightarrow 1$ over the *du4pred* conditions. To see it for all j, the ternary low 1s or low 0s are an offset

$$\begin{aligned} \mathit{g4pred}(j) &= 1 - \mathit{du4pred}\Big(j + \tfrac{1}{2}(3^k{+}1)\Big) \\ &\text{for any } k \text{ with } 3^k > 3j \end{aligned}$$

g4pred goes by ternary low 1s. Adding all 1s by $(3^k{-}1)/2$ sends them to 2s, and then +1 more sends them to 0s ready for *du4pred* to answer.

This $(3^k{+}1)/2$ offset means that in the alternate terdragon, *g4pred* is right turn predicate for the part of the curve commencing after the middle bridge of level k. Net turn is then bounded by direction $\log_9$ again (direction of the middle bridge is 0 or 1).

17.4.3 Rotated Fixed Point

Theorem 120. *The $-i$ rotated fixed points $\mathit{fpoint}(f) = -i.f$ of the dragon curve fractal are $f{=}0$ and $f = 1{-}\mathit{ffixed}$,*

$$\begin{aligned}\mathit{fpoint}(1{-}\mathit{ffixed}) &= -i\,.\,(1{-}\mathit{ffixed})\\ 1-\mathit{ffixed} &= 0.00101111000000101111\ldots \qquad \textit{binary}\\ &= 0.183639523\ldots\end{aligned}$$

Proof. 0 is clearly a fixed point of this type since $\mathit{fpoint}(0){=}0$. To see $1{-}\mathit{ffixed}$, consider a copy of the fractal directed up from $-i$ and expanded to one level of sub-curves.

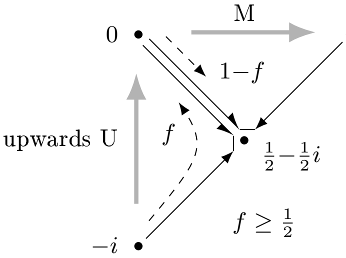


Figure 109

M is the usual curve 0 to 1. U is the upwards copy. The first half of M coincides with the second half of U. In these sub-curves, an f in U measures towards 0 but M measures away from 0 so $1{-}f$ in M for identity

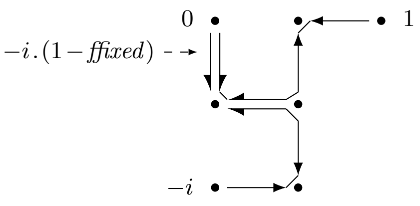
$$\mathit{fpoint}(1{-}f) = -i + i\,.\,\mathit{fpoint}(f) \qquad \text{for } f \geq \tfrac{1}{2}$$

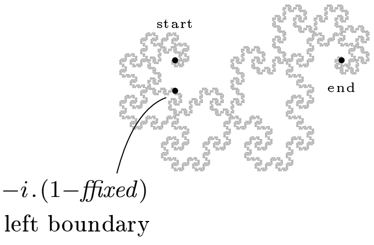
and at $f = \mathit{ffixed}$,

$$\begin{aligned}\mathit{fpoint}(1{-}\mathit{ffixed}) &= -i + i\,.\,\mathit{fpoint}(\mathit{ffixed})\\ &= -i + i\,.\,\mathit{ffixed}\\ &= -i\,.(1-\mathit{ffixed})\end{aligned}$$

To see no other rotated fixed points, the convex hull around the second half of M shows it does not touch or intersect the line 0 to $-i$. So $-i$ rotated fixed points can only be in the first half of M and therefore can only be plain fixed points in the second half of U. From theorem 115, they are only 1 and *ffixed* in U which are 0 and $1{-}\mathit{ffixed}$ in M. $\square$

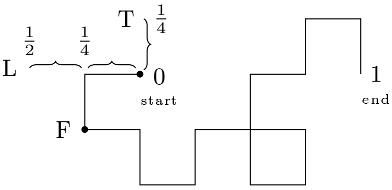
Expanding figure 109 a second time illustrates $1{-}\mathit{ffixed}$ in the first quarter sub-curve of M (last quarter of U). It is on the right boundary of U and so on the left boundary of M.





Theorem 121. *The only i or -1 rotated fixed point fpoint$(f) = i.f$ or fpoint$(f) = -f$ of the dragon fractal is f=0.*

Proof. Expand to x segments $\frac{1}{4}$ (with x directed i or -1),



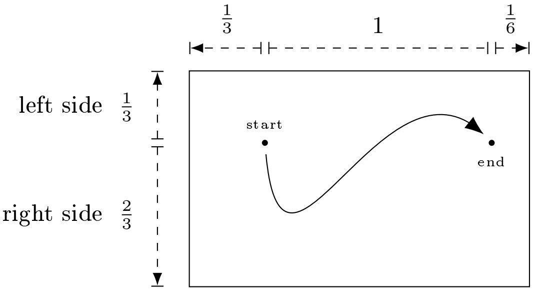
The only sub-curves whose convex hulls intersect the -1 direction towards L or the i direction towards T are the two up to F, so $f \le 2/2^4 = 1/2^3$. (These are the $k \equiv 0, 3$ orientations in figure 105.)

In the same manner as theorem 115, x is likewise constrained to $x \le 1/2^3$ and from there repeated constraints $2k-2$ restrict the only possible x and f to arbitrarily close to curve start 0. □

18 Computer Graphics

Drawing the dragon curve has been used as a graphics exercise or demonstration in various programming languages and computer systems. Often this is to demonstrate recursion. Quite simple code either recursive or iterative produces the intricate dragon curve shape.

Drawing extents follow from the convex hull in section 7. Relative to the endpoints, rectangular extents are



These limits are approached from below in all cases, so finite iterations have all segments entirely within these bounds.

If instead the first segment is held in a fixed direction, then the curve spirals around anti-clockwise.

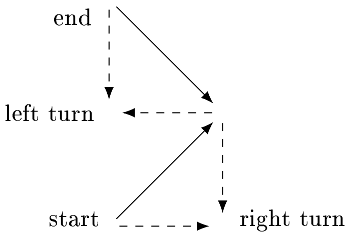
18.1 Drawing by Turn Sequence

The *turn* sequence of section 1.2 suits turtle drawing such as the Logo language or pen based plotters. Even in a graphics system without a notion of current direction it can be convenient to maintain a direction and follow the turn sequence.

```
for $n = 1$ to $2^k$
    draw line segment forward
    turn by $turn(n) \times 90°$
```

18.2 Recursion – Left and Right

The dragon curve can be drawn by mutually recursive routines which make either a right or left turn in between the sub-curves. In both cases, the sub-curves are a left-turning followed by a right-turning, they differ only in which turn is between the sub-curves. The curve starts from DragonLeft.

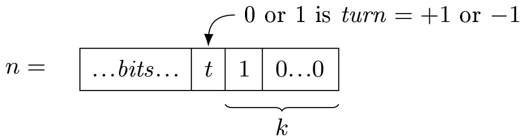


```
DragonLeft($k$)
    if $k = 0$  then  draw line segment forward
    else  DragonLeft($k-1$)
          turn left 90°
          DragonRight($k-1$)
DragonRight($k$)
    if $k = 0$  then  draw line segment forward
    else  DragonLeft($k-1$)
          turn right 90°
          DragonRight($k-1$)
```

The turn can be a parameter instead of mutual recursion,

```
Dragon($k$, $turn$)        initial $turn = +1$
    if $k = 0$  then  draw line segment forward
    else  Dragon($k-1$, $+1$)
          turn by $turn \times 90°$
          Dragon($k-1$, $-1$)
```

The effect of either a *turn* parameter or mutual recursion is to record whether drawing the first half or second half of the level above. The turn in the function is made after drawing the 2^{k-1} segments of sub-curve $k-1$ so is at a vertex number of the following form (with the origin reckoned n=0).

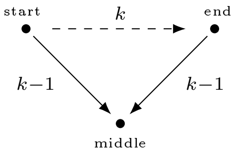


This means *turn* corresponds to *BitAboveLowestOne*(n) in the form *turn* $=$ $+1$ for 0-bit or -1 for 1-bit as per *turn*(n).

Each recursed call holds a *turn* which is the bit above so that the stack holds all the bits of vertex number n. Any recursion can be transformed to an iteration by holding parameters in an array instead of local variables. Here doing so and running up and down setting $+1$, -1 has the effect of successively incrementing n.

18.3 Recursion – Towards Midpoint

Dragon curve k comprises two $k{-}1$ directed to a common midpoint on the right of the curve. This suits a recursion which draws k by drawing sub-curves $k{-}1$ from start and end inwards to that midpoint.



```
Dragon($k$, $start$, $end$)
    if $k = 0$  then  draw line $start$ to $end$
    else  $middle = \frac{1}{2}(end + start) - i\,\frac{1}{2}(end - start)$
          Dragon($k{-}1$, $start$, $middle$)
          Dragon($k{-}1$, $end$, $middle$)
```

The recursion has no parameters except endpoints and level k. This can suit a graphics system with rotate, scaling and graphics object re-use but otherwise limited programmability. The SVG graphics file format is like this and an SVG dragon curve example by Kevin Reid at Rosetta Code uses this approach.

The effect of drawing towards the middle is to draw unit line segments in order of their binary reflected Gray code. Segment n of the curve is in the drawing sequence at position $j = Gray(n)$, so segments are not drawn sequentially. This is fine for off-line image creation but may look unusual if drawing progressively on screen.

This recursion is not well suited to a pen plotter since it moves to the end of each sub-curve to recurse back towards the middle. In the following diagram, the dashed lines show the pen-up moves between each drawn segment.

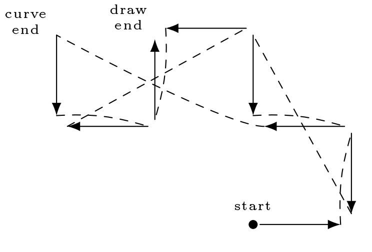


Figure 110: k=3 plotter drawing and moves
$End_3 = -1{+}2i$
$Move_3 = 10$

The amount of pen-up movement can be calculated. The end of recursions in level k is given by the end of recursion in the second $k{-}1$ sub-curve, directed inward back from the curve end,

$$\begin{aligned} \mathit{End}_k &= b^k - i\,\mathit{End}_{k-1} \qquad \text{starting } \mathit{End}_0 = 1 \\ &= \tfrac{3-i}{5} b^k + \tfrac{2+i}{5}(-i)^k \qquad (463) \\ &= 1,\ 1,\ i,\ -1{+}2i,\ -2{+}i,\ -3{-}2i,\ -2{-}5i,\ 3{-}6i, \dots \end{aligned}$$

The point number n of the end is the Jacobsthal numbers and falls within the middle biggest blob for $k \geq 4$.

$$\begin{aligned} \mathit{EndN}_k &= 2^k - \mathit{EndN}_{k-1} \qquad \text{starting } \mathit{End}_0 = 1 \\ &= \tfrac{2}{3}2^k + \tfrac{1}{3}(-1)^k \\ &= 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, \dots \qquad \text{A001045} \\ &= \text{binary } 1010\dots \text{ ending } 101 \text{ or } 11 \text{ for } k \text{ bits, } k{\geq}1 \end{aligned}$$

The move in level k is from the first half End_{k-1} to the curve end b^k ready to draw the second half inwards, or for k=0 no move. This move is the same as origin to End_{k+1}.

$$\mathit{MoveEnd}_k = \begin{cases} 0 & \text{if } k{=}0 \\ b^k{-}\mathit{End}_{k-1} \;\; = \mathit{End}_{k+1} & \text{if } k \geq 1 \end{cases}$$

Assume the plotter moves the pen x and y independently and at the same speed so time spent moving is the larger of x or y. On that measure, the total time moving is

$$\begin{aligned} \mathit{Move}_k &= 2\,\mathit{Move}_{k-1} + \mathit{MaxReIm}(b^k - \mathit{End}_{k-1}) \qquad \text{starting } \mathit{Move}_0 = 0 \\ &\quad \text{where } \mathit{MaxReIm}(z) = \max\left(|\operatorname{Re} z|, |\operatorname{Im} z|\right) \\ &= \tfrac{33}{17}2^k \; - [2, \tfrac{14}{5}].2^{\lfloor k/2 \rfloor} \; + \tfrac{1}{85}[5, -7, 20, 6,\;\; -5, 7, -20, -6] \qquad (464) \\ &= 0, 1, 4, 10, 23, 51, 108, 226, 465, 949, 1924, 3886, 7823, \dots \end{aligned}$$

So there is factor approaching $\frac{33}{17} = 1.941\dots$ of movement on top of the 2^k drawing, so nearly twice as much moving as drawing.

Figure 110 shows arcs where the level k=1 move goes across the line segment it is about to draw inwards. That duplication going along and back can be avoided by stopping the recursion at k=1 and drawing an L shape. Movement in the higher levels remains but reduces to factor approaching $\frac{45}{34} = 1.323\dots$.

The pen-up moves are similar to *PlusJump* and *MinusJump* of complex base $i{\pm}1$ from section 11.1 and section 11.2. If the moves here are measured by straight line distance then the mean of the $2^k{-}1$ many total moves is

$$\begin{aligned} \mathit{MoveDist}_k &= \sum_{j=0}^{k-1} 2^j \left| \mathit{MoveEnd}_{k-j} \right| \\ &= 0,\ 1,\ 2{+}\sqrt{5},\ 4{+}3\sqrt{5},\ 8{+}6\sqrt{5}{+}\sqrt{13},\ \dots \end{aligned}$$

$$\begin{aligned} \frac{\mathit{MoveDist}_k}{2^k{-}1} &= \frac{2^k}{2^k{-}1} \cdot \sum_{j=1}^{k} \left| \frac{(4{-}2i)/5}{b^j} + \frac{(1{+}2i)/5}{(-2i)^j} \right| \\ &= \frac{2^k}{2^k{-}1} \cdot \frac{2}{\sqrt{5}} \sum_{j=1}^{k} \frac{1}{\sqrt{2}^j} \sqrt{1 + \frac{[0,-1,1,-1,\,0,1,-1,1]}{2^{\lceil j/2 \rceil}} + \tfrac{1}{4}\frac{1}{2^j}} \qquad (465) \end{aligned}$$

$$\to 2.109117\ldots \quad \text{as } k{\to}\infty$$

This is bigger than base $i{+}1$ mean but smaller than base $i{-}1$ mean. The limit is the same if taken over 2^k (instead of $2^k{-}1$) to give fraction of the total curve distance like *MaxReIm* above.

The limit is bigger than $\frac{33}{17}$ from (464) since that is the longer leg of each x,y move triangle whereas here the straight line is the hypotenuse.

The root part in the sum (465) is rational when $j \equiv 2 \bmod 4$, and otherwise sometimes not.

18.4 Convex Hull Clip

The drawing methods above which recurse top-down can apply a clip region by comparing it to the convex hull of the prospective level. If the hull is entirely outside the desired clip then that curve sub-part can be skipped. If the hull is entirely inside the clip then all of that sub-part is to be drawn with no further clip testing.

Taking a rectangle around the curve may be easier to check against a clip than the convex hull shape. Such a rectangle is a little bigger than the hull and results in a little more recursing. In general, any shape enclosing the curve could be compared to the clip and the choice is a complex shape fitting the curve more closely but slower compare, versus a bigger simple shape with faster compare but more recursing.

An example of a simple shape is to notice that for either start or end the curve is at most $1{+}\frac{1}{3}$ in any x or y direction. So at the start of a recursion if the sub-curve x,y location is outside a clip by more than $1{+}\frac{1}{3}$ in either coordinate then that sub-part can be skipped. The effective shape is a square centred on start or end, so is much bigger than the actual curve, but no attention is needed to sub-curve orientation and direction forward/reverse.

If the curve might not be x,y aligned then its maximum from start or end at any angle is the bigger of *Sdist* and *Edist* from (216) in section 7.

$$\frac{\max(\mathit{Sdist}_k, \mathit{Edist}_k)}{\sqrt{2}^k} = \frac{\mathit{Edist}_k}{\sqrt{2}^k} \to \tfrac{1}{3}\sqrt{17} \;\; = 1.374368\ldots$$

The test at each recursion would be for either x or y coordinate more than this far outside the clip. The effective shape is a circle of this radius centred at start or end.

This sort of clipping can also be used in searches for some desired n in a given region (for example the minimum and maximum n occurring). Take the region as a "clip" and recurse down until a desired result is found. If a sub-part is entirely outside the clip then back-track to a higher level and search the second sub-part there.

18.5 Predicate

The *unpoint* of section 1.5 which determines n and arm at location $x{+}iy$ can be used as a predicate to decide whether to draw that point, and/or draw segments before and after it.

This suits panning or zooming to draw a particular region, perhaps interactively, without traversing or recursing through the whole curve. For example figure 106 was generated that way, being a few rows each side of a certain 112 bit x location.

The clip region methods above are likely to be faster overall, but the predicate has flexibility to draw in any sequence. For example a dot-matrix printer might require 8 or 16 dot columns across a row. Those dots can be tested and sent in that sequence with no memory etc needed for a full image.

Rollins [47] gave disk paging code to create and print images bigger than RAM on the TRS-80, including some dragon curve variations. Such a paging scheme is necessary for fully general drawing, but for just the dragon curve the predicate method can do direct printer output.

A predicate can also do progressive on-screen drawing, expanding out from the centre so as to show the part in the user's eye-line first. A square or diamond growing outwards is an easy loop. A diamond has the attraction of doing more of the middle before the corners. An expanding octagon would be possible too. In each case, on hitting the screen edge (or clip box) the loop can jump ahead to its next side.

18.6 Lindenmayer System

A Lindenmayer system (L-system) is a set of rules for expanding a symbol string which is then interpreted as instructions to go forward, turn left or right, etc.

The dragon curve can be expressed with two symbols. Both are draw forward and the "+" or "−" in between are turns by +90° or −90° respectively (or arbitrary turn angle). F is an even number segment and S is an odd number segment.

starting F
F → F + S
S → F − S

Some computer programs for L-systems have only a single drawing symbol and all others non-drawing. Non-drawing symbols are used to represent a state or location type and expansions of those symbols produce necessary actual drawing. The Fractint/Xfractint program is an example of this. Its included dragon curve rule by Adrian Mariano has symbols X and Y to represent even or odd locations in the curve (respectively).

starting X F
X → X − Y F −
Y → + F X + Y

F=draw forward, X,Y=noop

The curve boundary can be generated using the *RQsides* squares types of figure 20. Here again F is the sole drawing symbol,

starting RF

R → R F + L	type 1e
L → T F − F	type 1o
T → R F + U F −	type 2
U → V F − F − F +	type 3o
V → T F − F + V	types 3e,3o

Each symbol is the start of its square type and has the drawing for its sides following. V is two squares 3e followed by 3o. This is done since just 3e would have to reduce to 2 sides on its next expansion, so instead together 3e,3o → 2,3e,3o which is a new 2 preceding.

Other starting strings for boundary forms include,

RF++LF	whole curve boundary
RF+LF+RF+LF	4 sides for twindragon boundary
RF+RF	back-to-back for twindragon

The *Rpred* segment expansions of theorem 21 give an L-system too, with turns per the directions shown there. Like in *Rpred*, segment types 2eB and 3oC can be treated together for drawing, for 9 segment types. Segment 3eB is removed entirely (becomes non-boundary) on expansion, so for a one drawing symbol system it's necessary to F → (empty) to discard the drawing of the previous level, and re-insert Fs as appropriate after each new symbol.

An L-system can also have symbols which are fixed drawing directions, instead of making turns between. McWorter and Tazelaar[35] gave code in Basic for such a system.

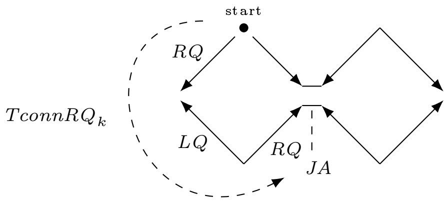
Their dragon curve rule has 4 symbols North, South, East, West. The curve always turns left or right so every East or West segment is even and expands on the right, and every North or South segment is odd and expands on the left.

Their "dragon interior" rule is one side of the twindragon area tree (here section 15.3) and effectively works by following the dragon curve and drawing between unit squares on the left of the curve. This is an implementation of Mandelbrot's description of following the river bank upstream and around. Their figure B is k=9 twindragon area tree.

Their "dragon boundary" rule is the complex base i+1 shape drawn in the manner of Dekking[15, section 4.4]. The states are N,S,E,W segments of 3 types each representing a configuration of neighbours for a total 12 states.

6 states of a, b and c unit sides from section 11.3 is also possible, starting from a 2×1 block. But then the starting point rotates around whereas with 12 states the start is fixed and each expansion shares a prefix with its successor, up to where the new second copy of the shape touches.

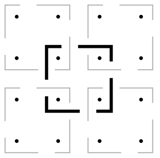
The length of that prefix can be calculated from how many boundary squares there are up to the twindragon connection point. Each boundary square corresponds to a complex base boundary corner and between each of those corners is a unit side of a boundary block.



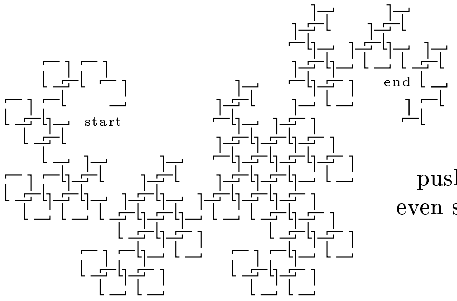
$$\begin{aligned} TconnRQ_k &= RQ_k + LQ_k + RQ_k - JA_k \\ &= BQ_k + JA_{k+2} + 1 \\ &= 3, 5, 8, 13, 22, 37, 62, 105, 178, 301, 510, \ldots \end{aligned}$$

L-system evaluation is often described in terms of full expansion of a list of symbols at each stage, producing potentially a very long string. But expansions can be done recursively and drawing performed on reaching the desired level. Any such recursion can be made iterative with an array of where and what expansion is in progress at each level.

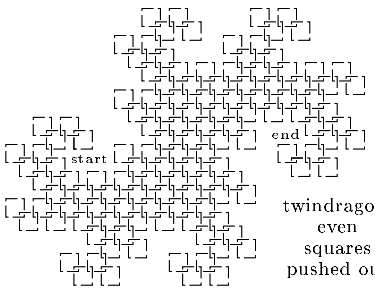
McWorter and Tazelaar also draw L-systems with little L shapes instead of line segments. The effect is to push out the sides of squares giving an interlocking appearance.



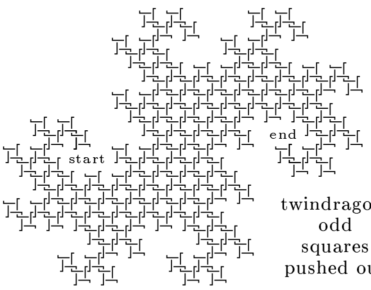
The sides can be pushed out from either odd or even squares (those with odd or even lower left corner $x+y$). The dragon curve turns 90° at each segment so the even squares are on the left of the curve and pushing them out gives a smooth right side.



push out
even squares



twindragon,
even
squares
pushed out



twindragon,
odd
squares
pushed out

18.7 Twindragon Skin Drawing

The twindragon skin of Mandelbrot (as from page 170 here) can be used to draw an approximation to the twindragon fractal.

```
DragonSkin($start$, $end$)
  if $|start - end| \le resolution$ then draw $start$ to $end$
  else
    $left = start + \frac{1}{4}(1+i)(end - start)$
    $right = end - \frac{1}{4}(1+i)(end - start)$
```

```
        DragonSkin($start$, $left$)
        DragonSkin($left$, $right$)
        DragonSkin($right$, $end$)
```

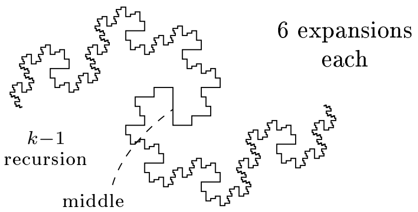
An expansion level parameter can limit the recursion instead of testing lengths. Then for example level 10 starting DragonSkin$(10, 0, 1)$.

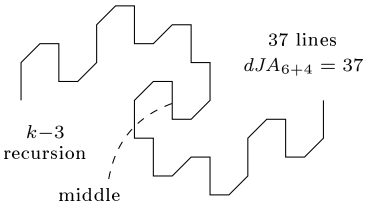
```
DragonSkin($k$, $start$, $end$)
    if $k \le 0$ then draw $start$ to $end$
    else
        $left = start + \frac{1}{4}(1{+}i)\,(end - start)$
        $right = end - \frac{1}{4}(1{+}i)\,(end - start)$
        DragonSkin($k-3$, $start$, $left$)                  (466)
        DragonSkin($k-1$, $left$, $right$)
        DragonSkin($k-3$, $right$, $end$)                   (467)
```

Notice the short lines *start*, *left* and *right*, *end* are reckoned as 3 levels down since they are factor $1/(\sqrt{2})^3$ shorter than start to end.

Both termination conditions result in up to 3 sizes of final lines, in ratios 1, $1/\sqrt{2}$ and $1/2$. The total number of lines goes as the dragon curve recurrence, $C_k = C_{k-1} + 2C_{k-3}$ starting $C_0 = C_{-1} = C_{-2} = 1$ which is a single line when $k \le 0$. This is $C_k = dJA_{k+4}$ from (163).

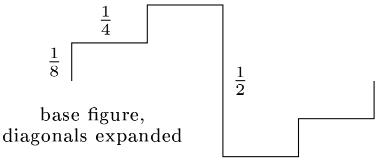
Any line expansion where parts of the expansion are different sizes has a choice whether to expand each to the same depth, or go in proportion to the lengths. Here, if the smallest line is reckoned just 1 expansion level, so $k{-}1$ in each recursion (466),(467), then the effect is to go to much smaller lines in some parts of the skin. This can be seen in a Logo example by Mike Horney [26].

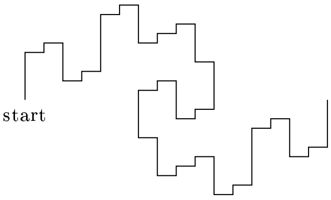




Notice the middle line is the same length in each form, but the end parts of $k{-}1$ have gone down to much smaller lines. Chang and Zhang [9] do similar (their DragonBorder2) for a boundary part corresponding to a join area section. The effect is interesting, but the resulting line lengths are much smaller than necessary for screen or printed resolution and a target minimum line length is a more uniform appearance.

Each expansion alternates straight or diagonal lines so a net odd number of expansions of a particular part is a diagonal line, as seen in the $k{-}3$ form above. If those parts are taken one further level then the result is all horizontal and vertical and final lengths in ratios 1, 1/2, 1/4. The effect is the same as expanding a shape

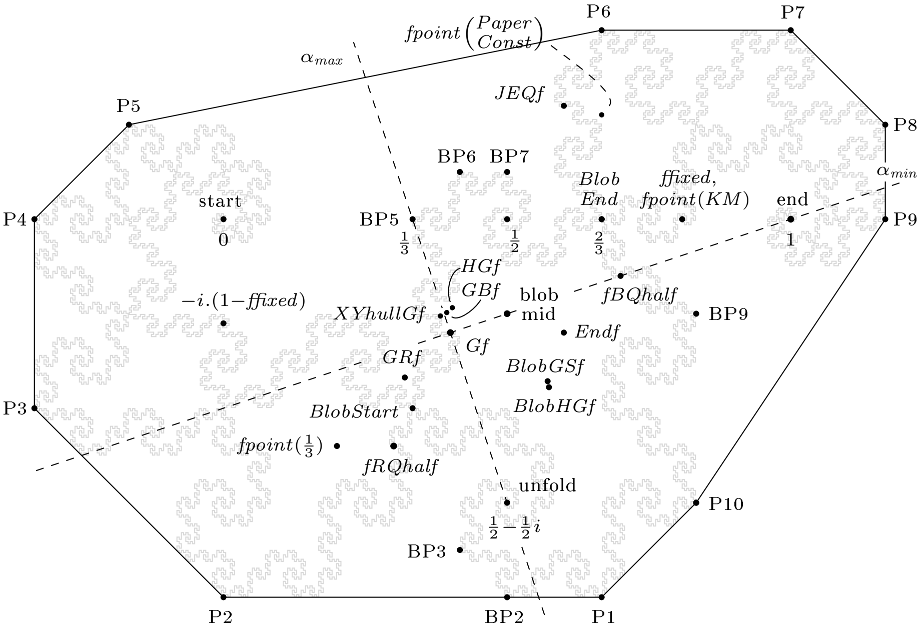




This is less smooth, but might suit a square grid of pixels or similar since all lengths are a multiple of the smallest. Note that the expansions give some consecutive line segments. For example on the right the line segment shown at “start” in the expansion is actually 2 separate segments for the purposes of subsequent expansion. It is the initial vertical $\frac{1}{8}$ of the base figure, and additional $\frac{1}{32}$ vertical from expansion of the $\frac{1}{4}$ horizontal.

19 Locations Summary

Various limit locations in the curve can be illustrated together,



$\mathit{fRQhalf}$ = right boundary half-way (130)
$\mathit{fBQhalf}$ = whole boundary half-way (131)
JEQf = join end (167)
Gf = segments centroid (229) $= \mathit{JEQf}/b$ halves join end
GBf = boundary segments centroid (239)
GRf = right boundary segments centroid (238)
$\alpha_{min}, \alpha_{max}$ = inertia principal axes, (252)
$\mathit{P1}, \ldots$ = convex hull vertices, section 7
HGf = hull centroid (231)
$\mathit{XYhullGf}$ = XY hull centroid (232)

$\mathit{BlobStart}, \mathit{BlobEnd} =$ blob start/end locations (305),(306)
$\mathit{BP1}, \ldots =$ blob convex hull vertices, section 12.8
$\mathit{BlobGSf} =$ blob segments centroid (353)
$\mathit{BlobHGf} =$ blob hull centroid (355)
$\mathit{fpoint}(\mathit{PaperConst})$ at (402)
$\mathit{fpoint}(\mathit{KM})$ at (401),(426)
$\mathit{ffixed} =$ fixed point, theorem 114, close to $\mathit{fpoint}(\mathit{KM})$ but not the same
$1 - \mathit{ffixed} =$ rotate $-i$ fixed point, theorem 120
$\mathit{Endf} =$ Gray code drawing end (463) $= \mathit{fpoint}(\frac{2}{3})$
$\mathit{fpoint}(\frac{1}{3})$ example (400) and 15ths (410)

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